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ON (σ, τ) DERIVATIONS WITH MODULE VALUES

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Abstract

Let R be a ring, $X \neq (0)$ an R-bi-module, $d : R \to X a(\sigma, \tau)$- derivation with module value such that $d\sigma = \sigma d, d\tau = \tau d$ and $U \neq (0)$ an ideal of R. Furthermore the following properties are also satisfied.

For $x \in X, a \in R \quad xRa = 0$ implies $x = 0$ or $a = 0 \ldots\ldots (G_1)$

For $a \in R, x \in X \quad aRx = 0$ implies $a = 0$ or $x = 0 \ldots\ldots (G_2)$

In this paper we have proved the following results; (1) If (G_1) (or (G_2)) is satisfied and for $a \in R, d(U)a = 0$ (or $ad(U) = 0$) then $d = 0$ or $a = 0$ (2) If (G_1) is satisfied and $[X, U] \subset C(X)$ or $[X, U]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ then R is commutative (3) Let X be $a2$-torsion free R-bi module, $d_1 : R \to X a(\sigma, \tau)$-derivation, $d_2 : R \to R$ a derivation such that $d_2(U) \subset U$. If (G_1) is satisfied and $d_1 d_2(U) = 0$ then $d_1 = 0$ or $d_2 = 0$ (4) Let X be $a2$-torsion free R-bi-module. If (G_1) and (G_2) are satisfied and for $a \in U, [d(U), a]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ then $a \in Z$ or $d = 0$.

1. Introduction

Let R be a ring, X be a left R-module and $d : R \to X$ an additive mapping. The definition of left derivation was given in [2] as fallows.

$$d(xy) = xd(y) + yd(x), \text{ for all } x, y \in R$$

Let X be an R-bi-module, $d : R \to X$ an additive mapping and σ, τ automorphisms of R. d is called (σ, τ)-derivation with module value, if for all $x, y \in R$

$$d(xy) = d(x)\sigma(y) + \tau(x)d(y)$$

The following results have been proved in ([6], [8], [1]) respectively.

(1) If Char $R \neq 2$ and $d \neq o$ is a derivation of R such that $d(R) \subset Z$ then R is commutative. (2) If a $\in R$ and ad(R)=0 then a=0 or d=0 (3) If $d \neq 0$ is a (σ, τ)-derivation of R, a $\in R$ and $[d(R), a]_{\sigma,\tau} \subset C_{\sigma,\tau}$ then a $\in Z$.

In this paper we have generalized the above reslts for a (σ,τ)- derivation with module value of R and an ideal of R.

Throughout R will represent a ring, X, R-bi module, U a nonzero ideal of R, $C(X) = \{x \in X \,|xr = rx, \forall r \in R\}$ and $C_{\sigma,\tau}(X) = \{x \in X|x\sigma(r) = \tau(r)x, \forall r \in R\}$. Further $d : R \to X$ will represent $a(\sigma,\tau)$-derivation and σ, τ automorphisims of R such that $d\sigma = \sigma d$, $d\tau = \tau d$. We shall often use the relations,

$$[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$$
$$[[x, y], z] = [x, [y, z]] + [y, [z, x]]$$
$$[[x, y]_{\sigma,\tau}, z]_{\sigma,\tau} = [x, [y, z]]_{\sigma,\tau} + [[x, z]_{\sigma,\tau}, y]_{\sigma,\tau}$$

Let us consider the follolwing properties.

If $x \in X, a \in R$ $\quad xRa = 0$ then $x = 0$ or $a = 0 \ldots\ldots (G_1)$

If $a \in X, x \in R$ $\quad aRx = 0$ then $a = 0$ or $x = 0 \ldots\ldots (G_1)$

2. Results

Lemma 1. *[3, Remark 5] let R be a ring and $X \neq (0)$ an R-bi-module. (i) If (G_1) is satisfied (or (G_2)) then R is prime. (ii) If (G_1) is satisfied (or (G_2)) and X is 2-torsion free R bi-module then R is too.*

Lemma 2 *Let R be a ring, $a \in R, X \neq (0)$ an R-bi- module and $U \neq (0)$ an ideal of R. If the condition (G_1) is satisfied (or (G_2)) and $xUa = 0$ (or $a\ Ux = 0$) for all $x \in X$ then $x = 0$ or $a = 0$.*

Proof. Let $xUa = 0$ for al $x \in X$. since U is an ideal of $R, xRUa \subset xUa = 0$ and so, $xRUa = 0$. It gives $x = 0$ or $Ua = 0$ by (G_1). since R is prime by lemma 1(i), we get $a = 0$ □

Lemma 3. *Let $U \neq (0)$ be an ideal of $R, X \neq (0)$ a 2- torsion free R-bi-module and (G_1) is satistied. (i) If $[X, U] \subset C(X)$ then R is commutative (ii) If $[X, U]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ then R is commutative*

Proof. (i) Let $[X, U] \subset C(X)$. Then, $0 = [[x, u], v]$ for all $u, v \in U, x \in X$. Using Jacobi identity and hypothesis we obtain $0 = [[x, u], v] = [x, [u, v]] + [u, [v, x]] = [x, [u, v]]$ that is

$$0 = [x, [u, v]], \forall x \in X, u, v \in U \ldots\ldots \qquad (1)$$

If we take $xr, r \in R$ instead of x in (1) and using (1) we obtain,

$$XR[R,[U,U]] = (0)\ldots\ldots \tag{2}$$

Using (G_1) in (2), since $X \neq (0)$, we obtain $[R,[U,U]] = 0$ that is $[U,U] \subset Z$. Thus by the [7, lemma 1.1.6] R is commutative.

(ii) $0 = [[x,u]_{\sigma,\tau}, v]_{\sigma,\tau} = [x,[u,v]]_{\sigma,\tau} + [[x,v]_{\sigma,\tau}, u]_{\sigma,\tau} = [x,[u,v]]_{\sigma,\tau}$ for all $x \in X$, $u, v \in U$. That is

$$[x,[u,v]]_{\sigma,\tau} = 0, \forall x \in X, u, v \in U\ldots\ldots \tag{3}$$

Replacing, x by $xr, r \in R$ in (3) we get, $x[r, \sigma([u,v]) = 0, \forall x \in X, r \in R, u, v \in U$.

Again, if we take $rs, s \in R$ instead of r, in the last equation we get

$$XR[R,\sigma([U,U])] = (0)\ldots\ldots \tag{4}$$

Using (G_1) in (4) $[U,U] \subset Z$ is obtained. Thus, R is commutative by [7, lemma 1.4] and [5, lemma 1.1.6] respectively. □

Lemma 4. *Let R be a ring, $U \neq (0)$ an ideal of $R, X \neq (0)$ an R-bi-modul $d : R \to Xa(\sigma,\tau)$-derivation and (G_1) is satisfied. (i) If $d(U) = 0$ then $d = 0$ (ii) If for $a \in R\, d(U)a = o$ then $a = 0$ or $d = 0$*

Proof. (i) By, the hypothesis $0 = d(ru) = d(r)\sigma(u) + \tau(r)d(u) = d(r)\sigma(u)$ for all $u \in U, r \in R$ that is,

$$d(r)\sigma(u) = 0, \forall r \in R, u \in U \tag{5}$$

If we replace r by $r\sigma^{-1}(s)$, $s \in R$ in (5) we get

$$d(R)R\sigma(U) = (0) \tag{6}$$

Since $U \neq (0)$, using (G_1) in (6), $d = 0$ is obtained.

(ii) By the hypohtesis $o = d(ru)a = d(r)\sigma(u)a + \tau(r)d(u)a = d(r)\sigma(u)$ a for all $u \in U, r \in R$ and so

$$d(R)\sigma(U)a = 0 \tag{7}$$

Since $\sigma(U) \neq (0)$ an ideal $a \in R$ using lemma 2, in (7) we get $a = 0$ or $d = 0$

If (G_2) is satisfied and ad(U)=0 then with same operations we obtain $a = 0$ or $d = 0$ □

Lemma 5. *Let R be a non-commutative ring, $X \neq (0)$ a 2- torsion-free R-bi-modul, $U \neq (0)$ an ideal of R and (G_1) is satisfied. If $d_1 : R \to X a(\sigma,\tau)$-derivation, $d_2 : R \to R$ a derivation such that $d_2(U) \subset U$ and $d_1 d_2(U) = 0$ then $d_1 = 0$ or $d_2 = 0$*

Proof. Let $d_1 d_2(U) = 0$ Then for all $u, v \in U$, $0 = d_1 d_2(uv) = d_1(d_2(u)v + ud_2(v)) = d_1 d_2(u)\sigma(v) + \tau(d_2(u))d_1(v) + d_1(u)\sigma(d_2(v)) + \tau(u)d_1 d_2(u) = \tau(d_2(u))d_1(v) + d_1(u)\sigma(d_2(v))$, that is

$$\tau(d_2(u))d_1(v) + d_1(u)\sigma(d_2(v)) = 0, \forall u, v \in U \tag{8}$$

If we replace v by $d_2(v)$ in (8) we get

$$d_1(U)\sigma(d_2^2(U)) = 0 \tag{9}$$

We use Lemma 4 (ii) in (9) to get $d_1 = 0$ or $d_2^2(U) = 0$ If $d_2^2(U) = 0$ then $d_2 = 0$, by [4, Theorem 1] □

Lemma 6. *Let R be a non-commutative ring, $X \neq (0)$ a 2- torsion-free R-bi-module, $U \neq (0)$ an ideal of R and (G_1) is satisfied. If $d : R \to X a(\sigma,\tau)$-derivation, $a \in R$ and $[d(U), a]_{\sigma,\tau} = 0$ than $a \in Z$ or $d = 0$*

Proof. Let $[d(U), a]_{\sigma,\tau} = 0$ Then for all $u, v \in U$,

$$\begin{aligned} 0 &= [d(uv), a]_{\sigma,\tau} = [d(u)\sigma(v) + \tau(u)d(v), a]_{\sigma,\tau} \\ &= d(u)[\sigma(v), \sigma(a)] + [d(u), a]_{\sigma,\tau}\sigma(v) + \tau(u)[d(v), a]_{\sigma,\tau} + [\tau(u), \tau(a)]d(v) \\ &= d(u)\sigma([v, a]) + \tau([u, a])d(v). \text{ From the last equation we obtain} \end{aligned}$$

$$d(u)\sigma([v, a]) + \tau([u, a])d(v) = 0, \forall u, v \in U \tag{10}$$

If we replace u by au in (10) we obtain

$$d(a)\sigma(U)\sigma[U, a] = 0 \tag{11}$$

Using Lemma 2 in (11) we obtain $d(a) = o$ or $[U, a] = 0$. If $[U, a] = 0$ then $a \in Z$ by [7, lemma 1.4]. Let $d(a) = 0$. Then, $d([u, a]) = [d(u), a]_{\sigma,\tau} - [d(a), u]_{\sigma,\tau} = 0$ for all $u \in U$ and so

$$d([u, a]) = 0, \forall u \in U \tag{12}$$

Taking vw, $w \in U$ instead of v in (10) we get

$$d(u)\sigma(v)\sigma([w, a]) + \tau([u, a])\tau(v)d(w) = 0, \forall u, v, w \in U \tag{13}$$

If we take $[w,a]\in U$ instead of w in (13) and use (12) we obtain

$$d(U)\sigma(U)\sigma([[U,a],a])=0 \tag{14}$$

Using Lemma 2 in (14) we obtain $d(U)=0$ or $[[U,a],a]=0$ If $d(U)=0$ then by lemma 4 (i) we get $d=0$. Let $[a,[a,U]]=0$. Since the mapping $I_a:R\to R$ defined by $I_a(x)=[a,x]$, is an inner derivation and $I_a^2(U)=0$. It gives $a\in Z$ by [4, Theorem 4]. Thus we obtain $d=0$ or $a\in Z$ by (14). □

Remark Let R be a ring, $X\neq(0)$ an R-bi-bodule

(i) If $a\in R, b\in C_{\sigma,\tau}(X), ab\in C_{\sigma,\tau}(X)$ and (G_2) is satisfied then $a\in Z$ or $b=0$

(ii) If $a\in C_{\sigma,\tau}(X), ab\in C_{\sigma,\tau}(X)$ and (G_1) is satisfied then $a=0$ or $b\in Z$

Proof. (i) Since $b, ab\in C_{\sigma,\tau}(X)$, for all $r\in R, 0=ab\sigma(r)-\tau(r)ab=a\tau(r)b-\tau(r)ab=[a,\tau(r)]b$ that is,

$$[a,\tau(r)]b=0 \tag{15}$$

If we replace r by $rs, s\in R$ in (15) we get $[a,R]Rb=0$. Thus a $\in Z$ or $b=0$ by (G_2)

(ii) We denote this as in (i) □

Theorem 7. *Let R be a non-commutative ring, $U\neq(0)$ an ideal of $R, X\neq(0)a2$-torsion-free R-bi-module and $(G_1),(G_2)$ are satisfied. If $a\in U$ and $d:R\to X$ is a (σ,τ)- derivation such that $[d(U),a]_{\sigma,\tau}\subset C_{\sigma,\tau}(X)$ then $a\in Z$ or $d=0$*

Proof. By the hypothesis we obtained, $C_{\sigma,\tau}(X)\ni[d(a^2),a]_{\sigma,\tau}=[d(a)\sigma(a)+\tau(a)d(a),a]_{\sigma,\tau}$ $=[d(a),a]_{\sigma,\tau}\sigma(a)+\tau(a)[d(a),a]_{\sigma,\tau}$. Since $[d(a),a]_{\sigma,\tau}\in C_{\sigma\tau}(X)$ this implies that $[d(a),a]_{\sigma,\tau}$ $\sigma(a)=\tau(a)[d(a)]_{\sigma\tau}$ and so from above we get $2\tau(a)[d(a),a]_{\sigma,\tau}\in C_{\sigma,\tau}(X)$. On the other hand X was 2-torsion free and so we obtain $\tau(a)]d(a),a]_{\sigma,\tau}\in C_{\sigma\tau}(X)$. This implies that $a\in Z$ or $[d(a),a]_{\sigma\tau}=0$ by Remark (i). If $[d(a),a]_{\sigma,\tau}=0$, then $C_{\sigma,\tau}(X)\ni[d([a,u]),a]_{\sigma,\tau}=[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau}-[[d(u),a]_{\sigma,\tau},a]_{\sigma,\tau}=[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau}$ for all $u\in U$ that is

$$[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau}\in C_{\sigma,\tau}(X), \text{ for all } \in U \tag{16}$$

If we replace u by au in (17) and use $[d(a),a]_{\sigma,\tau}=0$ we get

$$\tau(a)[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau}\in C_{\sigma,\tau}(X), \forall u\in U \tag{17}$$

This implies that $a \in Z$ or $[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau} = 0, \forall u \in U$, by Remark (i). Let $[[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau} = 0, \forall u \in U$. Then $0 = [[d(a),u]_{\sigma,\tau},a]_{\sigma,\tau} = [d(a),\ [u,a]]_{\sigma,\tau} + [[d(a),a]_{\sigma,\tau},u]_{\sigma,\tau}[d(a),[u,a]]_{\sigma,\tau}$, for all $u \in U$ and so,

$$[d(a),[a,U]]_{\sigma,\tau} = 0 \tag{18}$$

Since $I_a : R \to R$ defined by $I_a(x) = [a,x]$ is an inner derivation such that $I_a(U) \subset U$ and $I_{d(a)} : R \to X$, defined by $r \to [d(a),r]_{\sigma,\tau}, \forall r \in R$ is a (σ,τ)- derivation with module value, (18) becomes $I_{d(a)}I_a(U) = 0$. It gives $I_{d(a)} = 0$ or $I_a = 0$ by lemma 5, that is a $\in Z$ or $d(a) \in C_{\sigma,\tau}(X)$. If $d(a) \in C_{\sigma,\tau}(X)$, $C_{\sigma,\tau}(X) \ni [d(au),a]_{\sigma,\tau} = [d(a)\sigma(u) + \tau(a)d(u),a]_{\sigma,\tau} = d(a)\sigma([u,a]) + \tau(a)[d(u),a]_{\sigma,\tau}$ for all $u \in U$ that is

$$d(a)\sigma([u,a]) + \tau(a)[d(u),a]_{\sigma,\tau} \in C_{\sigma,\tau}(X), \forall u \in U \tag{19}$$

Using the definition of $C_{\sigma,\tau}(X)$ in (19) we get $0 = [d(a)\sigma([u,a]) + \tau(a)[d(u),a]_{\sigma,\tau},a]_{\sigma,\tau}$ $= d(a)\sigma([[u,a],a]) + [d(a),a]_{\sigma,\tau}\sigma([u,a]) + \tau(a)[[d(u),a]_{\sigma,\tau},a]_{\sigma,\tau} = d(a)\sigma([[u,a],a])$, for all $u \in U$. This implies that, $d(a)\sigma([[u,a],a]) = 0$. If we use that $d(a) \in C_{\sigma,\tau}(X)$ and $\sigma : R \to R$ is onto we obtain

$$d(a)R\sigma([u,a],a]) = 0 \tag{20}$$

Using (G_1) in (20) we obtain $d(a) = 0$ or $[a,[a,u]] = 0$. If $[a,[a,u]] = 0$; then since $I_a : R \to R$, defined by, $I_a(x) = [a,x]$ is an inner derivation and $I_a^2(U) = 0$ by the last equation and so $a \in Z$ by [4, Theorem 4]. If $d(a) = 0$; then we obtain, by (19),

$$\tau(a)[d(u),a]_{\sigma,\tau} \in C_{\sigma,\tau}(X), \forall u \in U \tag{21}$$

If we apply remark (i) to (21) we obtain, $a \in Z$ or $[d(U),a]_{\sigma,\tau} = 0$. If $[d(U),a]_{\sigma,\tau} = 0$; then we get $a \in Z$ or $d = 0$ by lemma 6. Consequently $a \in Z$ or $d = 0$ is obtained. □

Result 8 *Let $0 \neq d : R \to X$ be a (σ,τ)- derivation, $X \neq (0)$ a 2-torsion-free R-bi-module and (G_1) is satisfied. If $[d(U),U]_{\sigma,\tau} = 0$ then R is commutative.*

Proof. Let $[d(U),U]_{\sigma,\tau} = 0$. using Lemma 6 we obtain $U \subset Z$ and so R is commutative by [5,lemma 1.1.6]. □

Results 9 *Let $0 \neq d : R \to X$ a (σ,τ)- derivation $X \neq (0)$ a 2-torsion-free R-bi-module and $(G_1),(G_2)$ are satisfied. If $[d(U),U]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ then R is commutative*

Proof. Let $[d(U),U]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$, then $U \subset Z$ by Theorem 7 and so R is commutative by [5,lemma 1.1.6] □

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MODÜL DEĞERLİ (σ,τ)-TÜREVLER ÜZERİNE

Özet

R bir halka, $X \neq (0)$ bir R-bi-modül, $U \neq (0)$, R'nin bir ideali σ,τ R nin iki otomorfizmi ve $d : R \to X d\sigma = \sigma d$, $d\tau = \tau d$ olacak şekilde bir modül değerli (σ,τ) türevi olsun. Ayrıca: $a \in R, x \in X$ ler için

$$xRa = 0 \text{ ise } x = 0 \text{ veya } a = 0 \ldots\ldots (G_1)$$
$$aRx = 0 \text{ ise } a = 0 \text{ veya } x = 0 \ldots\ldots (G_1)$$

özellikleri bulunsun. Bu makalede aşağıdaki sonuçlar ispatlanmıştır.
(1) (G_1) özelliği var ve $d(U)a = 0$ ise $a = 0$ veya $d = 0$ dır.
(2) (G_1) özelliği var ve $[X,U]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ ise R komütatiftir.
(3) $(G_1)(G_2)$ özellikleri var olsun. $X \neq (0)$, 2-torsion-free R-bi-modül ve $a \in U[d(U),a]_{\sigma,\tau} \subset C_{\sigma,\tau}(X)$ ise $a \in Z$ veya $d = 0$.

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