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ON A DIFFERENTIAL SEQUENCE IN GEOMETRY

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Abstract

We construct an exact differential sequence which indicates certain relations between curvature, local flatness, torsion and simplicity of higher order connections. Our formulas are expressed explicity in terms of the Christoffel symbols of dual ε-connections.

1. Introduction

This note is the continuation of [7], [8] and [9]. In [7], we introduced in a completely elementary way a geometric object of order k and some differential expressions in terms of its components. The classical Christoffel symbols (CS) and the curvature tensor emerged from our expressions for $k = 2$. In [8], we showed that this geometric object is nothing but an ε-connection already defined by C. Ehresmann in 1956 in [2] in the case of semi-holonomic frame bundles and we gave the transformation rule of its components using the group operation of the jet group $GL_k(n,\mathbb{R})$. These objects are in one to one correspondence with linear connections, as shown in [4], [6], [12], [13]. We introduced in [8] also the dual object and two "differential operators". However, it has been communicated to us by the author of [10] that the operators in [8] are not well defined. In [9] we studied the relation of our formulas in [8] to the concepts of local flatness, curvature, torsion and simplicity of ε-connections. These concepts are used in relation to mechanics also in the recent works [1], [3] (see also the references therein). As an interesting fact, our mistake in [8] appears also in [3] in the framework of linear connections.

We construct in this note an exact differential sequence in terms of the differential expressions contained in [9], which also corrects the mistake in [8]. The exactness of this sequence indicates, in our opinion, that the interrelations between CS, local flatness, curvature, torsion and simplicity of higher order connections are rather intricated and not well understood. The sequence introduced here seems to be intimately related to the first nonlinear Spencer sequence defined in [11] and studied extensively, for instance, in [10]. We hope to clarify this relation in some future work.

2. Preliminaries

For more details on the notions used in this section, we refer the reader to [2], [6], [12], [13] and the references therein (where everything is done in terms of frame bundles).

Let M be a differentiable manifold of dimension n and $F^k(M) \to M$ be the coframe bundle of M of order k. The elements of $F^k(M)$ are k-jets of local diffeomorphisms with source in M and target at the origin of $\mathbb{R}^n$. The bundle $F^k(M) \to M$ is a left principle bundle with the structure group $GL_k(n,\mathbb{R})$, the jet group of order k. A dual ε-connection is a $GL_1(n,\mathbb{R})$ invariant section of $F^k(M) \to F^1(M)$, where we regard $GL_1(n,\mathbb{R})$ as a subgroup of $GL_k(n,\mathbb{R})$ by the canonical injection $GL_1(n,\mathbb{R}) \to (GL_1(n,\mathbb{R}), 0, \ldots, 0)$. ε- connections are first defined in [2] in the case of semi-holonomic frame bundles and studied further in [4], [6], [12], [13], [9], [3]. Let $(x^i, x^i_j, \ldots, x^i_{j_1 \ldots j_k})$ be local coordinates on $F^k(M)$ and Γ a dual ε-connection of order k. A straightforward computation using $GL_1(n,\mathbb{R})$ invariance shows that there exist functions $\Gamma^i_{j_1 j_2}(x), \ldots, \Gamma^i_{j_1 \ldots j_k}(x)$, called the CS of Γ, such that

$$\Gamma(x^a, x^a_b)^i_{j1 \ldots js} = x^i_m \Gamma^a_{j_1 \ldots j_s}(x^a) \quad 2 \le s \le k \tag{1}$$

The formulas (1) are contained in [12], p. 45 in the case of ε- connections. If $\nu = (j_1, \ldots, j_s)$ we will write (somewhat ambigiously) $|\nu| = s$ and $r\mu = (r, j_1, \ldots, j_s)$. Now (1) shows that the functions $\Gamma^i_\nu(x)$, $2 \le |\nu| \le k$ have a consistent transformation rule. It is straightforward to show that this rule is given by

$$\begin{aligned} &(\frac{\partial y^i}{\partial x^j}, 0, \ldots, 0) \bullet (\delta^i_j, \Gamma^i_{j_1 j_2}(x), \ldots, \Gamma^i_{j_1 \ldots j_k}(x)) \bullet \\ &(\frac{\partial x^i}{\partial y^j}, \ldots, \frac{\partial^k x^i}{\partial y^{j_1} \ldots \partial y^{j_k}}) = (\delta^i_j, \Gamma^i_{j_1 j_2}(y), \ldots, \Gamma^i_{j_1 \ldots j_k}(y)) \end{aligned} \tag{2}$$

where $\bullet$ denotes the group operation of $GL_k(n,\mathbb{R})$. Note that for $k = 2$, (2) gives the transformation rule of the classical CS. Now (2) implies the following

FACT: For $p \in M$, there exist local coordinates (x^i) around p such that the CS $\Gamma^i_\mu(x)$ of Γ vanish at p.

The above fact is mentioned in [6] for ε-connections. Following the classical terminology, we called such coordinates geodesic coordinates in [9].

Now let $\varepsilon^k(M) \to M$ be associated bundle of $F^k(M) \to M$ with respect to the right action given by (2). The CS become local coordinates on the natural bundle $\varepsilon^k(M) \to M$ and an ε-connection Γ becomes a section of $\varepsilon^k(M) \to M$. Note that CS considered in this note are symmetric which is essential for the above fact. However, this restriction can easily be removed using (2).

3. An exact differential sequence

It is easy to show that $\Gamma^i_\nu(x)$ appear linearly on the LHS of (2) and therefore $\varepsilon^k(M) \to M$ is an affine bundle (see step 2 in the proof of the Proposition below). Let $V(\varepsilon^k(M)) \to M$ denote the model vector bundle where V stands for vertical. For each open set $U \subseteq M$, consider the set $\Phi(U)$ of all diffeomorphisms $f : U \to \mathbb{R}^n$. Then $\{\Phi(U) : U \subseteq M\}$ defines a sheaf of sets on M which will be denoted by Diff $(M, \mathbb{R}^n)$. If $E \to M$ is a bundle, we will use the same notation E also for the sheaf of local sections of $E \to M$. We now have

Proposition There exists an exact differential sequence

$$\underset{(f^i)}{Diff(M,\mathbb{R}^n)} \xrightarrow{\ \rho\ } \underset{(\Gamma^i_\nu)}{\varepsilon^k(M)} \xrightarrow{\ D\ } T^*(M) \underset{(X^i_{r\mu})}{\otimes} V(\varepsilon^{k-1}(M)) \tag{3}$$

where $k \geq 3$ and the operators ρ, D are given locally by the formulas

$$\rho(f)^i_{j_1\ldots j_s} = g^u_a \partial_{j_1\ldots j_s} f^a \text{ where } g^i_m \partial_j f^m = \delta^i_j, \qquad 2 \leq s \leq k \tag{4}$$

$$D(\Gamma)^i_{rj_1\ldots j_t} = \partial_r \Gamma^i_{j_1\ldots j_t} + \Gamma^i_{ra}\Gamma^a_{j_1\ldots j_t} - \Gamma^i_{rj1\ldots jt} \tag{5}$$

where $2 \leq t \leq k-1$.

Using the above notation, we will define the formal symbol D_r by $D_r\Gamma^i_\mu = \partial_r\Gamma^i_\mu + \Gamma^i_{ra}\Gamma^a_\mu$ and rewrite (5) in the form $D(\Gamma)^i_{r\mu} = D_r\Gamma^i_\mu - \Gamma^i_{r\mu}, 2 \leq |\mu| \leq k-1$. Note that the operator ρ is well also for $k = 2$. This will be the case also for D if we ragard δ^i_j as CS as already suggested by (2). In this case it turns out that $D_r\delta^i_s = \Gamma^i_{rs}$ and $D(\Gamma) = 0$. However this point will not be important for our purpose here. Note that $D_{[r}\Gamma^i_{s]t}$ is the classical curvature tensor R^i_{trs} because $\Gamma^i_{[rs]t} = 0$.

We will carry out the proof of the proposition in 4 steps.

Step 1. ρ is well defined: Let (z) be local coordinates in $\mathbb{R}^n$ and (x), (y) be local coordinates on M. We need to show that if we substitute

$$\Gamma^i_{j_1\ldots j_s}(x) = \frac{\partial x^i}{\partial z^a}\frac{\partial^s z^a}{\partial x^{j_1}\ldots\partial x^{j_s}} \qquad 2 \leq s \leq k \tag{6}$$

into (2), then (2) becomes an identity. To see this, consider the group operation of $GL_k(n,\mathbb{R})$, that is, chain rule, which we will denote by $(\frac{\partial z}{\partial x})(\frac{\partial x}{\partial y}) = (\frac{\partial z}{\partial y})$. Let $p : GL_k(n,\mathbb{R}) \to GL_1(n,\mathbb{R})$ be the projection homomorphism. We have $p(\frac{\partial z}{\partial x})^{-1}(\frac{\partial z}{\partial x})(\frac{\partial z}{\partial y}) = p(\frac{\partial y}{\partial x})^{-1}p(\frac{\partial z}{\partial y})^{-1}(\frac{\partial z}{\partial y})$ or $p(\frac{\partial x}{\partial y})^{-1}p(\frac{\partial z}{\partial x})^{-1}(\frac{\partial z}{\partial x})(\frac{\partial x}{\partial y}) = p(\frac{\partial z}{\partial y})^{-1}(\frac{\partial z}{\partial y})$. Now it is easy to show that the components of $p(\frac{\partial z}{\partial x})^{-1}(\frac{\partial z}{\partial x})$ are the expressions on the RHS of (6), which establishes the claim.

As a very crucial observation, note that pointwise, $\Gamma^i_\nu(x)$ are always of the form (6) whereas (6) may not hold locally for some given Γ.

Step 2. D is well defined: We will write the above chain rule formula $(\frac{\partial z}{\partial y}) = (\frac{\partial z}{\partial x})(\frac{\partial x}{\partial y})$ in the form

$$y^i_\mu = x^i_\nu A^\nu_\mu \qquad 2 \leq |\mu| \leq k, \quad 1 \leq |\nu| \leq |\mu| \tag{7}$$

where we have written $(\frac{\partial z}{\partial y})^i_\mu = y^i_\mu, (\frac{\partial z}{\partial x})^i_\mu = x^i_\mu$ and A^ν_μ are expressions depending on $\frac{\partial x^i}{\partial y^j}, \dots, \frac{\partial^s x^i}{\partial y^{j_1} \dots y^{j_s}}, s = |\mu| - |\nu| + 1$. It is easy to show that

$$A^i_\mu = \frac{\partial^k x^i}{\partial y^{j_1} \dots \partial y^{j_k}} \qquad \text{for } |\mu| = k \tag{8}$$

In view of (2) and (7), we have

$$\Gamma^i_\mu(y) = \Gamma^a_\nu(x) \frac{\partial y^i}{\partial x^a} A^\nu_\mu + \frac{\partial y^i}{\partial x^a} \frac{\partial^{|\mu|} x^a}{\partial y^{j_1} \dots \partial y^{j_s}} \tag{9}$$

where $\mu = (j_1, \dots, j_s)$, $2 \leq |\mu| \leq k$ and, this time, $2 \leq |\nu| \leq |\mu|$ (see our remark on $\Gamma^i_j = \delta^i_j$ above).

We now assume that (x) is a geodesic coordinate system at some $p \in M$ and (y) is an arbitrary coordinate system around p. Differentiating (9) at p, we obtain

$$\begin{aligned} \partial_r \Gamma^i_\mu(y) &= \partial_a \Gamma^b_\nu(x) \frac{\partial x^a}{\partial y^r} \frac{\partial y^i}{\partial x^b} A^\nu_\mu + \frac{\partial^2 y^i}{\partial x^b \partial x^a} \frac{\partial x^b}{\partial y^r} \frac{\partial^{|\mu|} x^a}{\partial y^{j_1} \dots \partial y^{j_s}} + \\ & \frac{\partial y^i}{\partial x^a} \frac{\partial^{|\mu|+1} x^a}{\partial y^r \partial y^{j_1} \dots \partial y^{j_s}} \end{aligned} \tag{10}$$

It follows from (2) that we have $\Gamma^i_{j_1 \dots j_s}(y) = \frac{\partial y^i}{\partial x^a} \frac{\partial^s x^a}{\partial y^{j_1} \dots \partial y^{j_s}}$ at the point p, as can be seen by setting $\Gamma^i_{j_1 \dots j_t}(x) = 0$ in (2) for $2 \leq t \leq k$. Since $\frac{\partial^2 y^i}{\partial x^b \partial x^a} \frac{\partial x^b}{\partial y^r} = -\frac{\partial^2 x^c}{\partial y^r \partial y^b} \frac{\partial y^i}{\partial x^c} \frac{\partial y^b}{\partial x^a}$, substituting the last equality into (10), we find that the last two terms in (10) are $-\Gamma^i_{ra}(y)\Gamma^a_\mu(y) + \Gamma^i_{r\mu}(y)$ at p. Therefore (10) can now be written in the form

$$D_a \Gamma^i_\nu(y) - \Gamma^i_{r\mu}(y) = D_r \Gamma^b_\mu(x) - \Gamma^b_{a\nu}(x)) \frac{\partial x^a}{\partial y^r} \frac{\partial y^i}{\partial x^b} A^\nu_\mu \tag{11}$$

Omitting the last term on the RHS of (9), we get the transformation rules of the coordinates of the vector bundle $V(\varepsilon^k(M)) \to M$. Comparing these rules with (11), we see that the differential expressions $D_r \Gamma^i_\mu(x) - \Gamma^i_{r\mu}(x)$ transform from geodesic coordinates to arbitrary coordinates by the transition laws of the vector bundle $T^*(M) \otimes V(\varepsilon^{k-1}(M)) \to M$. Now let $(x; U)$ and $(y; V)$ be arbitrary coordinates neighbourhoods and $p \in U \cap V$. Choosing coordinates (z) which is geodesic at p, we have symbolically $(p; x) \leftarrow (p; z) \to (p; y)$. The above argument shows that each arrow in $(p; x) \leftarrow (p; z) \to (p; y)$ is induced by the transition laws of the bundle $T^*(M) \otimes V(\varepsilon^{k-1}(M))$. It follows that each arrow in $(p, x) \to (p; z) \to (p; y)$ and therefore also the the arrow in the composition $(p, x) \to (p, y)$ is induced by the transition laws of the bundle $T^*(M) \otimes V(\varepsilon^{k-1}(M))$, which establishes the claim.

Step 3. $Im(\rho) \subseteq Ker(D)$: This is an immediate consequence of the identity

$$\frac{\partial}{\partial x^r}\left(\frac{\partial x^i}{\partial z^a}\frac{\partial^s z^a}{\partial x^{j_1}\ldots\partial x^{j_s}}\right) = -\frac{\partial^2 z^a}{\partial x^r \partial x^b}\frac{\partial x^i}{\partial z^a}\frac{\partial x^b}{\partial z^c}\frac{\partial^s z^c}{\partial x^{j_i}\ldots\partial x^{j_s}} \tag{12}$$

$$\frac{\partial x^i}{\partial z^a}\frac{\partial^{s+1} z^a}{\partial x^r \partial x^{j_i}\ldots\partial x^{j_s}} \tag{13}$$

which becomes $\partial_r \Gamma^i_{j_1 \ldots j_s} = -\Gamma^i_{rb}\Gamma^b_{j_1\ldots j_s} + \Gamma^i_{rj_1\ldots j_s}$ when (6) is substituted.

This formal computation is contained in [7] and has been our starting point.

Step 4. Ker $(D) \subseteq Im(\rho)$: Suppose that $D(\Gamma) = 0$, that is,

$$D_j \Gamma^i_\mu - \Gamma^i_{j\mu} = 0 \qquad 2 \le |\mu| \le k-1 \tag{14}$$

We will show that there exists an invertible function $f = (f^i)$ such that $g^i_m \partial_\nu f^m = \Gamma^i_\nu, 2 \le |\nu| \le k$. Consider the following system of PDE

$$g^i_a \partial_j \partial_k f^a = \Gamma^i_{jk} \tag{15}$$

for the functions f^i where $g^i_s \partial_j f^s = \delta^i_j$. Now (14) is equivalent to the first order system

$$\partial_j f^i = f^i_j, \quad g^i_a \partial_j f^a_k = \Gamma^i_{jk} \tag{16}$$

for the functions f^i, f^i_j where $g^i_a f^a_j = \delta^i_j$. A straightforward (and classical) computation shows that the integrability conditions of (15) are given by $D_{[j}\Gamma^i_{k]m} = 0, \Gamma^i_{[jk]} = 0$, where the first one follows from (13) and the second one holds by the definition of $\varepsilon^k(M)$. Solving for f^i, f^i_j with arbitrary initial conditions we obtain (14) which starts the induction with $|\nu| = 2$. Assuming now we have $g^i_a \partial_\nu f^a = \Gamma^i_\nu$ for $|\nu| = k-1$, we differentiate the last equation which gives, after a straightforward computation in view of (13) and (14) $g^i_a \partial_{r\nu} f^a = D_r \Gamma^i_\nu = \Gamma^i_{r\nu}$, completing the inductive step and also the proof of the proposition.

Corollary Let $\Gamma \in \varepsilon^k(M), k \ge 3$. Then the following are equivalent:

1. $D(\Gamma) = 0$.
2. For an arbitrary coordinate system (x), we have $\Gamma^i_{j_1\ldots j_s}(x) = D_{j_1}D_{j_2}\ldots D_{j_{s-2}} \Gamma^i_{j_{s-1}j_s}(x), 3 \le s \le k$.
3. For $p \in M$, there exists a coordinate system $(U; x^i)$ around p such that $\Gamma^i_\nu(x)$ and $\frac{\partial \Gamma^i_\nu(x)}{\partial x^r}$ vanish at p.

4. For $p \in M$, there exists a coordinate system $(U; x^i)$ around p such that $\Gamma^i_\nu(x)$ vanish identically on U.

Note that on the level of Γ^i_{jk} the above corollary corresponds to a fundamental classical fact.

We would like to make now two brief remarks on the above Proposition.

1. As a peculiar fact, note that the image of D involves only 1-forms and not 2-forms.

2. If $D(\Gamma) = 0$, one is tempted to call Γ locally flat in view of 4 of the Corollary, but also simple in view of 2 of the Corollary (see [12], [13], [3], [5] for simplicity). However such names are inconsistent with their common usage within the framework of linear connections in view of Theorem II. 22 in [12] which states that a linear connections is locally flat if and only if it is simple and without torsion and curvature (see also [13]).

It also seems to us that there are some discrepancies between the above proposition and some assertions in [10] about the origin of the classical CS, but we will not enter into these matters here.

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GEOMETRİDE DİFERANSİYEL BİR ÇİZELGE ÜZERİNE

Özet

Yüksek mertebeli bağlantılarda yerel düzlük, büküm, burulma ve basitlik arasındaki bazı ilişkileri gösteren sağin bir çizelge inşa ediliyor. Kullanılan formüller ε-bağlantılarının Christoffel sembolleri cinsinden ifade ediliyorlar.

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