Tr. J. of Mathematics
20 (1996) , 463 – 465.

A NOTE ON GAMMA RINGS

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Abstract

Let M be a Γ-ring and D a non-zero left derivation on M. We show that if there exists an element m in M such that $D(m)$ is a right non-zero divisor, then M is commutative.

Introduction

Let M and Γ be additive groups. M is called a Γ-ring if the following conditions are satisfied.
For any $x, y, z \in M$ and $\alpha, \beta, \in \Gamma$,
(1) $x\alpha y \in M$,
(2) $(x+y)\alpha z = x\alpha z + y\alpha z,\ \ x(\alpha+\beta)z = x\alpha z + x\beta z$,
$x\alpha(y+z) = x\alpha y + x\alpha z$,
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

It was introduced by Nobusawa [1] and got some properties of Γ-ring. In this short note we show that a certain type of Γ-ring is commutative.

First, we define some notions. Let M be a Γ-ring and $D: M \to M$ be an additive map. D is called a *left derivation* if

$$D(x\alpha y) = x\alpha D(y) + y\alpha D(x) \quad (x, y \in M,\ \alpha \in \Gamma).$$

A Γ-ring M is called *commutative* if $x\alpha y = y\alpha x$ for any $x, y \in M$ and $\alpha \in \Gamma$. An element $m \in M$ is called a *right non-zero divisor* if $x\alpha m = 0$ implies $x = 0$ for any $\alpha \in \Gamma$. A left non-zero divisor is defined similarly.

Proposition *If there exists an element m in M such that $D(M)$ is a right non-zero divsor, then M is commutative.*

Proof. Let $a, b \in M$ and $\alpha \in \Gamma$ be arbitrary elements. First, we note that $D(a\alpha b) = D(b\alpha a)$. Since $D(m\alpha(a\alpha m)) = D((m\alpha a)\alpha m)$, we have

$$(m\alpha a - a\alpha m)\alpha D(m) = 0,$$

which shows $a\alpha m = m\alpha a$. Using this property, we get

$$D((a\alpha b)\alpha m) = D(a\alpha(b\alpha m)) - D((b\alpha m)\alpha a) = D((b\alpha a)\alpha m),$$

which implies

$$0 = D((a\alpha b - b\alpha a)\alpha m) = (a\alpha b - b\alpha a)\alpha D(m).$$

Therefore $a\alpha b = b\alpha a$.

We give some examples for Γ-rings with left non-zero divisors, right non-zero divisors and a left derivation. □

Example Let R be a ring with identity 1 and $M_{m,n}(R)$ the set of all $x \times n$-matrices with entries in R. Then it is easy to see that $M_{m,n}(R)$ is a $M_{n,m}(R)$-ring with respect to the matrix addition and multiplication I(cf. [1]). We set

$$M = \{(x,x)\} \subset M_{1,2}(R) \quad \text{and} \quad \Gamma = \{\begin{pmatrix} n\cdot 1 \\ 0 \end{pmatrix}\} \subset M_{2,1}(R),$$

where n is an integer. Then $M_{1,2}(R)$ is a Γ-ring and M is a Γ-subring. And if R is commutative, then M is commutative. Now we take R an integral domain of characteristic zero. Then $(1,y) \in M_{1,2}(R)$ is a left non-zero divisor and $M_{1,2}(R)$ has no right non-zero divisor. On the other hand, let

$$N = M_{2,1}(R) \text{ and } \Gamma = \{(n\cdot 1, 0)\}.$$

Then $\begin{pmatrix} 1 \\ y \end{pmatrix}$ is a right non-zero divisor in N. Finally we assume that $d : R \to R$ is a left derivation. The existence of left derivations on some ring was showed in [2]. Then the map $D : M \to M$ defined by $D((x,x)) = (d(x), d(x))$ is a left derivation on M. So many notions on the ring theory are extended to the Γ-ring and such examples exist.

References

[1] N. Nobusawa. On a generalization of the ring theory, Osaka J. Math. 1(1964), 81-89.

[2] M. Sapancı and A. Nakajima. A note on a left derivation. preprint.

GAMMA HALKALAR ÜZERİNE BİR NOT

Özet

M bir Gamma-halkası ve D, M üzerinde sıfırdan farklı bir sol türev olsun. Bu çalışmada D(m) bir sağ sıfırdan farklı bölen olacak şekilde M içinde bir m elemanı varsa o zaman M nin değişmeli olduğunu gösterdik. Buna uygun bir örnekte verdik.

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Received 9.11.1995