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ZEROS OF DERIVATIVES OF DIRICHLET L-FUNCTIONS*

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Abstract

In this paper diverse results on the location and number of zeros of derivatives of Dirichlet L-functions are proved.

1. Introduction

Many results concerning the zeros of derivatives of the Riemann zeta-function have accumulated over the last sixty years. The first study was by Speiser [7], showing that the Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < \frac{1}{2}$ (as usual $s = \sigma + it$). The works of Titchmarsh [12, Theorem 11.5(c)] and Spira [8],[9] demonstrated the existence of zero-free regions for $\zeta^{(k)}(s)$: A half-plane, $\sigma > \sigma_k$ $(2 < \sigma_1 < 3, \sigma_2 < 5, \sigma_k \leq \frac{7}{4}k + 2$ for $k \geq 3)$; a half-plane $\sigma < \alpha_k < 0$ where $\zeta^{(k)}(s)$ has only real zeros; in the strip $\alpha_k \leq \sigma < -\epsilon$ (for any $\epsilon > 0$) the set of points distant enough from the origin. Spira [8] also calculated the zeros of ζ' and ζ'' in the region $-1 \leq \sigma \leq 5, |t| \leq 100$. The zeros in Spira's list are all to the right of the critical line $\sigma = \frac{1}{2}$, except for a pair of zeros of $\zeta''(s)$ at approximately $-0.36 \pm 3.59i$. It is known after Berndt [1] that $\zeta^{(k)}(s)$ has $\frac{T}{2\pi}(\log \frac{T}{4\pi} - 1) + O(\log T)$ non-real zeros in $0 < t < T$, as $T \to \infty$, and after Spira [10] that most of these zeros lie in $0 \leq \sigma \leq \frac{1}{2} + \delta$ for $\delta > 0$. In a most comprehensive paper Levinson and Montgomery [5] obtained unconditional and conditional estimates as to how the zeros of $\zeta^{(k)}(s)$ are positioned with respect to the critical line. Further study in this direction was conducted by Conrey and Ghosh [3]. Levinson and Montgomery also showed, assuming RH, that $\zeta^{(k)}(s)$ has at most finitely many non-real zeros in $0 < \sigma < \frac{1}{2}$ (see also [13]). As the development of the theory of Dirichlet L-functions parallels that of the Riemann zeta-function, albeit with some differences, we have undertaken here to prove analogues of some of the results mentioned above for $L(s, \chi)$.

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Notation: Throughout this paper χ will be a primitive Dirichlet's character to the modulus q (therefore $q \geq 3$ and $q \not\equiv 2 \pmod 4$, see [11]) and

$$\mathfrak{a} = \begin{cases} 0 \text{ if } & \chi(-1) = 1, \\ 1 \text{ if } & \chi(-1) = -1. \end{cases}$$

Capital letters $C, D, ..$ will stand for fixed numbers
p : a prime number
ρ : a non-trivial zero of $L(s, \chi)$
$\rho_k = \beta_k + i\gamma_k$: any zero of $L^{(k)}(s, \chi)$, $k \geq 1$
γ : Euler's constant or $\Im\rho$ depending on the context.

2. Preliminaries on Dirichlet L-functions

In this section after recollecting the necessary basics (we refer the reader to Davenport's book [4] for an exposition of the theory of $\zeta(s)$ and $L(s, \chi)$) we shall present some lemmata. The Dirichlet L-function associated with χ is represented by the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 0) , \tag{1}$$

and satisfies the functional equation

$$L(1 - s, \chi) = \varepsilon(\chi)\, 2^{1-s}\, \pi^{-s}\, q^{s-\frac{1}{2}} \cos \frac{\pi}{2}(s - \mathfrak{a})\, \Gamma(s)\, L(s, \overline{\chi}), \tag{2}$$

where $|\varepsilon(\chi)| = 1$. Owing to the poles of the Γ factor in (2), $L(s, \chi)$ has so-called trivial zeros (all simple) at $0, -2, -4, ..$ if $\mathfrak{a} = 0$; at $-1, -3, -5, ..$ if $\mathfrak{a} = 1$. It is known that all other (i.e. non-trivial) zeros are in the critical strip $0 < \sigma < 1$. The Generalized Riemann Hypothesis states that all non-trivial zeros lie on $\sigma = \frac{1}{2}$. From the Weierstrass factorization of the entire function

$$\begin{aligned} \xi(s, \chi) &:= \left(\frac{q}{\pi}\right)^{\frac{s+\mathfrak{a}}{2}} \Gamma\left(\frac{s + \mathfrak{a}}{2}\right) L(s, \chi) \\ &= e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}, \end{aligned}$$

where $A = A(\chi)$ and $B = B(\chi)$, it follows that

$$\frac{L'}{L}(s, \chi) = -\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \frac{\Gamma'}{\Gamma}\left(\frac{s + \mathfrak{a}}{2}\right) + B(\chi) + \sum_{\rho} \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right). \tag{3}$$

On the other hand logarithmic differentiation in (2) gives

$$-\frac{L'}{L}(1-s,\chi)=\log\frac{q}{2\pi}-\frac{\pi}{2}\tan\frac{\pi}{2}(s-\mathfrak{a})+\frac{\Gamma'}{\Gamma}(s)+\frac{L'}{L}(s,\overline{\chi}). \tag{4}$$

One can obtain from the functional equation that

$$\Re B(\chi)=-\sum_{\rho}\Re\frac{1}{\rho}, \tag{5}$$

and, assuming GRH, (3) with $s=1$ gives

$$\Im B(\chi)=\Im\frac{L'}{L}(1,\chi). \tag{6}$$

The combination of (3) at $s=0$ and (4) at $s=1$ (if $\mathfrak{a}=0$ we consider $s\to 0$ in (3) and $s\to 1$ in (4)) leads to

$$B(\chi)=-\frac{1}{2}\log\frac{q}{\pi}+\frac{\gamma}{2}+(1-\mathfrak{a})\log 2-\frac{L'}{L}(1,\overline{\chi}). \tag{7}$$

Lemma 1. *Assuming GRH, as* $q\to\infty$,

$$\frac{L'}{L}(1,\chi)\ll\log\log q.$$

Proof.

$$-\frac{L'}{L}(1,\chi)=\sum_{n=2}^{\infty}\frac{\Lambda(n)\,\chi(n)}{n}=\sum_{n=2}^{N}+\sum_{n=N+1}^{\infty},$$

where N will be specified later. By the prime number theorem

$$\left|\sum_{n=2}^{N}\frac{\Lambda(n)\,\chi(n)}{n}\right|\le\sum_{n=2}^{N}\frac{\Lambda(n)}{n}=\sum_{p\le N}\frac{\log p}{p}+O(1)\ll\log N.$$

Applying partial summation to the consequence of GRH that

$$\sum_{n\le x}\Lambda(n)\,\chi(n)\ll x^{\frac{1}{2}}\log^2 qx,$$

we have

$$\sum_{n\ge N}\frac{\Lambda(n)\,\chi(n)}{n}\ll\frac{\log^2 qN}{\sqrt{N}}.$$

Choosing $N=\dfrac{\log^4 q}{(\log\log q)^2}$ we obtain the result. □

From Eqs. (4)-(7) we have

Corollary 1. *Upon GRH, as $q \to \infty$,*

$$\begin{aligned} B(\chi) &= -\frac{1}{2}\log q + O(\log\log q) \\ \Re B(\chi) &= -\sum_{\rho} \Re\frac{1}{\rho} = -\frac{1}{2}\log q + O(\log\log q) \\ \Im B(\chi) &= O(\log\log q) \\ \frac{L'}{L}(0,\chi) &= -\log q + O(\log\log q) \quad \textit{in case } \mathfrak{a} = 1. \end{aligned}$$

The proof of Lemma 1 may be carried out in a disk of radius $\ll \frac{1}{\log\log q}$ around $s = 1$, allowing us to state

Lemma 2. *Assuming GRH, as $q \to \infty$,*

$$\begin{aligned} \frac{L'}{L}(s,\chi) &\ll \log\log q, \quad \textit{for } |s-1| \le \tfrac{C}{\log q}, \\ \frac{L'}{L}(s,\chi) &\ll_D \log\log q, \quad \textit{for } |s-1| \le \tfrac{D}{\log\log q}. \end{aligned}$$

More generally, considering the Dirichlet series for $(\frac{L'}{L})^{(j-1)}, j \ge 1$, we have

Lemma 3. *Assuming GRH, as $q \to \infty$,*

$$\begin{aligned} \frac{L^{(j)}}{L}(s,\chi) &\ll_j (\log\log q)^j, \quad \textit{for } |s-1| \le \tfrac{C}{\log q}, \\ \frac{L^{(j)}}{L}(s,\chi) &\ll_{j,D} (\log\log q)^j, \quad \textit{for } |s-1| \le \tfrac{D}{\log\log q}. \end{aligned}$$

Lemma 4. *Upon GRH, for $k \ge 1$, as $q \to \infty$,*

$$\frac{L^{(k+1)}}{L^{(k)}}(0,\chi) = -(1 + \frac{1-\mathfrak{a}}{k})\log q + O_k(\log\log q).$$

Proof. By k-fold differentiation of both sides of (2)

$$(-1)^k L^{(k)}(1-s,\chi) = \frac{2\varepsilon(\chi)}{\sqrt{q}} \sum_{j=0}^{k} \binom{k}{j} \left\{ (\frac{q}{2\pi})^s \cos\frac{\pi}{2}(s-\mathfrak{a})\Gamma(s) \right\}^{(j)} L^{(k-j)}(s,\overline{\chi}). \quad (8)$$

We abbreviate $\log \frac{q}{2\pi} = \ell$. From (8) at $s = 1$, as $q \to \infty$,

$$(-1)^k L^{(k)}(0,\chi) = \begin{cases} \frac{\varepsilon(\chi)\sqrt{q}}{\pi} \sum_{j=0}^{k} \binom{k}{j} \ell^j (1 + O_j(\ell^{-1})) L^{(k-j)}(1,\overline{\chi}) & \text{if } \mathfrak{a} = 1, \\ \frac{-\varepsilon(\chi)\sqrt{q}}{2} \sum_{j=1}^{k} \binom{k}{j} j\ell^{j-1} (1 + O_j(\ell^{-1})) L^{(k-j)}(1,\overline{\chi}) & \text{if } \mathfrak{a} = 0. \end{cases}$$

This may be expressed in the form

$$(-1)^k L^{(k)}(0,\chi) = \begin{cases} \frac{\varepsilon(\chi)\sqrt{q}}{\pi} [(\ell + \frac{d}{ds})^k L(s,\overline{\chi})]_{s=1} (1 + O_k(\ell^{-1})) & \text{if } \mathfrak{a} = 1, \\ \frac{-\varepsilon(\chi)\sqrt{q}}{2} k\, [(\ell + \frac{d}{ds})^{k-1} L(s,\overline{\chi})]_{s=1} (1 + O_k(\ell^{-1})) & \text{if } \mathfrak{a} = 0. \end{cases}$$

Let $G_k(s,\ell) := (\ell + \frac{d}{ds})^k L(s,\overline{\chi})$, so that $G_{k+1} = \ell G_k + G_k'$, where the prime denotes differentiation with respect to s. Then

$$-\frac{L^{(k+1)}}{L^{(k)}}(0,\chi) = \begin{cases} (\ell + \frac{G_k'}{G_k}(1,\ell))(1 + O_k(\ell^{-1})) & \text{if } \mathfrak{a} = 1, \\ \frac{k+1}{k}(\ell + \frac{G_{k-1}'}{G_{k-1}}(1,\ell))(1 + O_k(\ell^{-1})) & \text{if } \mathfrak{a} = 0. \end{cases}$$

Now, for $k \geq 0$,

$$\frac{G_k'}{G_k}(1,\ell) = \frac{\frac{L'}{L} + \frac{k}{\ell}\frac{L''}{L} + \cdots + \frac{1}{\ell^k}\frac{L^{(k+1)}}{L}}{1 + \frac{k}{\ell}\frac{L'}{L} + \cdots + \frac{1}{\ell^k}\frac{L^{(k)}}{L}}(1,\overline{\chi}) \ll_k \log\log q,$$

□

by Lemmata 1 and 3. This completes the proof of Lemma 4.

Writing

$$\frac{L^{(k+1)}}{L^{(k)}}(s,\chi) = \frac{L^{(k+1)}}{L^{(k)}}(0,\chi) + \sum_{\rho_k} (\frac{1}{s - \rho_k} + \frac{1}{\rho_k}), \tag{9}$$

and differentiating both sides of (9) t times at $s = 0$ gives

$$(\frac{L^{(k+1)}}{L^{(k)}})^{(t)}(0,\chi) = -t! \sum_{\rho_k} \rho_k^{-t-1}, \quad (t \geq 1).$$

The left-hand side can be evaluated by repeated use of Lemma 4 yielding

Corollary 2. *Let ρ_k run through all zeros of $L^{(k)}(s,\chi), k \geq 1$. Upon GRH, as $q \to \infty$, for $t = 2, 3, ..$*

$$\sum_{\rho_k} \rho_k^{-t} = \begin{cases} \left(\frac{\log q}{k}\right)^t + O_k((\log q)^{t-1} \log\log q) & \text{if } \mathfrak{a} = 0, \\ O_k((\log q)^{t-1} \log\log q) & \text{if } \mathfrak{a} = 1. \end{cases}$$

Corollary 2 makes more sense upon searching for ρ_k close to the origin. Suppose $L^{(k)}(1-s,\chi) = 0$, where $|s-1| \ll \frac{1}{\log q}$. Arrange the right-hand side of (8) in powers of ℓ. It is enough to consider the two leading terms and to lump the rest with $\ell^j, j \leq k-2$, into an error term. In view of Lemma 3 we must have

$$\tan\frac{\pi}{2}(s-\mathfrak{a}) = \frac{2}{k\pi}\log q + O_k(\log\log q). \tag{10}$$

If $\mathfrak{a} = 1$, then $\tan\frac{\pi}{2}(s-\mathfrak{a})$ is nearly 0 for s close to 1 and (10) can't hold. Hence there are no zeros of $L^{(k)}(s,\chi)$ within a distance $O(\frac{1}{\log q})$ from the origin. If $\mathfrak{a} = 0$, then (10) has a solution which reveals a zero of $L^{(k)}(s,\chi)$ located at

$$\frac{k}{\log q} + O_k\left(\frac{\log\log q}{\log^2 q}\right). \tag{11}$$

This zero close to the origin for $\mathfrak{a} = 0$ gives rise to the main term in Corollary 2.

3. Zeros of $L'(s,\chi)$ in $0 \leq \sigma \leq \frac{1}{2}$

It turns out that the zero given by (11) is the only zero of L' in the left-half of the critical strip when q is sufficiently large.

Theorem 1. *Assume GRH. If $\mathfrak{a} = 0$ and $q \geq 216$, then $L'(s,\chi)$ has exactly one zero ρ_1 with $0 \leq \Re\rho_1 < \frac{1}{2}$, at $\frac{1}{\log q} + O(\frac{\log\log q}{\log^2 q})$. If $\mathfrak{a} = 1$ and $q \geq 23$, then $L'(s,\chi)$ has no zeros in the left-half of the critical strip.*

Proof. Taking real parts in (3), employing (5) and GRH ($\rho = \frac{1}{2} + i\gamma$), we have

$$\Re\frac{L'}{L}(s,\chi) = -\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right) + \left(\sigma - \frac{1}{2}\right)\sum_{\gamma}\frac{1}{(\sigma-\frac{1}{2})^2 + (t-\gamma)^2}. \tag{12}$$

Case 1: $\mathfrak{a} = 1$. As a contour take a large T and form the rectangle with corners at $\pm iT, \frac{1}{2} \pm iT$, making small left-semicircular indentations around the zeros of $L(s,\chi)$ on the critical line. On the horizontal sides of the contour all terms on the right-hand side of (12) are negative, so that $\Re\frac{L'}{L} < 0$. On the left edge the last term of (12) is negative. For $|t| \geq 3, -\Re\frac{\Gamma'}{\Gamma}(\frac{1+it}{2}) < 0$, and for $|t| < 3, \Re\frac{\Gamma'}{\Gamma}(\frac{1+it}{2}) \geq \Re\frac{\Gamma'}{\Gamma}(\frac{1}{2})$. So, if $q > \pi e^{|\frac{\Gamma'}{\Gamma}(\frac{1}{2})|}$ (i.e.

$q \geq 23$), then $\Re\frac{L'}{L} < 0$ on $[-iT, iT]$. On the right edge if $q > \pi e^{|\frac{\Gamma'}{\Gamma}(\frac{3}{4})|}$ (i.e. $q \geq 10$), then $-\frac{1}{2}\log\frac{q}{\pi} - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}(\frac{s+1}{2}) < 0$. Also $\Re\sum\frac{1}{s-\rho} = 0$ when $\sigma = \frac{1}{2}$, while $\Re\sum\frac{1}{s-\rho}$ can be made arbitrarily large negative on the indentations by taking the semicircles small enough. Hence $\Re\frac{L'}{L} < 0$ throughout the contour on and inside which $\frac{L'}{L}$ is analytic. Thus L' has no zeros in $0 \leq \sigma < \frac{1}{2}$.

Case 2: $\mathfrak{a} = 0$. We modify the contour by going around the origin on an arbitrarily small left-semicircle. On the horizontal sides and the critical line we argue as before. To guarantee that $\Re\frac{L'}{L} < 0$ we must take $q > \pi e^{|\Re\frac{\Gamma'}{\Gamma}(\frac{1}{4})|}$ (i.e. $q \geq 216$). On $\sigma = 0, \Re\frac{\Gamma'}{\Gamma}(it) > -\gamma, \forall t \neq 0$ and $\lim_{t\to 0}\Re\frac{\Gamma'}{\Gamma}(it) = -\gamma$. If $|t| \geq 2$, then $\Re\frac{\Gamma'}{\Gamma}(\frac{s}{2}) > 0$. On the left-semicircle around 0, $\Re\frac{\Gamma'}{\Gamma}(\frac{s}{2})$ assumes large positive values. Hence, for $q \geq 216, \Re\frac{L'}{L} < 0$ all along the contour. Since $\frac{L'}{L}$ has a pole at $s = 0$, by the argument principle there must be exactly one zero of L' inside the contour. We already know by (11) where this zero is. If χ is real, then this zero is also real. □

Remarks: (i) For small q the possibility remains that L' has zeros $\beta_1 + i\gamma_1$ with $0 < \beta_1 < \frac{1}{2}, |\gamma_1| < 3$.

(ii) We can see that $\Re\frac{L'}{L}(\frac{1}{2} + it, \chi) < 0$ if $L(\frac{1}{2} + it, \chi) \neq 0$. So L' may vanish on the critical line only at a multiple zero of L.

4. Zero-free regions on the right for $L^{(k)}(s, \chi)$

Theorem 2. *Let m be the smallest prime that doesn't divide q. Then $L^{(k)}(s, \chi) \neq 0$ for $\sigma > 1 + \frac{m}{2}\left(1 + \sqrt{1 + \frac{4k^2}{m\log m}}\right)$.*

Proof. Let $m = m(q)$ be the least $n > 1$ such that $\chi(n) \neq 0$. Equivalently m is the smallest prime that doesn't divide q. By the prime number theorem $m = O(\log q)$. Now

$$L^{(k)}(s, \chi) = (-1)^k \sum_{n=m}^{\infty} \frac{\chi(n)(\log n)^k}{n^s} \quad (\sigma > 0),$$

so that

$$\begin{aligned} |L^{(k)}(s, \chi)| &\geq \frac{(\log m)^k}{m^\sigma} - \sum_{n=m+1}^{\infty} \frac{(\log n)^k}{n^\sigma} \quad (\sigma > 1) \\ &> \frac{(\log m)^k}{m^\sigma} - \int_m^\infty \frac{(\log x)^k}{x^\sigma} dx \quad (\sigma > \frac{k}{\log m}) \\ &= \frac{(\log m)^k}{m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma - 1)^{k+1}} \sum_{j=0}^{k} \frac{(\log m)^j(\sigma - 1)^j}{j!}. \end{aligned}$$

Setting $z = (\sigma - 1)\log m$, we would have $|L^{(k)}(s,\chi)| > 0$, if

$$\frac{z^{k+1}}{k!m\log m} - \sum_{j=0}^{k} \frac{z^j}{j!} \geq 0.$$

For this type of polynomial there is only one $z \in \mathbb{R}^+$ at which equality is achieved (Polya and Szegö [6, Vol.I,Pt.III,Problem 16]). We can assume $\sigma > 1 + \frac{k}{\log m}$, so that $z \geq k$. Then,

$$\sum_{j=0}^{k} \frac{z^j}{j!} \leq \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!} \leq \frac{z^{k+1}}{k!m\log m},$$

the last inequality being true for $z \geq \frac{m\log m}{2}(1+\sqrt{1+\frac{4k^2}{m\log m}})$. Writing this condition on z in terms of σ completes the proof. □

5. Zero-free regions in the left half-plane for $L^{(k)}(s,\chi)$

Theorem 3. *Given any $\epsilon > 0, \exists K = K(k,\epsilon,\mathfrak{a})$ such that there is no zero of $L^{(k)}(s,\chi)$ in the region $|s| > q^K, \sigma < -\epsilon, |t| > \epsilon$.*

Proof. From (8) we can say that $|L^{(k)}(1-s,\chi)| > 0$ provided

$$|\Gamma^{(k)}(s)\cos\frac{\pi}{2}(s-\mathfrak{a})\,L(s,\overline{\chi})| > \sum_{j=0}^{k-1}\binom{k}{j}\Gamma^{(j)}(s)\sum_{l=0}^{k-j}\binom{k-j}{\ell}(\log\frac{q}{2\pi})^l\frac{d^{k-j-l}}{ds^{k-j-l}}\big(\cos\frac{\pi}{2}(s-\mathfrak{a})L(s,\overline{\chi})\big).$$

Suppose $|t| > \epsilon$, so that $|\tan\frac{\pi}{2}(s-\mathfrak{a})|$ is bounded. For $\Re s \geq 1+\epsilon$, $L^{(n)}(s,\chi)$ and $\frac{1}{L(s,\chi)}$ are all bounded. Since

$$\Gamma^{(j)}(s) = \Gamma(s)\{(\log s)^j + \sum_{n=0}^{j-1} E_{nj}(s)(\log s)^n\}, \quad E_{nj}(s) = O(\frac{1}{s}), \tag{13}$$

we see that if $|\log s| > K\log q$, for sufficiently large $K = K(k,\epsilon,\mathfrak{a})$, then (13) will be satisfied.

It can also be seen by Rouché's theorem that there is $\alpha_k = \alpha_k(q,\epsilon,\mathfrak{a}) < 0$ such that $L^{(k)}(s,\chi)$ has exactly one zero whose real part is in $(-1-2n-\mathfrak{a}, 1-2n-\mathfrak{a})$ for $1-2n < \alpha_k$. If χ is a real character, then all such zeros would be real. For complex χ these zeros will tend to the negative real axis as one goes leftward. □

6. The number of zeros of $L^{(k)}(s,\chi)$

Equipped with the information in the two preceeding sections we shall calculate $N_k(T,\chi), k \geq 1$, the number of zeros of $L^{(k)}(s,\chi)$ in a region $-q^K < \sigma < \sigma_k, |t| \leq T$ as $T \to \infty$. Here T is chosen so that there are no zeros of $L^{(k)}(s,\chi)$ on the lines $t = \pm T$ and σ_k is taken large enough so that
(i) $L^{(k)}(s,\chi)$ has no zeros in $\sigma \geq \sigma_k$.
(ii) $\sum_{n=m+1}^{\infty} \frac{(\log n)^k}{n^{\sigma_k}} \leq \frac{1}{2}\frac{(\log m)^k}{m^{\sigma_k}}$.
(The smallest such σ_k is $\leq 1 + m(1+\sqrt{\frac{1+2k^2}{m\log m}})$.)

By the argument principle,

$$\begin{aligned} N_k(T,\chi) &= \frac{1}{2\pi i}\{\int_{\sigma_k - iT}^{\sigma_k+iT} + \int_{\sigma_k+iT}^{-q^K+iT} + \int_{-q^K+iT}^{-q^K-iT} + \int_{-q^K-iT}^{\sigma_k-iT}\}\frac{d}{ds}\log L^{(k)}(s,\chi)\,ds \\ &= \frac{1}{2\pi i}(I_1+I_2+I_3+I_4), \end{aligned}$$

say. First,

$$\begin{aligned} I_1 &= \log L^{(k)}(s,\chi)|_{\sigma_k-iT}^{\sigma_k+iT} \\ &= \log\frac{(-1)^k\chi(m)(\log m)^k}{m^s}|_{\sigma_k-iT}^{\sigma_k+iT} + \log(1+g(s))|_{\sigma_k-iT}^{\sigma_k+iT}, \end{aligned}$$

where

$$g(s) = \sum_{n=m+1}^{\infty} \frac{(\frac{\log n}{\log m})^k \chi(n)\overline{\chi(m)}}{(\frac{n}{m})^s}.$$

By condition (ii) on σ_k, $|g(s)| \leq \frac{1}{2}$ on $\sigma = \sigma_k$. So $\Delta\arg(1+g(s))|_{\sigma_k-iT}^{\sigma_k+iT} \leq \pi$, and

$$I_1 = -2iT\log m + O(k\log\log m) + O(\sigma_k\log m) + O(1)\ .$$

Now, to estimate I_2 put

$$\phi_k(s,\chi) = (-1)^k\overline{\chi(m)}e^{iT\log m}L^{(k)}(s,\chi)\ ,$$

so that the leading term of $\phi_k(\sigma_k+iT,\chi)$ is positive. Let ν be the number of zeros of $\Re\phi_k$ on $(-q^K+iT, \sigma_k+iT)$. Then this interval is divided into at most $\nu+1$ subintervals in each of which $\Re\phi_k$ is of constant sign. Hence

$$|\Im I_2| = \Delta \arg \phi_k(s)|_{\sigma_k+iT}^{-q^K+iT} \leq (\nu+1)\pi ,$$

and we need to estimate ν. Let

$$f(z) = \frac{1}{2}\{\phi_k(z+iT) + \overline{\phi_K(\overline{z}+iT)} ,$$

and $n(\omega)$ be the number of zeros of $f(z)$ in the disk $|z-\sigma_k| < \omega$. Take $r = 2(\sigma_k + q^K)$. We have

$$\int_0^r \frac{n(\omega)}{\omega}\, d\omega \geq n(\frac{r}{2}) \int_{\frac{r}{2}}^r \frac{d\omega}{\omega} = n(\frac{r}{2}) \log 2,$$

so that by Jensen's formula

$$n(\frac{r}{2}) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |f(re^{i\theta} + \sigma_k)|\, d\theta - \frac{1}{\log 2} \log |f(\sigma_k)|.$$

To put a bound for the terms on the right-hand side note that

$$\begin{aligned} |f(\sigma_k)| &= \Re\phi_k(\sigma_k + iT) \\ &= \Re\{\frac{(\log m)^k}{m^{\sigma_k}} + \overline{\chi(m)} m^{iT} \sum_{n=m+1}^{\infty} \frac{(\log n)^k \chi(n)}{n^{\sigma_k+iT}}\} \\ &\geq \frac{1}{2}\frac{(\log m)^k}{m^{\sigma_k}}, \end{aligned}$$

by the choice of σ_k. Also $L^{(k)}(s,\chi) = O(|qs|^{Mq^K})$, where $M = M(k)$, as $t \to \infty$ in our strip. (For $L(s,\chi)$ the estimate $|L(s,\chi)| \leq 2q|s|$, for $\sigma \geq \frac{1}{2}$, ([4, Chapter 12]) along with the functional equation implies such a bound. Then, for $L^{(k)}(s,\chi)$ one can proceed inductively via Eq.(8).) This means $\nu \leq n(\frac{r}{2}) \ll q^K \log q|s|$, and

$$\Im I_2 = O_k(q^K \log qT) .$$

The same reasoning and estimate also holds for I_4.

It remains to estimate $I_3 = \log L^{(k)}(s,\chi)|_{-q^K+iT}^{-q^K-iT}$. From (8) we write

$$L^{(k)}(s,\chi) = \frac{\varepsilon(\chi)\sqrt{q}}{\pi} (\frac{2\pi}{q})^s e^{\frac{\pi}{2}i(1-\mathfrak{a}-s)} \Gamma(1-s)\{R_1 + R_2\} ,$$

where R_1 is the contribution coming from

$$\{(\frac{2\pi}{\sqrt{q}})^s \cos\frac{\pi}{2}(1-\mathfrak{a}-s)\,\Gamma(1-s)\}^{(k)} \frac{\overline{\chi(m)}}{m^{1-s}},$$

and R_2 is the contribution coming from

$$\{(\frac{2\pi}{\sqrt{q}})^s \cos\frac{\pi}{2}(1-\mathfrak{a}-s)\Gamma(1-s)\}^{(k)} \sum_{n=m+1}^{\infty} \frac{\overline{\chi(n)}}{n^{1-s}} +$$
$$+\sum_{j=0}^{k-1}\binom{k}{j}\{(\frac{2\pi}{\sqrt{q}})^s \cos\frac{\pi}{2}(1-\mathfrak{a}-s)\Gamma(1-s)\}^{(j)} L^{(k-j)}(1-s,\overline{\chi}).$$

If $\Re s$ is sufficiently large negative (i.e. if K is sufficiently large), then $|\frac{R_2}{R_1}(s)| < 1$, so that $\arg|1+\frac{R_2}{R_1}(s)|$ varies less than π as s goes on the path of I_3. Then

$$\begin{aligned} I_3 &= \{\log(\frac{2\pi}{q})^s + \log\Gamma(1-s) + \log R_1 + \log(1+\frac{R_2}{R_1}) + \frac{i\pi}{2}(1-\mathfrak{a}-s)\}|_{-q^K+iT}^{-q^K-iT} \\ &= 2iT\log\frac{qT}{2\pi e} + O_k(q^K) + O_k(\log\log T). \end{aligned}$$

Combining the estimates for I_1, I_2, I_3, I_4 we have proved

Theorem 4. $N_k(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi e m} + O(q^K \log T)$, *as* $T \to \infty$.

In this connection we remind that by a theorem of Bohr and Landau [2], a Dirichlet series which converges for $\sigma > 0$ (in particular $L^{(k)}(s,\chi), k \geq 0$) has $O(T)$ zeros in $\sigma > \sigma_0 > \frac{1}{2}, |t| \leq T$.

7. Zeros of $L^{(k)}(s,\chi)$ in $0 \leq \sigma < \frac{1}{2}$

We have seen that zeros of $L^{(k)}(s,\chi)$ can be classified in the following way:
(i) For $\sigma < -q^K$, what we may call trivial zeros are situated within ϵ of the negative real axis (they are on the axis for real χ).
(ii) There may be vagrant zeros in $|s| \leq q^K, \Re s < -\epsilon$.
(iii) There are zeros in $-\epsilon < \Re s$; these will be called non-trivial.
Here $\epsilon > 0$ can be any small fixed number. In this section we prove

Theorem 5. *Assuming GRH, there are at most finitely many zeros of* $L^{(k)}(s,\chi)$ *in the strip* $-\epsilon \leq \sigma < \frac{1}{2}$.

Proof. The proof will be by induction on k. For $k = 0$, GRH says it all. Assume that there are finitely many zeros ρ_k of $L^{(k)}(s,\chi)$ with $-\epsilon \leq \Re\rho_k < \frac{1}{2}$. Recalling (9) and Lemma 4,

$$\frac{L^{(k+1)}}{L^{(k)}}(s,\chi) = -(1+\frac{1-\mathfrak{a}}{k}+o(1))\log q + \sum_{\rho_k}(\frac{1}{s-\rho_k}+\frac{1}{\rho_k}),$$

where ρ_k runs through all zeros of $L^{(k)}(s,\chi)$. We split the last sum into three: $\sum_1$ will be over those ρ_k with $\Re\rho_k < -q^K$; $\sum_2$ will cover $-q^K \le \Re\rho_k < \frac{1}{2}$; $\sum_3$ will be over those zeros with $\Re\rho_k \ge \frac{1}{2}$.

For $-\epsilon < \sigma < \frac{1}{2}$ and $|t|$ large

$$\sum\nolimits_1 = \sum_{\Re\rho_k < -q^K} \left(\frac{1}{s - \Re\rho_k} + \frac{1}{\Re\rho_k}\right) + O\left(\frac{\epsilon}{q^K}\right) .$$

For trivial ρ_k, call $a_j(k) := -\Re\rho_k = (q^K + 2j) + O(1), j = 0, 1, 2, \ldots$ Clearly,

$$\Re \sum_{a_j > q^K} \left(\frac{1}{s + a_j} - \frac{1}{a_j}\right) = -|s|^2 \sum_{a_j > q^K} \frac{1}{a_j|s + a_j|^2} - \sigma \sum_{a_j > q^K} \frac{1}{|s + a_j|^2}.$$

The very last summation term is

$$\sigma \sum_{a_j > q^K} \frac{1}{|s + a_j|^2} \ll \sum_{a_j > q^K} \frac{1}{a_j^2 + t^2} \ll \frac{1}{t}.$$

Examine the other sum in two parts as $|a_j| \ge \frac{|s|}{2}$ and $|a_j| < \frac{|s|}{2}$. We have

$$|s|^2 \sum_{a_j \ge \frac{|s|}{2}} \frac{1}{a_j|s + a_j|^2} \ll |s|^2 \sum_{a_j \ge \frac{|s|}{2}} \frac{1}{a_j^3} = O(1).$$

If $|a_j| < \frac{|s|}{2}$, then $|s + a_j| < \frac{3|s|}{2}$, so that

$$|s|^2 \sum_{a_j < \frac{|s|}{2}} \frac{1}{a_j|s + a_j|^2} \le -\frac{2}{9} \log \frac{|s|}{2} + O(K \log q).$$

Hence for sufficiently large $|t|$, $\sum_1 \le -\frac{1}{9} \log |s|$. By the induction hypothesis, $\sum_2$ consists of finitely many terms. Lastly,

$$\Re \sum\nolimits_3 = \sum_{\beta_k \ge \frac{1}{2}} \left(\frac{\sigma - \beta_k}{|s - \rho_k|^2} + \frac{\beta_k}{|\rho_k|^2}\right) < \sum_{\beta_k \ge \frac{1}{2}} \frac{\beta_k}{|\rho_k|^2}$$

so that $\Re \sum_3$ is a bounded quantity (its value depending on χ) by Theorem 2. Thus, for $-\epsilon < \sigma < \frac{1}{2}$ and $|t|$ sufficiently large,

$$\Re \frac{L^{(k+1)}}{L^{(k)}}(s, \chi) < 0$$

ending the proof. □

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Dirichlet L-Fonksiyonlarının Türevleri

Özet

Bu çalışmada Dirichlet L-fonksiyonlarının türevleri, bunların sıfırlarının sayısı ve yerleri hakkında muhtelif önermeler ispatlanmıştır.

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