Tr. J. of Mathematics
20 (1996) , 389 – 394.

WEAK FORMS OF OPENNESS BASED UPON DENSENESS

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Abstract

A function is defined to be hardly open provided that the inverse image of each dense subset of the codomain that is contained in a proper open set is dense in the domain. This form of weak openness is shown to be strictly between near feeble openness and somewhat openness. Characterizations and properties of hardly open functions are presented.

Key Words and Phrases. hardly open, nearly feebly open, somewhat open. 1991 Mathematics Subject Classification. 54 C 10.

1. Introduction

The fact that open mappings inversely preserve dense sets has recently been used to develop weak forms of openness. In [2] Frolik developed feebly open functions which are surjections that inversely preserve dense sets. Gentry and Hoyle [3] dropped the surjective requirement of feeble openness and studied the resulting condition under the name of somewhat openness. Recently Jankovic and Konstadilaki-Savvopoulou [3] introduced the notion of nearly feebly open functions which are characterized by having dense inverse images of open dense sets. The purpose of this paper is to introduce a weak form of openness based upon denseness that is between somewhat openness and near feeble openness.

2. Preliminaries

Throughout this paper X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. For a subset A of a space X, the closure, interior and boundary of A are signified by $Cl(A), Int(A)$, and $Bd(A)$, respectively. A set A is called condense (nowhere dense) provided that $Int(A) = \phi(Int(Cl(A)) = \phi)$ and A is said to be semi-open [4] if $A \subseteq CI(Int(A))$.

Definition 1. *[3]. A function $f : X \to Y$ is said to be somewhat open provided that if U is a nonempty open subset of X, then there is a nonempty open subset V of Y such that $V \subseteq f(U)$.*

We also shall use the following obvious characterization of somewhat openness.

Theorem 1. *A function $f : X \to Y$ is somewhat open if and only if for every $A \subseteq X, Int(A) \neq \phi$ implies that Int $(f(A)) \neq \phi$.*

Definition 2. *[3]. A function $f : X \to Y$ is said to be nearly feebly open if $Int(Cl(f(U))) \neq \phi$ for every open subset U of X.*

3. Hardly Open Functions

In [3] Gentry and Hoyle showed that a function $f : X \to Y$ is somewhat open if and only if for each dense subset A of Y, $f^{-1}(A)$ is dense X. Similarly, Jankovic and Konstadilaki-Savvopoulou [4] characterized nearly feebly open functions as those functions for which inverse images of open dense sets are dense. With these characterizations in mind we make the following definition.

Definition 3. *A function $f : X \to Y$ is said to be hardly open provided that for each dense subset A of Y that is contained in a proper open set, $f^{-1}(A)$ is dense in X.*

Obviously hardly openness is between somewhat openness and near feeble openness. The following examples show that it is strictly between these two conditions.

Example 1.

Let X be any set with two or more points , $\mathcal{D}$, the discrete topology on X, and I, the indiscrete topoloy on X. Since there are no proper open subsets of (X, I), the identity mapping $f : (X, D) \to (X, I)$ is vacuously hardly open. However, since f^{-1} does not preserve dense sets, f is not somewhat open.

Example 2.

Let $X = \{a, b, c\}$ have the topology $\mathcal{T} = \{\mathcal{X}, \phi\{\dashv, \lfloor\}\}$ and let $f : X \to X$ be given by $f(a) = c, f(b) = b$, and $f(c) = a$. Since f^{-1} preserves the denseness of $\{a, b\}$, the only proper open dense set, f is nearly feebly open. However, since $f^{-1}(\{a\})$ fails to be dense, f is not hardly open.

For T_1-spaces every proper set is contained in a proper open set. Therefore we have the following result.

Theorem 2. *If Y is a T_1-space, then a function $f : X \to Y$ is hardly open if and only if it is somewhat open.*

Theorem 2 can be strengthened by replacing the T_1-requirement with the following weaker condition.

Definition 4. *[6]. A subset A of a space X is said to be g-closed provided that $Cl(A) \subseteq U$ whenever U is open and $A \subseteq U$. A space X is called a $T_{\frac{1}{2}}$ -space if every g-closed set is closed.*

The following result due to Dunham [1] provides a useful characterization of $T_{\frac{1}{2}}$ -spaces.

Theorem 3. *[1]. A space X is a $T_{\frac{1}{2}}$ -space if and only if every singleton set is either open or closed.*

Theorem 4. *If Y is a $T_{\frac{1}{2}}$ -space, then a function $f : X \to Y$ is hardly open if and only if it is somewhat open.*

Proof. Assume $f : X \to Y$ is hardly open. Let A be a dense subset of Y. Suppose $y \in Y - A$. Since A is dense, $Int(Y - A) = \phi$ and therefore $\{y\}$ is not open. Since Y is a $T_{\frac{1}{2}}$ -space, $\{y\}$ is closed. Therefore A is contained in the proper open subset $Y - \{y\}$ and, since f is hardly open, $f^{-1}(A)$ is dense in X. □

3. Characterizations of Hardly Open Functions

The first theorem of this section characterizes hardly open functions in trems of codense sets.

Theorem 5. *A function $f : X \to Y$ is hardly open if and only if $Int(f^{-1}(A)) = \phi$ for each set $A \subseteq Y$ having the property that $Int(A) = \phi$ and A contains a nonempty closed set.*

Proof. Assume f is hardly open. Let $A \subseteq Y$ such that $Int(A) = \phi$ and let F be a nonempty closed set contained in A. Since $Int(A) = \phi, Y - A$ is dense in Y. Because $F \subseteq A, Y - A \subseteq Y - F \neq Y$. Therefore $f^{-1}(Y - A)$ is dense in X. Thus $X = Cl(f^{-1}(Y - A)) = Cl(X - f^{-1}(A)) = X - Int(f^{-1}(A))$ which implies that $Int(f^{-1}(A)) = \phi$.

For the converse implication assume that $Int(f^{-1}(A)) = \phi$ for every $A \subseteq Y$ having the property that $Int(A) = \phi$ and A contains a nonempty closed set. Let A be a dense subset of Y that is contained in the proper open set U. Then $Int(Y - A) = \phi$ and $\phi \neq Y - U \subseteq Y - A$. Thus $Y - A$ contains a nonempty closed set and hence $Int(f^{-1}(Y - A)) = \phi$. Then $Int(f^{-1}(Y - A)) = Int(X - f^{-1}(A)) = X - Cl(f^{-1}(A))$ and hence $f^{-1}(A)$ is dense in X. □

Corollary 1. *If $f : X \to Y$ is hardly open, then for each closed codense set A, $f^{-1}(A)$ is codense.*

Corollary 2. *If $f : X \to Y$ is hardly open, then for each open set $V \subseteq Y, Int(f^{-1}(Bd(V))) =$ ϕ.*

It seems reasonable that hardly open functions should be characterizable in terms of noncodense sets in much the same way as nearly feebly open functions and somewhat open functions. However, we shall see that this type of characterization holds only for surjective hardly open functions.

Theorem 6. *Let $f : X \to Y$ be a function. If $Int(f(A)) \neq \phi$ for every $A \subseteq X$ having the property that $Int(A) \neq \phi$ and there exists a nonempty closed set F for which $f^{-1}(F) \subseteq A$, then f is hardly open.*

Proof. Let $B \subseteq U \subseteq Y$ where B is dense and U is a proper open set. Let $A = f^{-1}(Y - B)$ and $F = Y - U$. Obviously $f^{-1}(F) = f^{-1}(Y - U) \subseteq f^{-1}(Y - B) = A$. Also $Int(f(A)) = Int(f(f^{-1}(Y - B))) \subseteq Int(Y - B) = \phi$. Therefore we must have that $\phi = Int(A) = Int(f^{-1}(Y - B)) = Int(X - f^{-1}(B))$ which implies that $f^{-1}(B)$ is dense. It follows that f is hardly open. □

Theorem 7. *If $f : X \to Y$ is hardly open, then $Int(f(A)) \neq \phi$ for every $A \subseteq X$ having the property that $Int(A) \neq \phi$ and f(A) contains a nonempty closed set.*

Proof. Let $A \subseteq X$ such that $Int(A) \neq \phi$ and let F be a nonempty closed set for which $F \subseteq f(A)$. Suppose $Int(f(A)) = \phi$. Then $Y - f(A)$ is dense in Y and $Y - f(A) \subseteq Y - F$ which is a proper open set. Since f is hardly open, $f^{-1}(Y - f(A))$ is dense in X. But $f^{-1}(Y - f(A)) = X - f^{-1}(f(A))$ and hence $Int(f^{-1}(f(A)) = \phi$ It follows that $Int(A) = \phi$ which is a contradiction. □

Theorems 6 and 7 are reversible provided that f is surjective. Thus we have the following characterizations for surjective hardly open functions.

Theorem 8. *If $f : X \to Y$ is surjective, then the following conditions are equivalent:*
(a) f is hardly open.
(b) $Int(f(A)) \neq \phi$ for all $A \subseteq X$ having the property that $Int(A) \neq \phi$ and there exists a nonempty closed set $F \subseteq Y$ such that $F \subseteq f(A)$.
(c) $Int(f(A)) \neq \phi$ for all $A \subseteq X$ having the property that $Int(A) \neq \phi$ and there exists a nonempty closed set $F \subseteq Y$ such that $f^{-1}(F) \subseteq A$.

Proof. (c) $\Rightarrow$ (a) Theorem 6.
(a) $\Rightarrow$ (b). Theorem 7.
(b) $\Rightarrow$ (c). Since f is surjective, $f^{-1}(F) \subseteq A$ implies that $F \subseteq f(A)$. □

The next theorem characterizes semi-continuous, surjective, hardly open functions. Recall that a function is semi-continuous [5] provided that the inverse image of each open set is semi-open.

Theorem 9. *If $f : X \to Y$ is a semi-condinuous surjection, then the following conditions are equivalent :*
(a) f is hardly open.
(b) If $A \subseteq Y$ contains a nonempty closed set and $Int(A) = \phi$, then $Int(f^{-1}(A)) = \phi$ and $f^{-1}(A)$ contains a nonempty nowhere dense set.

Proof. $(a) \Rightarrow (b)$. Let $A \subseteq Y$ such that $Int(A) = \phi$ and let F be a nonempty closed set such that $F \subseteq A$. Since f is semi-continuous, $f^{-1}(Y - F)$ is semi-open and hence $f^{-1}(Y - F) \subseteq Cl(Int(f^{-1}(Y - F)))$. Then $X - f^{-1}(F) \subseteq X - Int(Cl(f^{-1}(F)))$ and therefore $Int(Cl(f^{-1}(F))) \subseteq f^{-1}(F) \subseteq f^{-1}(A)$. Since f is hardly open, Theorem 5 implies that $Int(f^{-1}(A)) = \phi$ and therefore $Int(Cl(f^{-1}(F))) = \phi$. Thus $f^{-1}(F)$ is nowhere dense and since $F \neq \phi$ and f is surjective, $f^{-1}(F) \neq \phi$.

(b) $\Rightarrow$ (a). Follows from Theorem 5. □

4. Additional Properties

Theorem 10. *Let $f : X \to Y$ be a bijective hardly open function. If C is a closed subset of X such that f (C) is contained in a proper open set, then f(C) is contained in proper closed set.*

Proof. Let C be a closed subset of X and V, a proper open subset of Y for which $f(C) \subseteq V$. Since f is surjective, $\phi \neq Y - V \subseteq f(X - C)$. Thus $f(X - C)$ contains a nonempty closed set. Since $Int(X - C) \neq \phi$, Theorem 7 implies that $Int(f(X - C)) \neq \phi$. Since f is injective, $f(C) \subseteq Y - f(X - C) \subseteq Y - Int(f(X - C))$ and therefore $f(C)$ is contained in a proper closed set. □

The proofs of the next two theorems are straightforward and are omitted.

Theorem 11. *If $X = A \cup B$ and $f : X \to Y$ is a function for which both $f|A : A \to Y$ and $f|B : B \to Y$ are hardly open, then $f : X \to Y$ is hardly open.*

Theorem 12. *If $f : X \to Y$ is hardly open and W is an open subset of X, then $f/W : W \to Y$ is hardly open.*

The following example shows that the restriction of a hardly open function is not in general hardly open.

Example 3.

Let X be any space with a dense set B contained in a proper open set and A, any nonempty set disjoint from B. (For example, $X = \mathcal{R}, \mathcal{B} =$ the rationals, and $A=$ the irrationals.) The identity mapping $f : X \to X$ is hardly open (in fact open). However, $f|A : A \to X$ fails to be hardly open because $f|_A^{-1}(B) = B \cap A = \phi$ is certainly not dense in A.

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YOĞUNLUĞA BAĞLI ZAYIF AÇIKLIK KAVRAMLARI

Özet

Bir fonksiyona, eğer açık ve komplimanı boş olmayan açık kümeler içinde kalan ve yoğun bir kümenin ters görüntüsü olan kümeler o açık kümede yoğun ise, zayıf açık denir.

Bu makalede zayıf açık fonksiyonların çeşitli karakterizasyonları verilmiştir.

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Received 1.2.1995