Tr. J. of Mathematics
20 (1996) , 467 – 472.

FIXED POINTS AND COMPLETENESS

Z. Liu

Abstract

We give five necessary and sufficient conditions for a metric space to be complete.
Key words and phrases. Completeness, fixed point, stationary point.

1. Introduction

Characterizations of metric completeness have received much attention in recent years. Hu [1] showed that a metric space is complete if and only if any Banach contraction on closed subsets thereof has a fixed point. Taskovic [5] also obtained a result similar to Hu using the notion of diametral ψ-contraction. Although Subrahmanyam [4] pointed out that one cannot claim that a metric space is complete if any Banach contraction on it has a fixed point, Zhang [7] proved that a metric space is complete if and only if each Kannan type contraction on it has a fixed point. Weston [6] established the following.

Theorem 1.1. *Let (X,d) be a metric space. Then (X,d) is complete if and only if every uniformuly continuous function $h : X \to [0,\infty)$ has a d-point x in X; that is, $hx - hy < d(x,y)$ for all y in $X - \{x\}$.*

Park and Kang [3] gave characterizations of the metric completeness using single valued mappings and Weston's result.

In this paper we exetend the results of Zhang [7] and Park and Kang [3] in two directions. We replace Kannan type contraction with more general conditions and replace single valued mappings with multivalued mappings. In Section 2 we obtain five necessary and sufficient conditions for a metric space to be complete.

Throughout this paper, ω, N and Z denote the sets of nonnegative integers, positive integers and integers, respectively. Let f be a self mapping of a metric space (X,d). For x, y in X and $A \subset X$, define $O(x,f) = \{f^n x : n \in \omega\}$, $O(x,y,f) = O(x,f) \cup O(y,f)$ and $\delta(A) = sup\{d(x,y) : x,y \in A\}$. 2^X denote the power set of X. Define a family of functions as follows:

$\Psi = \{\psi : \psi : [0,\infty) \to [0,\infty)$ is nondecreasing, continuous on the right and $\psi(t) < t$ for all $t > 0\}$.

It is easy to see that the following result holds.

Theorem 1.2. *If a sequence* $\{x_n\}_{n\in N} \subset [0,\infty)$ *satisfies that* $x_{n+1} \le \psi(x_n)$ *for all* n *in* N *and some* ψ *in* Ψ, *then* $x_n \to 0$ *as* $n \to \infty$.

2. Characterizations of Completeness

Our results are as follows:

Theorem 2.1. *For a metric space* (X,d), *the following statements are equivalent:*

(1) (X,d) *is complete;*

(2) If f *is a self mapping of* X *satisfying for every* x,y *in* X *and some* ψ *in* Ψ

$$d(fx,fy) \le \psi(\delta(O(x,y,f))), \qquad \delta(O(x,y,f)) < \infty \tag{a}$$

then f *has a fixed point;*

(3) If f *is a self mapping of* X *satisfying for all* x,y *in* X *and some* r *in [0,1)*

$$d(fx,fy) \le r\delta(O(x,y,f)), \qquad \delta(O(x,y,f)) < \infty \tag{b}$$

then f *has a fixed point.*

Proof. (1) $\Rightarrow$ (2) For x,y in X and n in N, let $x_n = f^n x, y_n = f^n y$. From (2), for $k,m \ge n$ in N we have $d(fx_k, fy_m) \le \psi(\delta(O(x_k,y_m,f)))$, which implies that $\delta(O(x_{n+1},y_{n+1},f)) \le \psi(\delta(O(x_n,y_n,f)))$. It follows from Theorem 1.2 that $\delta(O(x_n,y_n,f)) \to 0$ as $n\to\infty$. Therefore $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are Cauchy sequences. By completeness of X there is a point u in X such that $x_n \to u$ as $n\to\infty$. Note that

$$d(y_n,u) \le d(y_n,x_n) + d(x_n,u) \le \delta(O(x_n,y_n,f)) + d(x_n,u)$$

Consequently $y_n \to u$ as $n\to\infty$.

We now assert that $\delta(O(u,f)) = 0$. Otherwise $\delta(O(u,f)) > 0$. Then for any n, m in N, we have

$$d(f^n u, f^m u) \le \psi(\delta(O(f^{n-1}u, f^{m-1}u, f))) \le \psi(\delta(O(u,f)))$$

which implies that

$$\delta(O(fu,f)) \le \psi(\delta(O(u,f))) < \delta(O(u,f))$$

It follows that

$$\delta(O(u,f)) = \max\{sup\{d(u,f^m u) : m \in N\}, \delta(O(fu,f))\} = sup\{d(u,f^m u) : m\in N\} \tag{c}$$

In view of $\lim_{n\to\infty} x_n = u$, for every $\epsilon > 0$ there exists k in N such that $d(x_n,u) < \epsilon$ for $n \ge k$. Consequently, for each m in N and $n\ge k$ we have

$$\begin{aligned} d(u, f^m u) &\leq d(u, f^n x) + d(f^m u, f^n x) \leq \epsilon + \psi(\delta(O(f^{m-1}u, f^{n-1}x, f))) \\ &\leq \epsilon + \psi(\max\{2\epsilon, \delta(O(u,f)) + \epsilon\}) \end{aligned}$$

which implies that

$$sup\{d(u, f^m u) : m \in N\} \leq \epsilon + \psi(\max\{2\epsilon, \delta(O(u,f)) + \epsilon\})$$

Letting $\epsilon \to 0$, by the above inequality and (c) we have

$$\delta(O(u,f)) = sup\{d(u, f^m u) : m \in N\} \leq \psi(\delta(O(u,f))) < \delta(O(u,f))$$

which is impossible. Hence $\delta(O(u,f)) = 0$; i.e., u is a fixed point of f.

(2) $\Rightarrow$ (3) Take $\psi(t) = rt$ for all t in $[0, \infty)$.

(3) $\Rightarrow$ (1) Suppose that (X, d) is not complete. Let $X^\star$ be an isometric completion of X. Then there exists a Cauchy sequence $\{x_n\}_{n \in N} \subset X$ and a point u in $X^\star - X$ such that $x_n \to u$ as $n \to \infty$. Take $b = 1/5$ and $r = 1/2$. Define $D_n = \{x : x \in X$ and $d(x,u) \leq b^n\}$ for all n in Z. It is evident that $X = \cup_{n \in Z} D_n$ and that D_n is nonempty for each n in Z. Put $n(x) = \max\{n : x \in D_n\}$ for all x in X. Since $\lim_{i \to \infty} x_i = u$, for each n in N there exists a smallest $k(n)$ such that $x_i \in D_n$ for $i \geq k(n)$. Define a self mapping f on X by

$$fx = \begin{cases} x_{k(2)}, & \text{if } n(x) \leq 0 \\ x_{k(n(x)+2)}, & \text{if } n(x) > 0 \end{cases}$$

for each x in X. Obviously, f has no fixed point. Since $fX \subset D_1, fX$ is bounded. Note that

$$d(x, f^n x) \leq d(x, fx) + d(fx, f^n x) \leq d(x, fx) + \delta(fX)$$

for each x in X and each n in N. It follows that

$$\delta(O(x,y,f)) \leq d(x,y) + d(x,fx) + d(y,fy) + \delta(fX) < \infty$$

for all x, y in X. It is easy to verify that fx is in $D_{n(x)+2}$ for each x in X. This means that

$$d(fx, u) \leq b^{n(x)+2} \leq bd(x,u) \leq b[d(x,fx) + d(fx,u)]$$

It follows that for each x in X

$$d(fx,u) \leq b(1-b)^{-1} d(x, fx)$$

Consequently, for each x, y in X we have

$$\begin{aligned} d(fx, fy) &\leq d(fx, u) + d(u, fy) \leq b(1-b)^{-1}[d(x, fx) + d(y, fy)] \\ &\leq r \bullet \max\{d(x, fx), d(y, fy)\} \leq r\delta(O(x, y, f)) \end{aligned}$$

that is, f satisfies (b). By (3), f has a fixed point. This is a contradiction.

This completes the proof.

As a consequence of Theorem 2.1 we have □

Theorem 2.2. *If f is a self mapping of a complete metric space (X, d) satisfying (a) for all x, y in X and some ψ in Ψ, then f has a unique fixed point u and $\lim_{n\to\infty} f^n x = u$ for each x in X.*

Proof. It follows from the proof of Theorem 2.1 that for each x in X, there exists a fixed point u of f such that $f^n x \to u$ as $n \to \infty$. Condition (a) ensures that f has a unique fixed point. This completes the proof. □

In case $\psi(t) = rt$, for bounded (X, d), Theorem 2.2 is due to Ohta and Nikaido [2]. The example below demonstrates that our Theorem 2.2 essentially extends the result of Ohta and Nikaido [2].

Example Let $X = (-\infty, \infty)$ with the usual metric and take $\psi(t) = rt$ for t in $[0, \infty)$, where r is in $[0, 1)$. Define a self mapping f on X by $fx = rx$ if $x \geq 0$ and $fx = 0$ if $x < 0$. It is easily seen that for all x, y in X,

$$d(fx, fy) \leq rd(x, y) \leq r\delta(O(x, y, f)) \text{ and } \delta(O(x, y, f)) \leq |x| + |y| < \infty.$$

Hence the conditions of Theorem 2.2 are satisfied. But Theorem 1 of Ohta and Nikaido [2] is not applicable since X is unbounded.

Theorem 2.3. *For a metric space (X, d), (1) is eqivalent to each of the following:*

(4) For every mapping f of X into 2^X with a uniformly continuous function $h : X \to [0, \infty)$ such that, for each $x \in X - fx$, there exists $y \in X - \{x\}$ satisfying $d(x, y) \leq hx - hy$, f has a fixed point;

(5) For every mapping f of X into $2^X - \{\Phi\}$ with a uniformly continuous function $h : X \to [0, \infty)$ such that $d(x, y) \leq hx - hy$ for each $x \in X$ and each $y \in fx - \{x\}$, f has a stationary point w in X, that is, $fw = \{w\}$;

(6) For every mapping f of X into $2^X - \{\Phi\}$ with a uniformly continuous function $h : X \to [0, \infty)$ such that $d(x, y) \leq hx - hy$ for each $x \in X$ and each $y \in fx$, f has a stationary point w.

Proof. (1) $\Rightarrow$ (4) By Theorem 1.1, h has a d-point x in X. Suppose that $x \bar{\in} fx$. Then there exists $y \in X - \{x\}$ such that $d(x,y) \leq hx - hy < d(x,y)$, which is a contradiction. Therefore f has a fixed point.

(4) $\Rightarrow$ (5) Suppose that f has no stationary point; i.e., $fx - \{x\} \neq \Phi$ for all x in X. Take $gx = fx - \{x\}$. Then for each $x \in X - gx \subset X$ there exists $y \in gx - \{x\}$ satisfying $d(x,y) \leq hx - hy$. In view of (4), g has a fixed point w; that is, $w \in gw = fw - \{w\}$, which is impossible.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1) Suppose that h has no d-point. Then for each x in X, there exists $y \in X - \{x\}$ with $hx - hy \geq d(x,y)$. Define a map f of X into $2^X - \{\Phi\}$ by $fx = \{y : d(x,y) \leq hx - hy$ and $y \in X - \{x\}\}$. It follows from (6) that f has a sationary point x in X; i.e., $\{x\} = fx \subset X - \{x\}$. This is a contradiction. Hence h has a d-point. By Theorem 1.1, (X,d) is complete.

This completes the proof.

□

References

[1] Hu, T.K.: On a fixed point theorem for metric spaces. Amer. Math Monthly **74**, 436-437 (1967).

[2] Ohta, M., Nikaido, G.: Remarks on fixed point theorems in complete metric spaces. Math. Japonica **39**, 287-290 (1994).

[3] Park, S., Kang, B.G.: Generalizations of the Ekeland type variational principles. Chinese J. Math. (Taiwan) **21**, 313-325 (1993).

[4] Subrahmanyam, P.V.: Completeness and fixed-points. Monatsh. Math. **80**, 325-330 (1975).

[5] Taskovic, M.R.: A characterization of complete metric spaces. Math. Japonica **29**, 107-113 (1984).

[6] Weston, J.D.: A characterization of metric completeness. Proc. Amer. Math. Soc. **64**, 186-188 (1977).

[7] Zhang, C.: The generalized set-valued contraction and completeness of the metric space. Math. Japonica **35**, 111-118 (1990).

TAMLIK VE SABİT NOKTA TEOREMLERİ

Özet

Bu çalışmada bir metrik uzayın tam olabilmesi için beş gerek ve yeter koşul verilmiştir.

Z. LIU
Department of Mathematics,
Lianoning Normal University,
Dalian, Lianoning 116029 P.R.
CHINA

Received 26.5.1995
.. ....