Kirby Calculus in Manifolds with Boundary

Justin Roberts

Suppose there are two framed links in a compact, connected 3-manifold (possibly with boundary, or non-orientable) such that the associated 3-manifolds obtained by surgery are homeomorphic (relative to their common boundary, if there is one.) How are the links related? Kirby's theorem [K1] gives the answer when the manifold is S^3, and Fenn and Rourke [FR] extended it to the case of any closed orientable 3-manifold, or $S^1 \tilde{\times} S^2$. The purpose of this note is to give the answer in the general case, using only minor modifications of Kirby's original proof.

Let M be a compact, connected, orientable (for the moment) 3-manifold with boundary, containing (in its interior) a framed link L. Doing surgery on this link produces a new manifold, whose boundary is canonically identified with the original ∂M. In fact *any* compact connected orientable N, whose boundary is identified (via some chosen homeomorphism) with that of M, may be obtained by surgery on M in such a way that the boundary identification obtained after doing the surgery agrees with the chosen one. This is because $M \cup (\partial M \times I) \cup N$ (gluing N on via the prescribed homeomorphism of boundaries) is a closed orientable 3-manifold, hence bounds a (smooth orientable) 4-manifold, by Lickorish's theorem [L1]. Taking a handle decomposition of this 4-manifold starting from a collar $M \times I$ requires no 0-handles (by connectedness) and no 1- or 3-handles, because these may be traded (surgered 4-dimensionally) to 2-handles (see [K1]). The attaching maps of the remaining 2-handles determine a framed link L in M, surgery on which produces N.

The framed link representation is not at all unique, and the natural question is: given framed links L_0 and L_1 in M such that the surgered manifolds M_0, M_1 are homeomorphic relative to their boundary (there is a canonical identification between these boundaries which we must not change), how are L_0 and L_1 related? If M is the 3-sphere, the answer was given by Kirby [K1]: there is a finite sequence of (isotopy classes of) links, the first being L_0 and the last L_1, such that each is obtained from its predecessor by a move of type $\mathcal{O}_1$ or $\mathcal{O}_2$ or its reverse. The move $\mathcal{O}_1$ is supported in a 3-ball in M: it is simply disjoint union with a ± 1-framed unknot. The move $\mathcal{O}_2$ is supported in a genus-2 handlebody in M: it is any embedded image of the pattern depicted in figure 1, which is a modification of zero-framed links occurring inside a standard unknotted handlebody in S^3. (This is probably easier than thinking about it as a parallel-and-connect-sum operation.)

Fenn and Rourke proved in [FR] that the theorem could be extended to any closed orientable M by allowing an additional move $\mathcal{O}_3$, supported in a solid torus, and shown
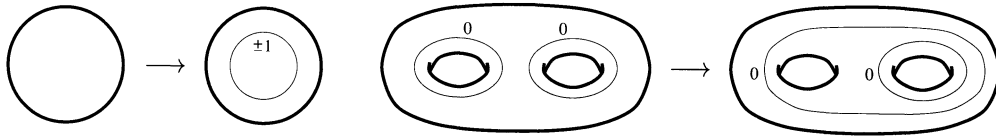
FIGURE 1. *Blow-up* $(\mathcal{O}_1)$ and *Handleslide* $(\mathcal{O}_2)$ moves.
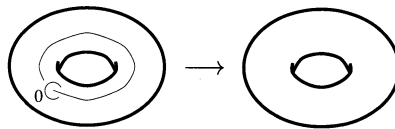
FIGURE 2. *Circumcision* $(\mathcal{O}_3)$ move.

in figure 2; its amusing name is due to Kauffman. (It is well-known that in S^3 this move follows from $\mathcal{O}_1$ and $\mathcal{O}_2$.)

Theorem 1. *For an arbitrary compact, connected, orientable 3-manifold with boundary, moves* $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ *still suffice.*

Proof. The proof follows from a careful reading of the original proof in Kirby's paper [K1]: little additional effort is required. However, to convince the reader, a short summary of this proof, recalling the main stages and noting the differences which arise, will be presented.

Stage 1. Let W_0 and W_1 be 4-manifolds formed by taking $M \times I$ and attaching 2-handles to the top surface along the links L_0, L_1. They have common boundaries $\partial W_0 = \partial W_1 = M \cup (\partial M \times I) \cup N$, so cross this with I and use it to glue them together to get a closed smooth 4-manifold. After connect-summing W_0 with some '$\mathbb{C}P^2$'s or their reverses (move $\mathcal{O}_1$) the signature of this closed manifold may be taken to be zero, so it bounds a smooth 5-manifold V. Form a handle decomposition of V, founded on $W_0 \times I$, by taking a generic Morse function on V which restricts to 0 on W_0, 1 on W_1, and t on the slice $\partial W_0 \times \{t\}$ of the connecting product 4-manifold. This decomposition can be chosen to have no 0- or 5-handles (by connectedness and non-closedness), and all its handles' attaching maps land in W_0, because of the form of the Morse function on the boundary. Just as in [K1], surger the 1- and 4-handles to 3- and 2-handles, reorder them in the natural way and view V as having two "ends" W_0 and W_1, and a "middle slice" $W_{1/2}$ (the level set of $\frac{1}{2}$) which is reached, working inwards from a collar on either end, by the attachment of 5-dimensional 2-handles. Such things attach by framed '$S^1 \times B^3$'s lying in whichever end W_i ($i = 0, 1$) is being considered. Since each W_i is obtained by attaching just 2-handles to $M \times I$, the images of the attaching maps may be isotoped into $M \times I$. Now comes a crucial difference: in [K1], each $S^1 \times B^3$ lies in $S^3 \times I$, so it

is null-homotopic and may be unknotted; the effect of surgery is thus to connect-sum W_i with $(S^4 - S^1 \times B^3) \cup (B^2 \times S^2)$, which is either $S^2 \times S^2$ or the twisted bundle $S^2 \tilde{\times} S^2$, depending on which of the two framings of $S^1 \times B^3$ is used. Such a connect-sum can be achieved by using only $\mathcal{O}_1$ and $\mathcal{O}_2$ (although of course it may be regarded as just the reverse of $\mathcal{O}_3$). In the general situation, the homotopy class of the knot in $M \times I$ may be non-trivial, and it is clear that the attaching cannot in general be achieved using just $\mathcal{O}_1$ and $\mathcal{O}_2$, because these moves preserve the subgroup of $\pi_1(M)$ generated by the components of the framed link. Instead, if the $S^1 \times B^3$ is pictured as lying on top of the top surface of $M \times I$, meeting it in $S^1 \times B^2$ (some framed knot whose "integral" framing stabilised to a "mod 2" one for $S^1 \times B^3$ agrees with the given one), then replacing it by $B^2 \times S^2$ may be achieved by decomposing $B^2 \times S^2 = B^2 \times B^2 \cup_{B^2 \times S^1} B^2 \times B^2$ and adding these pieces one at a time: the first attaches as a 2-handle to the framed knot $S^1 \times B^2$, and the second as a 2-handle to the framed core of that handle. Pushing it off the core into the top surface of M, results in a small zero-framed meridional curve linking the original one once. This is exactly the reverse of the move $\mathcal{O}_3$, occurring in the framed solid torus $S^1 \times B^2$.

Stage 2. The previous stage showed that by using just $\mathcal{O}_1$ and $\mathcal{O}_3$, the links in M can be altered to the point where they represent diffeomorphic 4-manifolds, namely the common "middle slice" $W_{1/2}$, henceforth renamed simply W. Note that nothing in that stage perturbed the canonical decomposition of the boundary of W into $M \cup (\partial M \times I) \cup N$, since all the handles were attached in the interior of M. Now apply Cerf theory: regard the two different handle decompositions of W as being induced by Morse functions f_0, f_1 each taking the value 0 on M, t on $\partial M \times \{t\}$, 1 on N, and having no critical points near the boundary. These may be connected by a generic path of functions (all satisfying the same boundary conditions) in the usual way. There is *no* difference at this stage between the original proof (where, at this stage, W has two disjoint closed boundary components M, N, assigned the values 0 and 1) and this refined case (where M, N have boundary and there is a connecting collar $\partial M \times I$ with projection to I as a Morse function) since Cerf's theory of genericity of functions on manifolds with boundary (see for example [HW]) requires only that there are never critical points near the boundary: this is ensured by the restriction of the Morse function. Consequently, everything can be worked as in [K1]. The arc of functions defines a graphic depicting the indices and heights of the critical points; there are isolated moments at which a handle pair is born or dies, or when one handle's descending manifold slides over another's critical point, instead of descending to the "base" M, but generically all that is occurring is isotopy of the attaching maps. (Other than at handleslides, all descending manifolds still reach the base because the Morse function on the "sides" is arranged to stop them landing there.)

Recall that the idea of Kirby's proof is to alter the graphic so as to remove all critical points of index other than 2: for then, what remains is a graphic depicting isotopy of 2-handle attaching maps and occasional slides of one 2-handle over another, which correspond to the move $\mathcal{O}_2$. Much progress can be made by applying the manouevres of Cerf theory (dovetail lemma, beak lemma, principle of independent trajectories). First,

the 0-1-handle births and deaths are put in a nested form. By introducing trivial 1-2-pairs and cancelling their 1-handles with the 0-handles (formally, introducing two dovetails and then moving beaks and independent trajectories), all 0-handles can be removed from the graphic, leaving new 2-handles in their place. A similar procedure replaces the 4-handles by 2-handles. Next, the 1-2- and 2-3-pairs are nested in a similar fashion: it is advantageous at this stage to apply the above procedure once again, in order to replace the 3-handles by 1-handles (this was not used in [K1]). So at the end of this stage, the graphic has only 1- and 2-handles.

Stage 3. There remains only the problem of "continuously surgering a 1-handle to a 2-handle" all the way along the arc of functions between the birth and death.

Consider the outermost 1-2-pair in the nest. Just after the birth, the attaching maps may be visualised as a pair of 3-balls in M, representing the feet of the 1-handle, together with a (framed) attaching curve $A \cup C$ for the 2-handle, where C is an arc in M and A is the core of the 1-handle. The surgery will be specified by picking any (framed) curve consisting of another core A' of the 1-handle and an arc B_0 in M (choose it to miss all the other handle attachments). During the period of time between the birth and death of the 1-handle under consideration, the feet of the 1-handle are isotoped around, as are the other attaching maps, and there are births and deaths of pairs and 2-handle slides. Notice that 1-handles do *not* slide on 1-handles: this is part of the outcome of the Cerf theory in stage 2. The isotopy of the feet extends to an ambient isotopy of M which drags the arc B_0 through a family B_t.

"Continuous surgery" on $A' \cup B_t$ means replacing its neighbourhood by $B^2 \times S^2$ in some standard fashion, independent of t. A concrete way of realising this 5-dimensional cobordism is to attach a pair of 2-handles (as described more explicitly in stage 1) onto W, one running along $A' \cup B_t$ and one (dual to it) on a small 0-framed meridian of this curve. Then perform a collapse of the 1-handle across the new 'long' 2-handle onto a regular neighbourhood of B_t, removing the pair. The effect on the handle decomposition is to delete the 1-handle, connect up all the other attaching curves which run over it using arcs running parallel to B_t in its regular neighbourhood, and encircle all these with a 0-framed unknot. This operation is obviously continuous with respect to isotopy of attaching maps (and to other births and deaths and handleslides, which may be chosen to happen away from the feet and arc) except when a 2-handle attaching curve crosses the arc B_t, but this requires only a slide move $\mathcal{O}_2$. This procedure may thus be used to eliminate the 1-handles, but it remains to examine the change in framed links which occurs in the time interval spanning the birth and the surgery. (The case of a death is exactly the reverse of this situation, once any attaching curves which go over the 1-handle have been slid off over the cancelling 2-handle, using move $\mathcal{O}_2$.)

The birth of such a pair can be explicitly realised (via a homeomorphism) as an expansion (inverse collapsing) move from a regular neighbourhood of the arc $C \subseteq M$ (which is assumed to be disjoint from any other attaching curves.) The changes in the link are shown in figure 3; the initial and final configurations differ by handleslides and then a circumcision $\mathcal{O}_3$, finishing the proof. $\qquad\square$
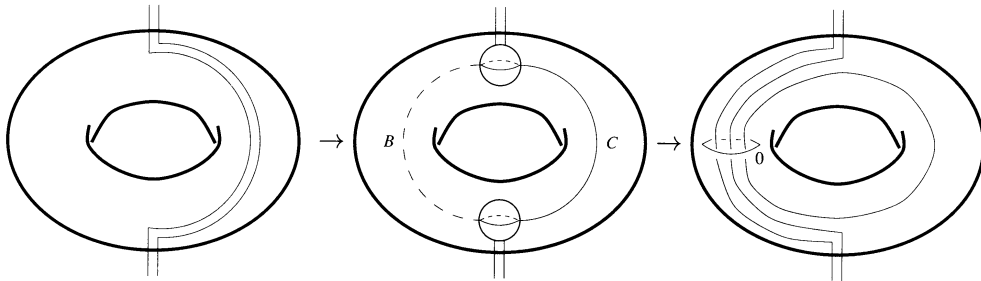
FIGURE 3. Birth of a 1-2-handle pair via expansion from C, followed by the surgery defined by B and a collapse to its regular neighbourhood.
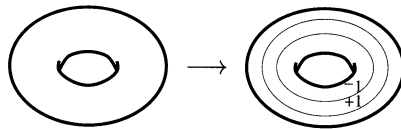

FIGURE 4. *Antigeny* $(\mathcal{O}_3')$ move.

Remark 1. It is easily shown that the moves $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ generate the same equivalence relation on framed links as do $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3'$, where $\mathcal{O}_3'$ is the alternative operation shown in figure 4.

Remark 2. In this formulation of the theorem, the question of "reparametrisation of the boundary" has been deliberately avoided. It can be brought in by gluing on a mapping cylinder, defined as $(F \times [0, \frac{1}{2}]) \cup_f (F \times [\frac{1}{2}, 1])$, where F is a closed surface, and f is an automorphism: in this form, its boundaries are still canonically identified. Expressing f as a product of Dehn twists gives a presentation of the cylinder as $F \times I$, surgered on a sequence of curves in successive F-slices of the cylinder. The move $\mathcal{O}_3'$, in this context, expresses cancellation of a Dehn twist and its inverse in the mapping class group. (It seems to me possible, though unlikely, that Kirby calculus in $F \times I$ could be used to derive a presentation of the mapping class group of F.)

The non-orientable case

If M is non-orientable then it is still possible to give such a classification theorem. Once again, if M and N are two compact, connected, non-orientable manifolds with identified boundaries, then gluing them via a collar on the boundary gives a closed non-orientable manifold. By Lickorish's theorem [L2], it bounds a (smooth) non-orientable 4-manifold. A handle decomposition of this 4-manifold built on $M \times I$ can have its 1- and 3-handles removed by surgering them to 2-handles (surgery is performed on a circle consisting of

115
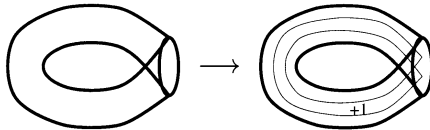
FIGURE 5. *Möbius* ($\mathcal{O}_4$) move.

the core union an arc in M, which has to be chosen so that the circle has a trivial normal bundle; since M itself contains orientation-reversing loops, this will always be possible). Consequently, it is possible to get from M to N (*rel* the boundary) by surgery on a framed link. It should be noted that since 2-handles attach along solid tori rather than twisted disc bundles, the homotopy classes of all attaching curves lie in the orientation-preserving subgroup of $\pi_1(M)$. (Note also that adding 2-handles cannot change orientability of the 3-manifold, so this is a totally separate case.)

To generate the equivalence relation on links requires another move $\mathcal{O}_4$, similar to the μ-move which was introduced by Fenn and Rourke [FR]. It is supported in a solid Klein bottle $S^1 \tilde{\times} B^2$ contained in M, and shown schematically in figure 5. Take the simple closed curve on the bounding Klein bottle which runs twice around the S^1 direction (this is unique up to isotopy, in fact Lickorish [L2] demonstrates that there are only two non-trivial isotopy classes of unoriented, orientation-preserving simple closed curves on a Klein bottle). Give this curve a framing $+1$ relative to the surface of the Klein bottle, and push it slightly into the solid bottle. The move $\mathcal{O}_4$ consists of doing surgery on this framed curve. It is easy to see that the surgered manifold is homeomorphic (*rel* its boundary) to the same solid Klein bottle by considering the equivalence between surgery on a curve in a surface (with relative framing $+1$) and cutting-and-regluing via a (negative) Dehn twist on that curve. Since the curve on the Klein bottle bounds a Möbius strip, and the Dehn twist parallel to the boundary of such a strip is isotopic to the identity, the homeomorphism is clear. (This also makes it clear that parity of the framing is all that matters.)

An alternative interpretation of this move at the 4-manifold level is useful. Consider the standard decomposition of $\mathbb{R}P^4$ with one handle in each dimension. Its 2-skeleton is $S^1 \tilde{\times} B^3$ union a 2-handle along the framed curve above. (The framing is checked by computing the mod 2 self-intersection of $\mathbb{R}P^2$, which is the core of the 2-handle union the Möbius strip used above.) The other two handles form $S^1 \tilde{\times} B^3$. An $S^1 \tilde{\times} B^3$ contained in a 4-manifold may be cut out and replaced with the complement of the 1-skeleton of $\mathbb{R}P^4$ by using the 5-cobordism $(S^1 \tilde{\times} B^3) \times I$. If the original $S^1 \tilde{\times} B^3$ is a 4-dimensional neighbourhood of an orientation-reversing curve in $M \subseteq M \times I$, then performing this replacement corresponds to attaching the 2-handle to the curve in the 3-dimensional solid Klein bottle neighbourhood in M, as in move $\mathcal{O}_4$.

Theorem 2. *If M is a compact, connected, non-orientable manifold then moves $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ suffice.*

Proof. The start of the proof is the same as before: connect the 4-manifolds via a collar. The resulting closed 4-manifold bounds a smooth 5-manifold if and only if its Stiefel-Whitney numbers w_2^2 and w_4 to vanish. This can be achieved by connect-summing with $\mathbb{C}P^2$ ($w_2^2 = 1, w_4 = 1$) and $\mathbb{R}P^4$ ($w_2^2 = 0, w_4 = 1$) if necessary. The first is effected by move $\mathcal{O}_1$. In the second case, use the 5-cobordism described above instead of a 1-handle: the effect on the characteristic classes is the same, and this is move $\mathcal{O}_4$.

Now take a Morse function on the 5-manifold whose restriction to the boundary is as before. Once again, replace the 1- and 4-handles by 3- and 2-handles using surgery, having first chosen suitable arcs to get trivial normal bundles. The effects of the 2- and 3-handles are, as before, to add pairs of 2-handles corresponding to move $\mathcal{O}_3$, starting from each end of the 5-manifold, to reach the common $W_{1/2}$. Now the Cerf theory works as before, and lack of orientability does not play a part in the remainder of the proof. $\square$

Acknowledgement. Recently Kerler [Ke1] and Sawin [S], motivated by the desire to give clean constructions of Topological Quantum Field Theories à la Reshetikhin-Turaev, obtained related presentations and 'moves' for general orientable 3-manifolds. Neither of these papers dealt with the natural generalisation of the original framed link calculus of Kirby, as presented here, although Kerler checked in [Ke2] that his bridged link calculus (essentially, allowing 1-handles as well as 2-handles) implied it. (The rather different version of Sawin does not seem to have this property.) Neither did these papers deal with the non-orientable case (usually neglected in TQFT), although presumably they would generalise too. I am grateful to Rob Kirby and Steve Sawin for the encouragement to write this version up.

References

[FR]   R. A. Fenn, C. P. Rourke. *On Kirby's calculus of links*, Topology 18 (1979), 1-15.

[HW]   A. E. Hatcher, J. B. Wagoner. *Pseudo-isotopies of compact manifolds*, Astérisque 6 (1973).

[Ke1] T. Kerler. *Bridged links and tangle presentations of cobordism categories*, Adv. Math. (to appear).

[Ke2] T. Kerler. *Equivalence of a bridged link calculus and Kirby's calculus of links on non-simply connected 3-manifolds*, preprint (1994).

[K1]   R. C. Kirby. *A calculus for framed links in S^3*, Invent. Math. 45 (1978), 35-56.

[K2]   R. C. Kirby. *Topology of 4-manifolds*, Springer Lect. Notes in Maths. 1374 (1989).

[L1]   W. B. R. Lickorish. *A representation of orientable combinatorial 3-manifolds*, Ann. Math. 76 (1962), 531-540.

[L2]   W. B. R. Lickorish. *Homeomorphisms of non-orientable 2-manifolds*, Proc. Cam. Phil. Soc. 59 (1963), 307-317.

[S]   S. Sawin. *Extending the Kirby calculus to manifolds with boundary*, preprint (1994).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA

Current address: Department of Mathematics and Statistics, University of Edinburgh, Edinburgh EH9 3JZ, Scotland

E-mail address: justin@maths.ed.ac.uk