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Critical Point Theory in Mathematics and in Mathematical Physics

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Introduction

At last year's Gökova Conference I reported on a "topological approach" to the "new" knot invariants, which Cliff Taubes and I had worked out along lines initiated by Axelrod and Singer [AS] and Kontsevich [K] in their work on 3-manifold invariants.

This year I thought I would comment on how these invariants arise out of physics-inspired considerations, but in a manner which might be more palatable to matnematicians than other accounts I can point to in the literature.

At the heart of the matter, as I see it, is the quite different role critical point theory plays in topology and in quantum physics, and I will therefore start with a comparison of these roles in the simplest instance: that of a compact oriented manifold M equipped with a "Morse function" $f : M \to R$.

1. The topological case

Recall that the critical points of a Morse function f are assumed to be nondegenerate. That is,

$$df_p = 0 \Rightarrow \det \left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p \neq 0,$$

where the x^i are smooth coordinates centered at p. This condition can also be invariantly expressed by demanding that the natural Hessian quadratic form $H_p f$ of f, defined on the tangent space of M at a critical point p, be nondegenerate.

The Morse index $\sigma_f(p)$, of f at p is then defined as the number of negative eigenvalues of $\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p$, or equivalently as the dimension of a maximal linear space on which $H_p f$ is negative definite.

It follows immediately that a Morse function has only a finite number of critical points on M, so that in particular the "Morse polynomial" of f

$$\mathcal{M}_t(f) \equiv \sum_p t^{\sigma(p)}$$

is well-defined.

The famous Morse inequalities arise by comparing this polynomial to the Poincaré polynomial $P_t(M)$ of M, relative to any field K, where

$$P_t(M) \equiv \sum \dim H^i(M;K)t^i.$$

They can be expressed via the formula

$$\mathcal{M}_t(f) - P_t(M;K) = (1+t)Q_f(t) \tag{1.1}$$

and the condition that the coefficients of the polynomial $Q_f(t)$ are all nonnegative:

$$Q_f(t) = a_0 + a_1 t + \dots \qquad a_i \geq 0.$$

These inequalities constitute the basic topological constraint on the size and disposition of the critical set of a generic smooth function on M, and the most elementary approach to proving them remains Morse's original one: one studies the growth of the half-spaces

$$M_f^\alpha = \{m \in M | f(m) < \alpha\}$$

and finds that their topological type changes only when α passes a critical value of f, and that if p is the only critical point with critical value $f(p) = c$, then for suitably small ε one has the "crossing formula"

$$M_f^{c+\varepsilon} \cong M_f^{c-\varepsilon} \cup e^{\sigma(p)}. \tag{1.2}$$

Here $\sigma(p)$ is the index of p, the right-hand side indicates $M_f^{c-\varepsilon}$ with a $\sigma(p)$-cell attached, and $\cong$ indicates homotopy equivalence.

The deformations needed to carry out this program of course depend on the general existence theorems for the "flow lines" of the negative gradient of f taken relative to some auxiliary Riemann structure g on M. Thus one studies the ordinary differential equations

$$\frac{dx}{dt} = -\nabla f \tag{1.3}$$

and pushes M in the direction of their solutions.

It was Smale who, in the '60s, rethought Morse theory in terms of the dynamics of the differential equations (1.3), and his work implicitly leads to the following refinement of the Morse inequalities: for a *generic* metric g — relative to f! — the solutions of (1.3) can be used to give a chain complex which is very economical for computing the cohomology of M (relative to any coefficients!), and the inequalities then follow by quite standard arguments.

The recipe is as follows: let C_f be the set of critical points of f, graded by the index $\sigma(p)$ of f at p, and let C_f^* be the corresponding graded free abelian group generated by C_f. Now under the "f-generic" assumption on g, it follows that the number of solutions of (1.3) which connect a critical point p of index λ and a critical point q of index $\lambda+1$ is finite. Furthermore, there is a natural sign assigned to each such flow-line, or "instanton", as the physicists call these. Hence one can define an operator

$$\delta_f : C_f^p \to C_f^{p+1}$$

by setting

$$\delta_f(p) = \sum \operatorname{sign}(\mu) q, \tag{1.4}$$

where μ runs over the "instantons" from p to all critical points q of index $\sigma(p) + 1$.

More precisely, the μ are solutions of

$$\frac{d\mu}{dt} = -\nabla f,$$

subject to the boundary conditions

$$\lim_{t \to -\infty} \mu(t) = p, \qquad \lim_{t \to +\infty} \mu(t) = q,$$

and the term "instanton" is apt because such a solution spends most of its time near p, then in a *finite time* — i.e. an instant — moves to the vicinity of q and then hovers there for all large t.

In any case, the crucial assertion is that *the cohomology of this complex computes the cohomology of M*:

$$H^*(M) \simeq H^*(C_f). \tag{1.5}$$

The Morse inequalities (1.1) are now seen to be the same ones that always exist between the dimensions of the finite-dimensional chain complex and its cohomology.

Let me warn you though that this formulation cannot be found explicitly in the earlier literature. A detailed self-contained account of why $\delta_f^2 = 0$ can only be found in more recent papers. Papers written after the work of Witten and Floer focused our attention on the Morse theory in a quite new and original manner. See for instance [AB].

So much then for the "baby version" of critical point theory in its topological-geometric setting.

2. Classical mechanics

On the physics side, this critical point theory played a central role from the very start in the Lagrangian and Hamiltonian formulation of classical mechanics, but immediately in an infinite-dimensional context, that is, in the framework of the calculus of variations.

Recall that in the Lagrangian formulation, the mechanical problem of interacting particles is subsumed in a single function $L(q, \dot{q})$ defined on the tangent bundle TM of the "configuration space" M underlying the problem.

The motion of the mechanical system in time thus corresponds to a path on M, and Hamilton's principle asserts that a path $\mu(t)$ joining the configuration $p_1 \in M$ to $p_2 \in M$ in time T will extremize the action functional

$$S(\mu) = \int_0^T L(\mu(t), \dot{\mu}(t))\, dt \tag{2.1}$$

and be subject to the boundary conditions

$$\mu(0) = p_1, \qquad \mu(T) = p_2. \tag{2.2}$$

Put differently, let $\Omega = \Omega^T_{p_1,p_2}$ be the space of smooth paths on M subject to these boundary conditions. Then (2.1) defines S as a real-valued function on Ω, and an extremal path $\mu(t)$ corresponds to a critical point of S on Ω. In short,

$$dS|_\mu = 0. \tag{2.3}$$

In this context, Morse theory predicts solutions of an extremal problem in terms of the topology of the *path space* Ω, and the theory remains the same as in the finite-dimensional situation, except that the counting function $\mathcal{M}_t(S)$ and the Poincaré polynomial $P_t(\Omega M)$ are now usually found to be infinite series. For instance, when M is the n-sphere, and $\Omega = \Omega S^n$ is the space of paths with fixed endpoints on S^n, Morse computed $P_t(\Omega)$ to be given by

$$P_t(\Omega) = \frac{1}{1 - t^{n-1}} = 1 + t^{n-1} + t^{2(n-1)} + \dots$$

and concluded from that computation that there is an infinite number of geodesics joining two fixed points on S^n endowed with *any* Riemann structure!

In short, classical mechanics fits quite naturally into the theory of critical points, and indeed was one of the principal driving forces for the work of both M. Morse and J. D. Birkhoff.

3. The quantum case

In quantum mechanics, however, critical points and their effects appear in a quite new and at first unexpected guise. This transformation comes about by what I like to call the "Dirac-Feynman paradigm".

In classical physics, the tangent bundle TM plays the role of the "phase space" of the system. Each point of TM describes a state of the system, and in favorable circumstances the laws of motion given by the extremal condition on S translate into a vector field X_S on TM such that the exponential e^{tX_S} describes the time evolution of the system.

In the simplest instances of quantum mechanics, the phase space — i.e. the "space of states" — is taken to be $H = L^2(M)$, and the time evolution of the system is given by a unitary operator U_T on H.

Now ideally — if "quantization" were a functor — one would therefore hope to pass from a classical Lagrangian L on TM to a unitary operator U^L_T on H in a constructive manner. Unfortunately such a procedure can not really be hoped for, but the Dirac-Feynman paradigm nevertheless gives us a remarkably suggestive clue as to the nature of this relationship. Namely, it proposes the following formula for $U = U^L_T$. First note that because this operator acts on functions, it can, in principle, be written in the form:

$$(Uf)(x) = \int U(x,y) f(y) dy,$$

where $U(x, y)$ corresponds to the Schwartz kernel of U. This granted, Feynman, pressing an analogy remarked earlier by Dirac, writes the following formula for $U(x, y)$:

$$U(x,y) = \int e^{2\pi i/h \cdot S(\mu)} \mathcal{D}\mu. \tag{3.1}$$

Here μ ranges over all paths on M subject to the boundary conditions

$$\mu(0) = x, \qquad \mu(T) = y,$$

and

$$S(\mu) = \int_0^T L(\mu, \dot{\mu})\, dt.$$

The integral here is to be taken over all these paths, and is of course mathematically ill-defined. Still it is a formula of great beauty and has pointed the way of all of modern quantum physics.

First of all, it correctly predicts the effect of "slit-experiments" with electrons. Here then there are only a finite number of paths involved, and (3.1) weights them with *complex numbers* — the "amplitudes" of quantum mechanics rather than the real probabilities which common sense and causality would have predicted.

Secondly, (3.1), together with the fact that h is very small, leads to the principle that (3.1) yields classical mechanics in the limit as $h \to 0$ and that quantum corrections should be computable by perturbation expansions in terms of h!

Thirdly, (3.1) has the correct "functorial form" with respect to t and partitions of the space M.

Here let me take the second point as the main lesson that the mathematician has to learn from this paradigm, and in our finite-dimensional setting this amounts to the following:

Given a function $S : M \to \mathbb{R}$, pass to the function $e^{i\lambda S}$, but considered as a smooth distribution (on top-dimensional forms on M with compact support):

$$T_\lambda\{S\} : \omega \mapsto \left(\frac{1}{2\pi}\right)^{n/2} \int_M e^{i\lambda S}\omega, \qquad \omega \in \Omega_c^n(M) \tag{3.2}$$

and derive an asymptotic formula for $T_\lambda\{S\}$ as $\lambda \to +\infty$.

That the critical points of S are again decisive in such an expansion is seen quite easily. For suppose that the support of ω is disjoint from the critical set C_S. It is then clearly possible to construct a vector field X on M such that

$$X \cdot S = 1 \qquad \text{on } \operatorname{Supp}\omega.$$

Differentiating in the direction of X, one obtains the identity

$$\begin{aligned} 0 = \int X(e^{i\lambda S}\omega) &= i\lambda \int (XS) e^{i\lambda S}\omega + \int e^{i\lambda S} X\omega \\ &= i\lambda \int e^{i\lambda S}\omega + \int e^{i\lambda S} X\omega, \end{aligned}$$

and, taking absolute values, one has the inequality

$$|\lambda|\left|\int e^{i\lambda S}\omega\right| \leq \int |X\omega|.$$

Thus $\left|\int e^{i\lambda S}\omega\right| < c_1/|\lambda|$, for some constant c_1, and iterating this procedure we see that $\left|\int e^{i\lambda S}\omega\right|$ *decreases faster than any power of* λ^{-1} *as* $\lambda \to \infty$. Hence we have the

Proposition. *If the support of ω is disjoint from the critical set C_S of S, then $T_\lambda\{S\}(\omega)$ decays faster than any power of $1/\lambda$:*

$$|T_\lambda\{S\}(\omega)| \leq \frac{c_k(\omega)}{\lambda^k} \qquad \textit{for } k = 0, 1, 2 \ldots \,.$$

It follows that the asymptotics of $T_\lambda\{S\}$ are concentrated on C_S, so that if S is Morse, $T_\lambda(S)$ should be asymptotically equivalent to a distribution consisting of a sum of δ-functions and their derivatives at the points of C_S.

To explain this expansion, we consider the simplest possible example:

$$S = x^2/2, \qquad M = \mathbb{R}, \tag{3.3}$$

so that $0 \in \mathbb{R}$ is the only critical point of S. As a first computation, observe that the test form

$$\varphi_\alpha = e^{-\frac{\alpha^2 x^2}{2}}\,dx \qquad \alpha > 0$$

does not vanish at $0 \in \mathbb{R}$, and that although φ_α is not of compact support, it decays sufficiently fast for $T_\lambda\{S\}$ to be well-defined on φ_α.

In fact, if we write T_λ for $T_\lambda\{S\}$, so that

$$T_\lambda(\varphi_\alpha) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int e^{\frac{i\lambda x^2}{2}} e^{-\frac{\alpha^2 x^2}{2}}\,dx, \tag{3.4}$$

then (3.4) can be explicitly evaluated in the following manner. The integral (3.4) is the limit of the integral over the interval $[-r_1, r_2]$ as r_1 and r_2 tend to $+\infty$, taken in the orientation indicated in Figure 1. By Cauchy's theorem, we can deform this integral into the dotted contour of Figure 1, where β is the indicated square root of $-\frac{1}{-\alpha^2+i\lambda}$, that is, $\beta = \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}}\left(1 + \frac{i\alpha^2}{\lambda}\right)^{-\frac{1}{2}}$. For large λ it will point in direction close to $e^{i\pi/4}$. Now the contributions of c_1 and c_3 to this integral are seen to decay as $e^{-\frac{\alpha^2 r_1}{2}}$ and $e^{-\frac{\alpha^2 r_2}{2}}$ respectively, as soon as $\lambda \geq \alpha$. Indeed, on the quarter circle $x = r_2(\cos\theta + i\sin\theta)$ and $0 \leq \theta \leq \pi/4$, the exponent of (3.4) is given by

$$\frac{1}{2}\{(-\alpha^2 + i\lambda)r_2^2(\cos(2\theta) + i\sin(2\theta))\}$$

and hence has real part

$$-\frac{r_2^2}{2}(\alpha^2\cos 2\theta + \lambda\sin 2\theta).$$

For $\lambda \geq \alpha$, we have the upper bound

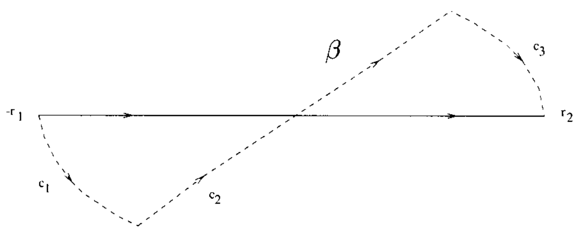

FIGURE 1

$$-\frac{r_2^2}{2}\alpha^2(\cos 2\theta + \sin 2\theta) \le -\frac{r_2^2\alpha^2}{2}.$$

Hence our integral is bounded in absolute value by $e^{-r_2^2\alpha}$ on c_3. Replacing θ by $\theta+\pi$ and r_2 by r_1 preserves this inequality; hence, this estimate also holds on c_1.

It follows that in the limit as $r_1, r_2 \to \infty$ our integral is given by the integral along the line $\beta\cdot r$, $r \in \mathbb{R}$. But if we set $x = \beta r$, then (3.4) is transformed into

$$\begin{aligned} T_\lambda(\varphi_\alpha) =& \left(\frac{1}{2\pi}\right)^{1/2} \beta \int_{-\infty}^{\infty} e^{-r^2/2}\, dr \\ =& \beta. \end{aligned}$$

For large λ, $T_\lambda(\varphi_\alpha)$ therefore has the expansion

$$\begin{aligned} T_\lambda(\varphi_\alpha) =& \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \cdot \left\{1 + \frac{i\alpha^2}{\lambda}\right\}^{-\frac{1}{2}} \\ =& \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \cdot \left\{1 - \frac{i\alpha^2}{2\lambda} + \dots\right\}. \end{aligned} \tag{3.5}$$

Thus the leading term of the asymptotic expansion of

$$T_\lambda(\varphi_\alpha) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\frac{x^2}{2}} \varphi_\alpha(x)\, dx$$

is given by $\frac{e^{i\pi/4}}{\sqrt{\lambda}}$.

We next test the behavior of T_λ on functions which vanish at 0 to order n, and for this purpose we compute the value of T_λ on the test forms

$$x^n \varphi_\alpha(x) = e^{-\frac{\alpha^2 x^2}{2}} x^n\, dx.$$

But, just as in the previous case, we now find that

$$T_\lambda(x^n \varphi_\alpha(x)) = \left(\frac{\beta}{\lambda}\right)^{n+1} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} r^n \, dr \tag{3.6}$$

We will denote the integral in (3.6) by $\langle r^n \rangle$:

$$\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2}} r^n \, dr \equiv \langle r^n \rangle \tag{3.7}$$

because in probability terminology it represents the n-th moment of the probability measure $\left(\frac{1}{2\pi}\right)^{1/2} e^{-r^2/2} \, dr$ on $\mathbb{R}$.

This granted, we have

$$T_\lambda(x^n \varphi_\alpha(x)) = \begin{cases} \frac{e^{\frac{i\pi}{4}}(i)^m}{\lambda^m \sqrt{\lambda}} \left(1 + \frac{i\alpha^2}{\lambda}\right)^{-(m+\frac{1}{2})} \langle r^{2m} \rangle & \text{for } n = 2m \\ 0 & \text{for } n = 2m+1. \end{cases} \tag{3.8}$$

Again the leading term of this expression is independent of α and vanishes for n odd. For $n = 2m$ even, it is given by:

$$T_\lambda(\varphi_\alpha x^n) \sim \frac{e^{\frac{i\pi}{4}}(i)^m \langle r^{2m} \rangle}{\lambda^m \sqrt{\lambda}} + \dots \tag{3.9}$$

With these preliminaries out of the way, we are ready to write down the complete asymptotic expansion of T_λ.

Proposition. *The distribution* $T_\lambda\left\{\frac{x^2}{2}\right\} = T_\lambda$ *on* $\mathbb{R}$ *has the asymptotic expansion:*

$$T_\lambda \sim \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \left\{ \sum_{m=0}^{\infty} \frac{(i)^m}{\lambda^m} \frac{\langle r^{2m} \rangle}{2m!} \, \delta^{(2m)}(x) \right\} \tag{3.10}$$

where $\delta(x)$ *denotes the Dirac function and* $\delta^{(k)}(x)$ *its kth derivative.*

Explicitly, (3.10) states that if $\varphi = g(x)\, dx$ is a C^∞ 1-form on $\mathbb{R}$ with compact support, then

$$T_\lambda \sim \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \left\{ \sum \frac{(i)^m}{\lambda^m} \frac{\langle r^{2m} \rangle}{2m!} \, g^{(2m)}(0) \right\} \tag{3.11}$$

Sketch of Proof. Note that if $\varphi(x) = g(x)\, dx$ is a test form with compact support, then $T_\lambda(\varphi)$ is bounded:

$$|T_\lambda(\varphi)| \le M \; l, \tag{3.12}$$

where M is the maximum value of f and l is the length of the support of φ.

We now use an integration by parts to show that

$$|T_\lambda(x\varphi)| \le \frac{c(\varphi)}{\lambda}. \tag{3.13}$$

Indeed if X denotes the Lie derivative in the direction $\frac{\partial}{\partial x}$, we have

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} X\left(e^{\frac{ix^2\lambda}{2}} g(x)\,dx\right) \\ &= i\lambda \int_{-\infty}^{\infty} e^{\frac{ix^2\lambda}{2}} xg(x)\,dx + \int_{-\infty}^{\infty} e^{\frac{ix^2\lambda}{2}} g'(x)\,dx, \end{aligned}$$

which implies (3.13) with $c(\varphi) = \int_{-\infty}^{\infty} |g'(x)|\,dx$.

Now then, given $\varphi(x) = g(x)\,dx$, consider the expression $\varphi - g(0)\varphi_\alpha$, where φ_α is our test form $\varphi_\alpha(x) = e^{-\frac{\alpha^2 x^2}{2}}\,dx$ of old. Clearly this expression vanishes at 0 and hence can be written as $x\psi$ with ψ smooth and also of compact support:

$$\varphi - g(0)\varphi_\alpha = x\psi. \tag{3.14}$$

Applying T_λ and using (3.13) yields

$$\left| T_\lambda\varphi - g(0)\,\frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}}\left(1 - \frac{i\alpha^2}{2\lambda}\cdots\right)\right| \le \frac{c(\psi)}{\lambda},$$

which proves that $T_\lambda\varphi \sim \frac{g(0)e^{\frac{i\pi}{4}}}{\sqrt{\lambda}}$ up to order $\frac{1}{\lambda}$. Induction now yields the general case: a truncation of the right-hand side at degree $m + \frac{1}{2}$ in λ approximates $T_\lambda(\varphi)$ up to degree $m + 1$. Q.E.D.

Remarks. 1. Formally, the proposition can be remembered by simply making the substitution $x = \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} y$ in the integral $\int_{-\infty}^{\infty} e^{\frac{i\lambda x^2}{2}} g(x)\,dx$, to obtain $\frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} g\left(\frac{ye^{\frac{i\pi}{4}}}{\sqrt{\lambda}}\right) dy$, and then *expanding g in a Taylor series about 0 in the y parameter.*

2. On a less formal level, we see that the smoothness and compact support of the test forms allows us to compute the asymptotics of $T_\lambda\left\{\frac{x^2}{2}\right\}$ by computing those of $T_{\lambda + i\alpha^2}$ and then sending α to 0.

3. Note next that the analysis goes through unchanged for the function $S = \frac{\rho x^2}{2}$ as long as $\rho > 0$. One simply rescales x to $\sqrt{\rho}x$; choosing $\sqrt{\rho} > 0$. On the other hand, to compute the asymptotics of the function $S = -\frac{x^2}{2}$ we must *conjugate* T_λ*:*

$$T_\lambda\left\{-\frac{x^2}{2}\right\} = \overline{T_\lambda\left\{\frac{x^2}{2}\right\}},$$

so that (3.10) yields

$$T_\lambda\left\{-\frac{x^2}{2}\right\} = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{\lambda}}\left\{\sum_{m=0}^{\infty} \frac{(-i)^m}{\lambda^m}\,\frac{\langle r^{2m}\rangle}{2m!}\,\delta^{2m}(x)\right\}. \tag{3.15}$$

In view of this state of affairs, we see that the distribution $T\left\{\frac{\rho x^2}{2}\right\}$ has the asymptotic expansion

$$T_\lambda\left\{\frac{\rho x^2}{2}\right\} \sim \frac{e^{\frac{i\pi}{4}\operatorname{sign}(\rho)}}{\sqrt{\lambda}\sqrt{|\rho|}}\left\{\sum_{m=0}^{\infty}\left(\frac{i}{\rho\lambda}\right)^m \frac{\langle r^{2m}\rangle}{2m!}\,\delta^{(2m)}(x)\right\}. \tag{3.16}$$

Note finally that due to the vanishing of the odd moments, the expression in the curly brackets on the right is rational in ρ, and that ρ is of course the Hessian of the function $\frac{\rho x^2}{2}$ relative to x.

We are now ready to treat the case of a general Morse function S on $\mathbb{R}$ with a finite number of critical points. It is clear from the preceding that $T_\lambda\{S\}$ is determined by the behavior of S near its critical point set C_S, and if we use a "Morse coordinate" x_p near each $p \in C_S$, so that S is of the form $S = S(p) + \frac{x_p^2}{2}$ at the local minima and $S = S(p) - \frac{x_p^2}{2}$ at the local maxima, then each p contributes according to (3.10), multiplied by $e^{i\lambda S(p)}$. Thus the leading term of $T_\lambda\{S\}$ is given by:

$$T_\lambda\{S\} \sim \frac{1}{\sqrt{\lambda}}\left(\sum_p e^{i\lambda S(p)}\delta(x_p)e^{\frac{i\pi}{4}\operatorname{sign}(H_pS)}\right) + \dots\,, \tag{3.17}$$

where H_pS denotes the Hessian of S at p and $\operatorname{sign}(H_pS)$ denotes its signature.

This leading term therefore carries the topological information in the phase shifts that occur at the various critical points. In terms of our earlier Morse index $\sigma(p)$, we can also write (3.17) as:

$$T_\lambda\{S\} \sim \frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}}\left\{\sum_p e^{i\lambda S(p)}\;\delta(x_p)i^{-\sigma_p}\right\} + \dots\,. \tag{3.18}$$

The higher order terms of course depend on the total germ of S at p, because they involve the derivatives of $\delta(x_p)$ *taken with respect to a "Morse coordinate"* x_p of S at p.

Now as it is in general quite difficult to find a Morse coordinate x_p near a critical point p, it is essential to have an algorithm for computing the asymptotics at $p \in C_S$ in terms of a general coordinate system. In principle this can be done by using the transformation laws of $\delta^{(k)}(x)$, but in practice it is easier to proceed in the following manner.

Suppose then that the Taylor series of S at $p \in C_S$ takes the form:

$$S = S(p) + \frac{a_2x^2}{2!} + \frac{a_3x^3}{3!} + \dots\,,$$

with $a_n = \frac{d^nS}{dx^n}\Big|_p$. We write

$$S = S(p) + \frac{a_2x^2}{2} + Q(x), \qquad \text{with } Q(x) = \sum_{k\geq 3}\frac{a_kx^k}{k!},$$

so that

$$e^{i\lambda S} = e^{i\lambda S(p)}e^{i\lambda\frac{a_2x^2}{2}}e^{i\lambda Q(x)}. \tag{3.19}$$

Now in our computation of $T_\lambda(S)$ near p, notated $T_\lambda\{S\}_p$, we have seen that the decisive parameter is not x, but $y = e^{-\frac{i\pi}{4}}\sqrt{\lambda|a_2|}x$, and the expansion of $i\lambda Q(x)$ in this parameter starts with a term $O\left(\frac{1}{\sqrt{\lambda}}\right)$ and then proceeds in higher powers of $\frac{1}{\sqrt{\lambda}}$.

In view of the foregoing, it is not surprising that we can compute the contribution to $T_\lambda(S)$ at p by simply expanding $e^{i\lambda Q(x)}$ in the exponential series and then computing $T_\lambda(S)$ term by term. In short, we have:

$$T_\lambda\{S\}_p(\varphi) = e^{i\lambda S(p)} \sum_{k=0}^{\infty} T_\lambda \left\{\frac{a_2 x^2}{2}\right\} \left(\frac{(i\lambda Q)^k}{k!}\varphi\right), \tag{3.20}$$

and this formula furnishes us with the desired algorithm.

For example, if

$$S = \frac{x^2}{2} + x^3,$$

then the asymptotics of $T_\lambda\{S\}_0$ are obtained from those of $\frac{x^2}{2}$, by premultiplying φ with the formal series

$$e^{i\lambda x^3} = 1 + i\lambda x^3 - \frac{\lambda^2 x^6}{2!} + \dots .$$

Thus the problem is reduced to rewriting the formal expression:

$$\frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \cdot \sum_{k=1}^{\infty} \frac{\langle r^{2k}\rangle i^k \delta^{(2k)}(x)}{2k!\ \lambda^k} \cdot \sum_{s=1}^{\infty} \frac{(i\lambda x^3)^s}{s!} \tag{3.22}$$

in terms of linear combinations of the $\delta^{(2k)}(x)$. This works in spite of the fact that these series proceed in $\frac{1}{\lambda}$ and λ respectively, because $\delta^{(2k)}(x)$ vanishes on all powers of x greater than $2k$. Thus, for instance, the first term of (3.22) is

$$\frac{e^{\frac{i\pi}{4}}}{\sqrt{\lambda}} \cdot \left(\sum_{m=1}^{\infty} \frac{\langle r^{6m}\rangle i^{3m}}{\lambda^m (2m)!}\right) \delta(x)$$

and corresponds to the case when $\delta^{(2k)}(x)$ differentiates x^{3s} entirely, i.e. when $2k = 3s = 6m$. The coefficients of the higher $\delta^{(2k)}(x)$ are correspondingly more complicated but clearly computable.

The 1-dimensional model we have discussed — quite possibly, in greater detail than is welcome — generalizes to the n-dimensional case without difficulty. In the vicinity of a critical point p of a Morse function S on M, we can always introduce Morse coordinates x^j, $j = 1, \dots, n$, so that near p,

$$S = S(p) + \frac{1}{2}\sum \varepsilon_j (x^j)^2, \tag{3.23}$$

with $\varepsilon_j = \pm 1$, so that the index of S at p counts the number of ε_j equal to -1.

In these coordinates, $e^{i\lambda S} = e^{i\lambda S(p)} \prod_{j=1}^{n} e^{\frac{i\lambda \varepsilon_j x_j^2}{2}}$, so that, by Fubini's theorem, the integral

$$\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int e^{i\lambda S} \varphi \, dx^1 \cdots dx^n$$

can be carried out one variable at a time. In short, one simply iterates the 1-dimensional case n times. It follows that

$$T_\lambda(S) \sim \frac{1}{\lambda^{\frac{n}{2}}} \sum_{p \in C_f} e^{i\lambda S(p) + \frac{i\pi}{4} \cdot \mathrm{sign}(H_p S)} \frac{\delta_p(x)}{\sqrt{|\det H_p S|}} + \dots , \tag{3.24}$$

and the leading term again carries the purely topological information in the phase-shift.

Remark. In the physics context, one is supplied not only with a Morse function S, but also a specified volume form $v \in \Omega_c^n(M)$, and it is the specific integral

$$\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_M e^{i\lambda S} v$$

whose asymptotics are of importance. One can then speak of a *perfect Morse function relative to v*. For such a function S, only the leading order terms contribute, and they actually *compute the integral.* That is, one has the formula:

$$\int_M e^{i\lambda S} v = \frac{1}{\lambda^{\frac{n}{2}}} \sum_{p \in C_f} e^{i\lambda S(p) + \frac{i\pi}{4} \cdot \mathrm{sign}(H_p(S))} \frac{\delta(x_p)\; v}{|\det H_p(S)|^{\frac{1}{2}}} . \tag{3.25}$$

A famous instance of this "perfection" occurs in the S^1-equivariant theory of *symplectic manifolds.* In that framework, M is compact, with symplectic form ω, and has a circle action preserving ω "in the equivariant sense." That is, there is a function S on M such that if the S^1-action is generated by the vector field X, then

$$\iota_X \omega = -dS.$$

(Such an S is called a *moment map* for the action). In any case, these data guarantee that if this S is Morse, then it is also perfect in the above sense, relative to the natural volume element $v = \frac{\omega^n}{n!}$ on M. This assertion is known as the *Duistermaat-Heckman formula* (see [DH]), and it has many striking applications in both physics and mathematics. By the way, the functions arising in this manner are always also perfect Morse functions in the topological sense.

This concept can also be formulated in the Riemannian category. For this purpose, note first that a Riemann structure g on an orientable manifold M canonically defines a distribution on the compactly supported functions $f \in \Omega_C^0(M)$ by the formula:

$$f \mapsto \int_M f v_g .$$

Indeed, g determines two sections of $\Lambda^n T^*$ with norm 1. We choose one of these and call it v_g, and then orient M so as to make the above integral non-negative whenever f is non-negative.

We may therefore speak of a Morse function S being perfect relative to a given Riemann structure g, meaning thereby that the leading term of the asymptotic expansion of $\frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{i\lambda S}v_g$ computes the integral.

For example, if M is the unit sphere $S^2 \subset \mathbb{R}^3$ and S is any linear function, then S is perfect in this sense relative to the inherited Riemann structure on S^2. In fact, this example is the paradigm for the whole Duistermaat-Heckman theory.

4. Graphical Methods.

Computing the higher order contributions to the asymptotic expansion (3.24) is in general a daunting task, and a global insight into their disposition would be well-nigh impossible were it not for the basic properties of the "moments" of Gaussian integrals and their graphical representations.

For this purpose one first needs an n-dimensional understanding of the the expectation values $\langle r^k\rangle$ we have already encountered in dimension one. More precisely, recall that if A is a positive definite $n\times n$ matrix, then

$$\mu_A = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} (\det A)^{\frac{1}{2}} e^{-\frac{1}{2}\langle Ax,x\rangle}\, dx_1\cdots dx_n$$

defines a Gaussian probability measure on $\mathbb{R}^n$, and if f is a measurable function (relative to μ_A), then its "expectation value" is defined by

$$\langle f\rangle_A = \int_{\mathbb{R}^n} f\,\mu_A.$$

In particular, the expectation values of $\langle x^\alpha\rangle$ monomials

$$x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$$

are called the "moments" of μ_A.

The reconstruction of a measure from its moments is in general an interesting question, but it is very easy in the Gaussian case. Namely, one has the following

Proposition. *The moments of a Gaussian integral relative to the Gaussian measure*

$$\mu_A = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} |\det A|^{\frac{1}{2}} e^{-\frac{1}{2}\langle Ax,x\rangle}\, dx_1\cdots dx_n \qquad \text{on } \mathbb{R}^n$$

are given by

$$\langle x^i, x^j\rangle = B_{ij},$$

where $B = A^{-1}$ is the inverse matrix to A. Furthermore, all the moments $\langle x^a\rangle$ of μ_A are given by universal polynomials in the B_{ij} according to the following algorithm.

To compute $\langle x_1^{\alpha_1}\cdots x_n^{\alpha_n}\rangle$, consider first the integer $\alpha_1+\alpha_2\ldots+\alpha_n = |\alpha|$. If this number is odd then *the expectation value $\langle x^\alpha\rangle$ is 0!*

If $|\alpha|$ is even, construct a graph consisting of n vertices v_i, and equip v_i with α_i "incipient arms". Thus, for instance, corresponding to $\langle x_1^3 x_2^3 x_3^4\rangle$ we have Figure 2.

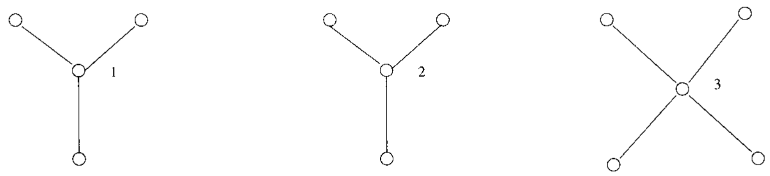

FIGURE 2

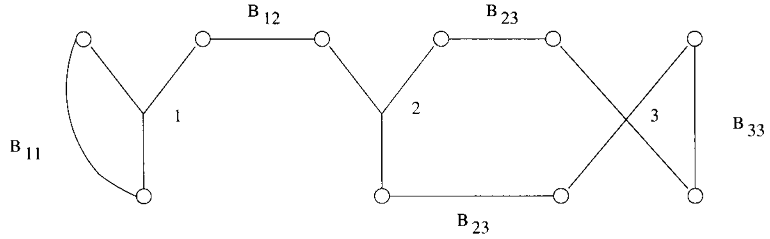

FIGURE 3

At this stage, one has $|\alpha|$ "open vertices", as indicated by the circles above.

Next, choose a partition of the "open vertices" into two disjoint sets A and B *of equal cardinality* and also choose a one-to-one correspondence $\Phi : A \to B$. (We call such a choice a *pairing*.) Finally connect an open vertex a to $\Phi(a)$ by an edge e_a and mark it with B_{ij} if e_a "connects" v_i to v_j, or v_j to v_i. Thus, in our example, this procedure might lead to Figure 3.

If Γ denotes the labeled graph determined at this stage, set $B(\Gamma)$ = product of all the labels in Γ. With this understood one has the formula:

$$\langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \rangle_A = \sum_{\Gamma} \frac{B(\Gamma)}{|\operatorname{Aut}\Gamma|}. \tag{4.2}$$

Here Γ ranges over the graphs arising from all possible "pairings" Φ as described above, and $|\operatorname{Aut}\Gamma|$ denotes the order of the group of automorphisms of Γ.

Warning. In the literature my "incipient arms" are usually dispensed with, so that the diagram on the previous page would appear simply as in Figure 4.

The derivation of this algorithm is based on the following identity:

$$-\frac{1}{2}\langle Ax, x\rangle + \langle x, y\rangle = -\frac{1}{2}\langle A(x - A^{-1}y), (x - A^{-1}y)\rangle + \frac{1}{2}\langle A^{-1}y, y\rangle. \tag{4.3}$$

If we set $u = x - A^{-1}y$ and $A^{-1} = B$, then (4.3) takes the form:

$$-\frac{1}{2}\langle Ax, x\rangle + \langle x, y\rangle = -\frac{1}{2}\langle Au, u\rangle + \frac{1}{2}\langle By, y\rangle, \tag{4.4}$$

where $y \in \mathbb{R}^n$ is an auxiliary parameter ranging over $\mathbb{R}^n$.

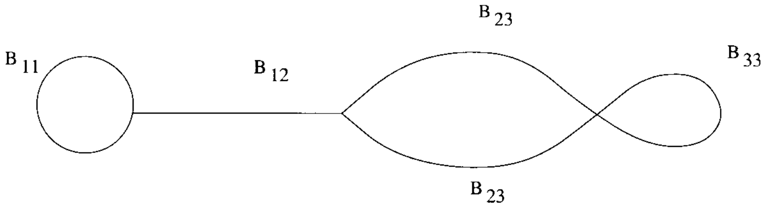

FIGURE 4

Examining the left-hand side of (4.4), we have

$$\frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\dots\partial y_n^{\alpha_n}}\int e^{-\frac{1}{2}\langle Ax,x\rangle+\langle x,y\rangle}\,dx_1\cdots dx_n\Big|_{y=0}=\int e^{-\frac{1}{2}\langle Ax,x\rangle}x^\alpha dx^1\cdots dx^n. \qquad (4.5)$$

On the other hand, applying the translation $x\mapsto x-A^{-1}y=u$ does not change the measure $dx_1\cdots dx_n$, so that (4.5) implies the relation

$$\int e^{-\frac{1}{2}\langle Ax,x\rangle+\langle x,y\rangle}\,dx^1\cdots dx^n=e^{\frac{1}{2}\langle By,y\rangle}\int e^{-\frac{1}{2}\langle Ax,x\rangle}\,dx^1\cdots dx^n.$$

It follows that

$$\langle x^\alpha\rangle=\frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}\dots\partial y_n^{\alpha_n}}e^{\frac{1}{2}\langle By,y\rangle}\Big|_{y=0}. \qquad (4.6)$$

In particular, $\langle x_i,x_j\rangle=B_{ij}$, as remarked earlier, and it is now clear that all the $\langle x^\alpha\rangle$ are universal polynomials in the B_{ij}'s. As for the translation of the combinatorics of (4.6) into graphical terms, we will content ourselves with checking it in the simple example of the expectation values $\langle x^{2m}\rangle$ in dimension 1.

$$\langle x^{2m}\rangle=\frac{\partial^{2m}}{\partial y^{2m}}e^{\frac{1}{2}y^2}\Big|_{y=0}=\frac{2m!}{m!\cdot 2^m}=(2m-1)(2m-3)\cdots(1)\equiv(2m-1)!! \qquad (4.7)$$

Thus $\langle x^2\rangle=1$, $\langle x^4\rangle=1\cdot 3$, $\langle x^6\rangle=1\cdot 3\cdot 5$, etc. To verify the graphical interpretation of these numbers, let us compute, for example, $\langle x^4\rangle$. According to the algorithm we should count all diagrams arising from "pairings" of Figure 5, and there are clearly 3 of these: $(1\leftrightarrow 2, 3\leftrightarrow 4)$, $(1\leftrightarrow 3, 2\leftrightarrow 4)$, and $(1\leftrightarrow 4, 2\leftrightarrow 3)$. Because Γ has only one vertex, $|\operatorname{Aut}\Gamma|=1$. By induction, $\langle x^{2m}\rangle=(2m-1)!!$, which agrees with (4.7).

In conclusion, note that the identity (4.6) enables us to rewrite (3.16) in the closed form:

$$T_\lambda\left(\rho\frac{x^2}{2}\right)\sim\frac{e^{\frac{i\pi}{4}\operatorname{sign}\rho}}{\sqrt{\lambda}\sqrt{|\rho|}}e^{\frac{\partial}{\partial y}\frac{\partial}{\partial x}}\delta_p(x)e^{\left(\frac{i}{\rho\lambda}\right)^{\frac{1}{2}}\frac{y^2}{2}}\Big|_{y=0}. \qquad (4.8)$$

More generally, we therefore have the

Proposition. *Let S be a Morse function on M and p be a critical point of f, with Morse coordinates x^i vanishing at p, so that*

$$S=S(p)+\frac{1}{2}\sum\varepsilon_i(x^i)^2 \qquad \textit{near } p.$$

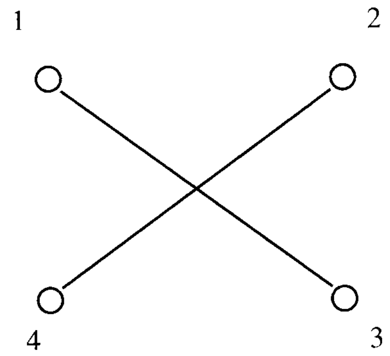

FIGURE 5

Then the contribution at p of $T_\lambda(S)$ *is given by*

$$T_\lambda \sim \frac{e^{i\lambda S(p)+\frac{i\pi}{4}\,\mathrm{sign}\, H_pS}}{\lambda^{\frac{n}{2}}|\det H_pS|^{\frac{1}{2}}} e^{\sum_\alpha \frac{\partial}{\partial y_\alpha}\frac{\partial}{\partial x_\alpha}} \delta_p(x) e^{\sqrt{\frac{i}{\lambda}}\langle (H_pS)^{-\frac{1}{2}} y, y\rangle}\Big|_{y=0}. \tag{4.9}$$

5. The equivariant case

Morse functions are generic in the space of C^∞-functions on M; however, they fail to be so in the realm of functions which exhibit infinitesimal symmetries, for then the critical set may contain nontrivial orbits. The appropriate concept in this equivariant situation is therefore the following one.

One calls a function S nondegenerate if and only if C_S falls into a union of smooth manifolds $C_S = \{N\}$ such that the Hessian of S restricts to a quadratic form $H_N(S)$ along N, whose kernel is precisely the tangent space to N:

$$\mathrm{Ker}\, H_N(S) = TN. \tag{5.1}$$

This concept is then clearly also preserved for the pull-back of functions under fiber projections $\widetilde{M} \xrightarrow{\pi} M$, and Morse functions on M pull back to nondegenerate ones on $\widetilde{M}$. Note also that since S is nondegenerate, $H_S(N)$ is also nondegenerate on any transverse surface of $T_pM|N, \quad p \in N$.

The extension of our asymptotics to $T_\lambda(S)$ when S is nondegenerate is an interesting question, but would take us too far afield here. Rather, let me concentrate on the simplest instance of the equivariant situation, namely when a Lie group G acts freely and locally on a manifold P. The quotient space P/G is then a smooth manifold M and $P \xrightarrow{\pi} M$ is a principal G-fibration.

Given a function S, invariant under the right action of G on P, it of course descends to a function $\underline{S}$ on M, such that $\pi^*\underline{S} = S$. On the other hand by Fubini's theorem, we have

$$\int_P \omega = \int_M \pi_*\omega,$$

where π_* denotes integration over the fiber. Hence the formula

$$\int_P e^{i\lambda S}\omega = \int_M e^{i\lambda \underline{S}}\pi_*\omega \tag{5.2}$$

reduces us to the previous situation.

However, the correspondence $\omega \mapsto \pi_*\omega$ is in general "nonlocal". If, though, ω is itself G-invariant — and note that this can happen only when G is compact! — then $\pi_*\omega$ can be computed in the following local manner. Consider the Lie algebra $\mathfrak{g}$ of G, and choose a nonzero element $e \in \Lambda^g\mathfrak{g}$, where $g = \dim G$. Such an element determines a volume for G by the formula

$$\mathrm{vol}_e(G) = \int_G \omega$$

where ω is the unique invariant volume form on G whose value at e is 1. This granted, one has the local formula

$$\pi^*\pi_*\omega = \mathrm{vol}_e(G)\cdot \iota(e)\cdot\omega, \tag{5.3}$$

valid for any invariant volume ω on P. Here $\iota(e)$ denotes the inner product with the "vertical" section of $\Omega^g(P)$ determined by e. Note also that π^* *imbeds* $\Omega^*(M)$ in $\Omega^*(P)$ as the subcomplex of "basic forms" in P (that is, the forms annihilated by ι_x and $\mathcal{L}_x$ for any vertical vector field induced by an $x \in \mathfrak{g}$), so (5.3) *characterizes* $\pi_*\omega$ *completely.*

In the physics literature, (5.3) is the point of departure for making sense of π_* for *noncompactly supported but invariant* volumes ω on P. One simply passes from ω to the basic form $\iota(e)\omega$. Thus to study the asymptotics of

$$T_\lambda(\omega) = \int_P e^{i\lambda S}\cdot\omega \tag{5.4}$$

in the equivariant context means to study the asymptotics of

$$\omega \mapsto \int_M e^{i\lambda\underline{S}}\cdot\iota(e)\cdot\omega. \tag{5.5}$$

In practice, the right-hand side is computed by restricting $e^{i\lambda S}\iota(e)\omega$ to any section $s : M \to P$. That is,

$$\int_M e^{i\lambda\underline{S}}\iota(e)\cdot\omega \equiv \int_M s^*e^{i\lambda S}\,\iota(e)\cdot\omega. \tag{5.6}$$

These "equivariant", or regularized, asymptotics are therefore well-defined only up to a choice of $e \in \Lambda^g\mathfrak{g}$, and hence depend on a parameter. However, the ratio of two such regularized series is intrinsically well-defined.

In certain situations, a "natural" choice for e presents itself. Notably, suppose V is a finite-dimensional Hilbert space and S is a quadratic form on V. Then there is a unique self-adjoint operator A on V, such that

$$S(x) = \frac{1}{2}\langle Ax, x\rangle \qquad x \in V,$$

and V splits into an orthogonal direct sum:

$$V = \operatorname{Ker} A \oplus \operatorname{Im} A$$

which is stable under A. In particular, $A' \equiv A|\operatorname{Im} A$ is nonsingular, and hence has a well defined inverse ${A'}^{-1}$ and determinant $\det(A')$, which we will denote by $\det' A$.

If $\operatorname{Ker} A \neq 0$, then S is clearly invariant under the additive group G of $\operatorname{Ker} A$. Hence, $\mathfrak{g}$ is naturally identified with $W = \operatorname{Ker} A$, and the induced metric on W determines two canonical elements $e \in \Lambda^{\text{top}} W$ of norm 1, so that with the appropriate one of these, the integral $\int e^{i\lambda S}\psi \operatorname{vol}_V$ is "regularized" in the equivariant sense by

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int_{W^\perp} e^{i\lambda S}\psi \operatorname{vol}_{W^\perp}, \qquad m = \dim W^\perp. \tag{5.7}$$

Note finally that we can define the asymptotics of

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int_{W^\perp} e^{i\lambda S} \operatorname{vol}_{W^\perp} \tag{5.8}$$

as those of (5.7) where ψ is locally constant and equal to 1 at 0. Thus

$$\frac{1}{(2\pi)^{\frac{m}{2}}} \int_{W^\perp} e^{i\lambda S} \operatorname{vol}_{W^\perp} \sim \frac{e^{\frac{i\pi}{4}\operatorname{sign} A}}{\lambda^{\frac{m}{2}} |\det'(A)|^{\frac{1}{2}}}. \tag{5.9}$$

It is the infinite-dimensional analogue of this situation that will now finally, in the next section, lead us to the "higher" linking invariants of knots. But it may be appropriate to close this section with some remarks on the Faddeev-Popov construction which treats the equivariant situation in a quite different manner.

In this construction, a section $s : M \to P$ is described as the zero set of an auxiliary function F, from P to a vector space V of dimension g. Thus,

$$F : P \to V$$

is assumed to have 0 as a regular value, so that the manifold $F^{-1}(0)$ defines a section s_F of π. This implies that for each $p \in F^{-1}(0)$, dF_p induces an isomorphism of the vertical tangent space to P at p with V:

$$dF_p : \mathfrak{g} \xrightarrow{\sim} V.$$

Such an F, usually called a gauge-fixing function, now determines a *new* function on $P \times V^*$:

$$\bar{S}(p, s) \equiv \underline{S}(p) + \langle F(p), \xi \rangle, \ p \in P, \ \xi \in V^*.$$

The first virtue of this procedure is that if $\underline{S}$ is *nondegenerate*, then $\bar{S}$ is also *nondegenerate*, but with critical set isomorphic to that of $\underline{S}$! More precisely, we have

$$C_{\bar{S}} = s_F(C_{\underline{S}}). \tag{5.10}$$

Thus in particular, if $\underline{S}$ is Morse, so is $\bar{S}$! Note next that the Hessians of $\underline{S}$ at p and $\bar{S}$ at $s_F(p)$ differ by a quadratic form of signature 0. Thus the phase shifts determined by $\bar{S}$ and $\underline{S}$ are equal. And in fact, the equivariant asymptotics of $T_\lambda(\underline{S})$ can be read directly from those of $T_\lambda(\bar{S})$. Precisely, the relation between $\underline{S}$ and $\bar{S}$ is given by the Fourier transform:

$$\int_M e^{i\lambda \underline{S}} \iota(e)\omega = \left(\frac{1}{2\pi\lambda}\right)^{\frac{g}{2}} \lim_{\varepsilon \to 0} \int_{P_x V^*} e^{i\lambda \bar{S} - \varepsilon|\xi^*|^2} \langle dF_* \cdot e, d\xi \rangle \cdot \sigma^* \omega \otimes d\xi^*, \tag{5.11}$$

where $\sigma : P_x V^* \to P$ is the projection, $d\xi$ is a volume on V, and $d\xi^*$ is the dual volume on V^* (so $d\xi \otimes d\xi^* = 1$, and the norm in V^* is taken relative to any positive definite inner product on V).

The ε-regularization is needed because the volume $\sigma^*\omega \otimes d\xi^*$ has infinite support in V^*, and this formula comes about because the push-forward $\frac{1}{(2\pi\lambda)}^{\frac{n}{2}} \pi_*^\sigma \, e^{i\lambda\langle F, \xi^* \rangle - \varepsilon|\xi|^2}$ approximates the δ-function $\delta(F)$ of F as ε tends to 0. On the other hand,

$$\int_P \delta(F)\, \omega \equiv \int_{F=0} \frac{\omega}{dF^* \cdot d\xi}, \tag{5.12}$$

so that the term $\langle dF_* \cdot e, d\xi \rangle$ converts this integral to the desired one:

$$\int_{F=0} \iota(e)\omega = \int_M s_F^* \iota(e)\omega. \tag{5.13}$$

In the physics literature, the determinantal factor $\langle dF_* \cdot e, d\xi \rangle$ is usually next expressed as a Berezin integral over some ghost variables, but I will not attempt to delve into these secrets here.

6. Relations with C^∞-invariants of Manifolds.

The primary link between topology and quantum physics is the de Rham complex $\Omega^*(M)$, and the simplest and most intrinsic functions on $\Omega^*(M)$ arise from its ring structure. Thus, if M is compact and oriented, we have two natural bilinear forms on $\Omega^*(M)$:

$$\langle \varphi, \psi \rangle_1 = \int_M \varphi \wedge \psi \tag{6.1}$$

and

$$\langle \varphi, \psi \rangle_2 = \int_M \varphi \wedge d\psi. \tag{6.2}$$

The quadratic form associated with the first of these descends to cohomology, and for $4k$-dimensional manifolds induces a symmetric quadratic form on $H^{2k}(M)$ which plays a fundamental role in all aspects of topology. The signature of the form is called the signature of M and, according to a celebrated theorem of Hirzebruch, can be expressed in terms of the characteristic classes of M.

The relations with topology of the quadratic form corresponding to (6.2) are more subtle. Note that $\langle \varphi, \psi \rangle_2$ becomes symmetric when restricted to Ω^{2q-1} on manifolds of dimension $4q-1$, so that its manifestation occurs for the first time in three dimensions. This form provides us with an interesting function S on the q-forms of a $(4q-1)$-manifold:

$$S(\varphi) = \frac{1}{2}\langle \varphi, \varphi \rangle_2, \qquad \varphi \in \Omega^{2q-1}(M) \quad \dim M = 4q-1, \tag{6.3}$$

and it is this S which is the abelian antecedent of the "new" knot invariants in three dimensions. Also, the relation of this S to the Ray-Singer torsion was already pointed out by A. Schwarz some 20 years ago [S].

We restrict our remarks to the case where $\dim M = 3$ from now on. Note then that S clearly vanishes on the image of $\Omega^0(M)$ under d. Hence if $H^1(M) = 0$, then the image of d precisely describes the null-space of S.

This follows at once from a Riemannian reinterpretation of S. Indeed, let g be a Riemann structure on M and $*$ the usual Hodge duality operator induced by g, mapping Ω^q to Ω^{n-q}, but renormalized by the the volume of M relative to g. That is, assume

$$\int_M *1 = 1. \tag{6.4}$$

In terms of $*$, each $\Omega^q(M)$ inherits an L^2-structure from g, given by

$$(\varphi, \psi) = \int_M \varphi \wedge *\psi. \tag{6.5}$$

Note that we have $*^2 = 1$ on 3-manifolds, so that our action S is given by

$$S(\varphi) = \frac{1}{2}(\varphi, *\, d\varphi), \qquad \varphi \in \Omega^1, \tag{6.6}$$

and is thus the quadratic form associated to the self-adjoint operator $A = *d$. Hence the null-space of S is the kernel of A. But $*d\varphi = 0$ if and only if $d\varphi = 0$. Q.E.D.

At this stage, we see that this S is precisely the infinite-dimensional analogue of the equivariant situation described in the previous section with Ω^0/(constant functions) playing the role of the symmetry group. If sense can be made of the terms in the asymptotics of $\int e^{i\lambda S}$ vol in this infinite-dimensional context, then one might hope that each term in this asymptotic expansion might be a differentiable invariant of M.

Purely on the formal level an analogue of (5.9) yields:

$$\int e^{i\lambda S}\,\mathrm{vol} \sim \frac{1}{(2\pi\lambda)^{\frac{\infty}{2}}} \cdot \frac{e^{i\frac{\pi}{4}\,\mathrm{sign}(*d)}}{|\det{}'(*d)|^{\frac{1}{2}}} \tag{6.7}$$

Now the $(2\pi\lambda)^{-\frac{\infty}{2}}$ can be discarded, renormalising $T_\lambda(S)$ to $\frac{1}{(2\pi\lambda)^{\frac{n}{2}}}T_\lambda(S)$ on n-dimensional manifolds, and then letting the dimension go to ∞. Similarly, one might try to exhaust $\Omega^1(M)$ by finite-dimensional subspaces and make sense of the other two terms. This does not quite work, but the "zeta-function regularizations" of these entities at least produce finite answers for both terms. This follows from deep results on the spectrum of elliptic operators, going back to Minakshi-Sundram and Pleyel [MP] and Seeley [Se], and put to wonderful use over the years by Ray and Singer [RS], as well as by Atiyah, Singer and Patodi [ASP].

This "zeta regularization" proceeds as follows: if $\lambda_1, \ldots, \lambda_n$ are positive real numbers, then their ζ-series is given by

$$\zeta(s) = \sum_{i=1}^{n} \lambda_i^{-s}. \tag{6.8}$$

Hence $\zeta'(s) = \sum - \ln \lambda^i \cdot \lambda_i^{-s}$, and $\zeta'(0) = -\sum \ln \lambda^i$. It follows that

$$e^{-\zeta'(0)} = \prod \lambda_i. \tag{6.9}$$

Similarly, given n real numbers λ_i, and setting

$$\eta(s) = \sum_{i=1}^{n} |\lambda_i|^{-s} \operatorname{sign}(\lambda_i), \tag{6.10}$$

we have

$$\eta(0) = \# \text{ of positive } \lambda\text{'s} - \# \text{ of negative } \lambda\text{'s}. \tag{6.11}$$

Now the operator

$$*d : \Omega^1 \to \Omega^1 \tag{6.12}$$

is both self-adjoint and elliptic and has a non-negative square $*d*d = d^*d$. Furthermore, $*d$ has a discrete spectrum, and if we set

$$\zeta_{d^*d}(s) = \sum \lambda_i^{-s}, \qquad \lambda_i \in \text{ Spectrum of } d^*d, \tag{6.13}$$

then this zeta function converges for $\operatorname{Re} s$ large. In addition, this function of s extends to a meromorphic function with poles along the real axis, but regular at 0! Thus, $\zeta_{d^*d}(0)$ is well-defined and serves to define the regularized value of the determinant of d^*d on $\operatorname{Im}(d^*a)$:

$$\det{}'(d^*d) \equiv e^{-\zeta'_{d^*d}(0)}. \tag{6.14}$$

Similarly,

$$\eta_{*d}(s) = \sum |\lambda_i|^{-s} \operatorname{sign}(\lambda_i) \tag{6.15}$$

serves to define the ζ-regularized signature of $*d$:

$$\operatorname{sign}(*d) \equiv \eta_{*d}(0). \tag{6.16}$$

Armed with these facts, one is led to conjecture that the asymptotic expansion of $e^{\frac{1}{2}i\lambda \int (*d\varphi, \varphi) \operatorname{vol}}$ (where the integral is taken over $\Omega^1/d\Omega^0$) is given by the single leading term

$$\frac{e^{i\frac{\pi}{4} \operatorname{sign}(*d)}}{|\det{}'(d^*d)|^{\frac{1}{4}}}. \tag{6.17}$$

Now the ζ-regularization has certain obvious functorial properties which enable one to express this quantity in terms of the regularized determinants of the Laplacian $\square =$

$dd^* + d^*d$ on Ω^0 and Ω^1. To keep track of these in greater detail, we write $d_i : \Omega^i \to \Omega^{i+1}$, so that $d_i^* : \Omega^{i+1} \to \Omega^i$, and

$$\begin{aligned}\Box|\Omega^0 &= d_0^* d_0\\ \Box|\Omega^1 &= d_1^* d_1 + d_0 d_0^*\\ (*d)^2 &= d_1^* d_1.\end{aligned}$$

Hence, $\det'(*d) = \det'(d_1^* d_1)^{\frac{1}{2}} = \left\{ \dfrac{\det'(\Box_1)}{\det'(\Box_0)} \right\}^{\frac{1}{2}}$.

On the other hand, the action of the symmetry group $\Omega^0/\mathbb{R}$ occurs via the differential d_0. That is, $S(\theta) = S(\theta + d_0 f)$, and d_0 is *not* an isometry of $\Omega^0/\mathbb{R}$ onto $\mathrm{Ker}(*d)$. The correction factor for this discrepancy introduces another $\det'(\Box_0)^{\frac{1}{2}}$ into the denominator. These heuristics therefore lead us to rewrite (6.17) in the form

$$e^{i\frac{\pi}{4}\operatorname{sign}'(*d)}\{\det'\Box_0\}^{\frac{3}{4}}\{\det'\Box_1\}^{-\frac{1}{4}}, \tag{6.18}$$

and one might hope that this expression is independent of the Riemannian metric g. Unfortunately this is not quite the case: the invariant $\mathrm{sign}(*d)$ does vary with g, but the absolute value of this number is independent of g, and its square is a torsion invariant the Ray-Singer type. See for instance [S].

7. The introduction of links in M.

An oriented knot K in M defines a distribution $\gamma(K)$ on $\Omega^1(M)$ by integration:

$$\gamma_K(\theta) = \int_K \theta, \tag{7.1}$$

and γ_K is clearly invariant under the gauge group $\Omega^0/\mathbb{R}$:

$$\int_K (\theta + d\alpha) = \int_K \theta. \tag{7.2}$$

If one thinks of a tangent vector X_p at $p \in M$ as a linear coordinate on Ω^1, its value $X_p(\theta)$ at θ being the value $\theta(X_p)$ of θ on X_p, then γ_K appears as a smeared out "sum" of these $\theta(X_p)$. This train of thought leads one to inquire whether the asymptotic expectation values of these "sums" relative to the quadratic form S associated with $*d$ can be defined meaningfully, and if so, what their topological significance is. In short, if $K_1, \ldots, K_l$ are knots in M, we seek to determine the asymptotic ratio:

$$\langle \gamma(K_1) \cdots \gamma(K_l) \rangle = \frac{\int e^{i\lambda S} \gamma(K_1) \cdots \gamma(K_l)\,\mathrm{vol}}{\int e^{i\lambda S}\,\mathrm{vol}} \tag{7.3}$$

In view of the foregoing, it is clear that the difficult constant term of the previous section now cancels out, and that these expectation values should then be computable in terms of the inverse of the operator $(*d)'$, that is, in terms of the operator which inverts $*d$ on its image.

This is the guiding analogy for all the infinite-dimensional constructions in field theory, and, as we will see in a moment, it leads to a satisfactory geometric and topological conclusion in this instance.

To explain these matters, we first recall some basic facts of elliptic theory, facts which become especially geometric and topological in the context of the Hodge theory.

Given a manifold M, we write Δ for the diagonal in $M \times M$, and $M\widehat{\times}M$ for $M \times M$ with Δ "blown up". Thus $M\widehat{\times}M$ maps naturally onto $M \times M$ and is one-to-one outside Δ, while in $M\widehat{\times}M$, Δ is replaced by the space of rays in the normal bundle $N\Delta$ of Δ in $M \times M$. We then have natural projections π_1 and π_2 of $M\widehat{\times}M$ onto the two factors of M, and the fiber of π_i at p consists of M_p, the space obtained by "blowing up" $p \in M$. Thus $M\widehat{\times}M$ is a manifold whose boundary is

$$E = \partial(M\widehat{\times}M), \tag{7.4}$$

the normal ray-bundle over Δ, which is an S^{n-1}-bundle over Δ. Note that $\pi_1 = \pi_2$ on E, so $\pi_i^E \equiv \pi_i|E$ is again a spherical fibration.

In terms of these concepts, the Hodge theory on a compact n-dimensional Riemannian manifold M with n odd provides one with a canonical Green's form in $\Omega^{n-1}(M\widehat{\times}M)$, which serves to invert the operator d as far that is possible. Precisely, we have:

Proposition. *The Green's form Θ associated to a Riemann structure g on M is a form $\Theta \in \Omega^{n-1}(M\widehat{\times}M)$ with the following three properties:*

$$\pi_*^E \iota_E^* \Theta = 1 \tag{7.5}$$

$$d\Theta = \sum_{\alpha_i} \pm \pi_1^* \alpha_i \wedge \pi_2^* * \alpha_i \tag{7.6}$$

$$T^*\Theta = (-1)^{n+1}\Theta, \tag{7.7}$$

where the α_i runs over an orthonormal basis for the harmonic forms of M relative to g and $T : M\widehat{\times}M \to M\widehat{\times}M$ acts by permuting the factors.

Remark. The Green's form Θ should be thought of as implementing the parametrix associated with the Green's operator G on $\Omega^*(M)$. Recall that the Laplacian of g

$$\Box = dd^* + d^*d \tag{7.8}$$

splits Ω^q canonically into three orthogonal pieces:

$$\Omega^q = \operatorname{Im} d \oplus \operatorname{Im} d^* \oplus \mathcal{H}^q, \tag{7.9}$$

with $\mathcal{H}^q = \operatorname{Ker} \Box_q$, so that $\Box_q$ is invertible on the other two pieces. The operator G is now defined by

$$\begin{gathered} G|\operatorname{Im} d = \Box^{-1}, \qquad G|\operatorname{Im} d^* = \Box^{-1} \\ G|\mathcal{H}^q = \text{ identity.} \end{gathered} \tag{7.10}$$

It follows that the operator

$$P_g = d^* \circ G \tag{7.11}$$

maps Ω^q to Ω^{q-1} and satisfies the relation:

$$dP_g + P_g d = 1 - \pi_{\mathcal{H}}, \tag{7.12}$$

where $\pi_{\mathcal{H}}$ is the orthogonal projection onto the harmonic forms $\mathcal{H} = \oplus \mathcal{H}^q$.

The relation (7.12) follows by comparing the two sides of the equation to each other while restricting to each summand of (7.9): first, it is clear that both sides annihilate $\mathcal{H}$. Next if $\omega = d\alpha \in \operatorname{Im} d$, the left-hand side sends ω to $dP_g d\alpha = dd^*\Box^{-1}d\alpha = d(\Box^{-1}d^*d\alpha) = d\alpha$, which agrees with the identity map on the right. Similarly, if $\omega = d^*\alpha$, then the left side sends ω to $d^*\Box^{-1}dd^*\alpha = d^*d\Box^{-1}d^*\alpha = d^*\alpha$. Q.E.D.

In general, an operator P with the property that

$$dP + Pd = 1 - S, \tag{7.13}$$

where S is a smoothing operator, is called a parametrix for Ω^*, so P_g is the parametrix on Ω^* which is especially attuned to the Riemann structure g.

The statement that the Green's form Θ implements P_g means precisely that P_g can be defined in terms of Θ by the formula:

$$P_g\alpha = \pi_*^1(\pi_2^*\alpha \wedge \Theta). \tag{7.14}$$

One checks that (7.14) implies (7.12) via the generalized Stokes formula for an oriented fibering $\pi : E \to M$, whose fiber is a manifold F with boundary ∂F. Under these circumstances, we have:

$$d\pi_* = (-1)^f \pi_* d + \pi_*^\partial \qquad f = \dim F, \tag{7.15}$$

where π_*^∂ denotes integration over the boundary of E.

In the case at hand, (7.14) therefore yields:

$$dP_g\alpha = -\pi_*^1\pi_2^*d\alpha\Theta \pm \pi_*^1\pi_2^*\alpha d\Theta + \pi_*^\partial\pi_2^*\alpha\Theta. \tag{7.16}$$

Now by (7.5) and (7.6) and the fact that $\pi_1^* = \pi_2^*$ on ∂E, this equals $-P_q d\alpha \pm \pi_{\mathcal{H}} + \alpha$. Q.E.D.

Remark. The Green's form Θ should be thought of as the Schwartz kernel of P_g, with the sign conventions arranged so as to fit a cohomological interpretation. The form Θ is constructed explicitly in dimension 3 by Axelrod and Singer in their ground-breaking paper [AS].

Our next observation is that, in our situation, Θ also serves to invert the operator $A = *d$ on its image in Ω^1. Observe first of all that $H^1(M) = 0$ implies

$$\text{image} *d = \operatorname{Ker} d* \qquad \text{in } \Omega^1(M). \tag{7.17}$$

Hence if we set

$$Q\alpha = \pi_*^1(\pi_2^*((*\alpha) \wedge \Theta) \qquad \alpha \in \operatorname{Ker} d*, \tag{7.18}$$

then the Stokes formula immediately yields

$$*dQ\alpha = \alpha. \tag{7.19}$$

Now the restriction $\iota^*\Theta$ of Θ to $(M \times M) - \Delta$ breaks into components according to the decomposition of $T_{p\times q}(M \times M) = T_pM \oplus T_qM$:

$$\iota^*\Theta = \Theta^{2,0} + \Theta^{1,1} + \Theta^{0,2}, \tag{7.20}$$

and it is clear for dimensional reasons that only $\Theta^{1,1}$ is involved in the definition of Θ. In short, the *Schwartz kernel of Q is represented by* $\Theta^{1,1}$, and, so interpreted, corresponds to the "propagator" in the physics literature. Essentially, it plays the role of the (i,j)-th entry of the matrix B of Section 4. Thus in the present context, where (i,j) should be thought of as two tangent vectors X_p and Y_q of M, the entry B_{ij} corresponds to $\Theta^{1,1}(X_p, Y_q)$.

At this stage, the expectation values $\langle \gamma_{K_1}, \gamma_{K_2} \rangle$ are readily identified. Namely, let $\mathcal{K}$ denote the space of smooth parameterized imbeddings

$$K : S^1 \hookrightarrow M. \tag{7.21}$$

The evaluation map then gives us an arrow

$$\mathcal{K} \times S^1 \xrightarrow{e} M, \tag{7.22}$$

and if K is a point of $\mathcal{K}$, we also write $K : S^1 \to M$ for e restricted to $K \times S^1$. Now given two *disjoint* knots K_1 and K_2 we have the mapping

$$K_1 \times K_2 : S^1 \times S^1 \to (M \times M) - \Delta, \tag{7.23}$$

whence the integral

$$\int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta^{1,1} \tag{7.24}$$

precisely corresponds to the "smeared out" sum of the propagator applied to a point on K_1 and a point on K_2.

In short, the asymptotics of $\langle \gamma_{K_1}, \gamma_{K_2} \rangle$ are given by:

$$\langle \gamma_{K_1}, \gamma_{K_2} \rangle \sim \frac{i}{\lambda} \int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta^{1,1}. \tag{7.25}$$

Observe now that $(K_1 \times K_2)^*$ preserves types of our forms, and that in dimension two, $S^1 \times S^1$ only has type $(1,1)$ forms. We can therefore also write

$$\langle \gamma_{K_1}, \gamma_{K_2} \rangle \sim \frac{i}{\lambda} \int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta, \tag{7.26}$$

and we have then arrived at a formula very much in the spirit of my lectures last year. There our starting point was the "tautological" form $\Theta_{12} \in \Omega^2((\mathbb{R}^3 \times \mathbb{R}^3) - \Delta)$ which we defined as the pull-back of the normalized spherical volume ω on S^2 under the map

$$\varphi : (x,y) \to \frac{x-y}{|x-y|}, \qquad x, y \in \mathbb{R},\ x \neq y. \tag{7.27}$$

Furthermore, we then determined the integral

$$l(K_1, K_2) = \int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta_{12} \tag{7.28}$$

to be precisely the Gauss integral for the linking number of two disjoint knots in $\mathbb{R}^3$.

Now it is not difficult to show that in fact Θ_{12} on $(\mathbb{R}^3 \times \mathbb{R}^3) - \Delta$ is also precisely the restriction of the Green's form Θ of $\mathbb{R}^3$ in its usual metric to $(\mathbb{R}^3 \times \mathbb{R}^3) - \Delta$. Of course, as $\mathbb{R}^3$ is not compact, Θ must now be interpreted as the "inverse" of d on the compactly supported forms.

Thus (7.26) identifies the coefficient of $1/\lambda$ in the quadratic expectation values $\langle \gamma_{K_1}, \gamma_{K_2} \rangle$ with the linking number of K_1 and K_2 when $M = \mathbb{R}^3$. Actually, this interpretation of $\langle \gamma_{K_1}, \gamma_{K_2} \rangle$ holds on any homology 3-sphere M, so Θ is indeed the proper generalization of Θ_{12} to all such manifolds.

Moreover, the integral

$$\int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta \tag{7.29}$$

will always describe the linking number of K_1 and K_2, provided Θ implements a parametrix for M.

Indeed, let K_1 and K_2 be two distinct knots in the homology 3-sphere M. In homology, the linking number $l(K_1, K_2)$ is given by the intersection number of K_1 with any chain bounding K_2. Dually, if u_1 and u_2 are the Poincaré duals of K_1 and K_2 respectively, this same linking number is given by the integral over M of $u_1 \cdot \alpha$, where α is any 1-form with $d\alpha = u_2$. Thus,

$$l(K_1, K_2) = \int u_1 \alpha, \qquad \text{if } d\alpha = u_2. \tag{7.30}$$

Now suppose the parametrix P is implemented by $\Theta \in \Omega^2(M \widehat{\times} M)$:

$$Pu = \pi_*^1(\pi_2^* u)\Theta. \tag{7.31}$$

Then $dP + Pd = 1$ implies that $dPu_2 = u_2$, so we may choose Pu_2 for α in (7.30). Hence:

$$\begin{aligned} l(K_1, K_2) &= \int_M u_1 \cdot \pi_*^1(\pi_2^* u_2 \Theta) \\ &= \int_M \pi_*^1(\pi_1^* u_1 \cdot \pi_2^* u_2 \Theta) \\ &= \int_{M \times M} \pi_1^* u_1 \cdot \pi_2^* u_2 \Theta \\ &= \int_{S^1 \times S^1} (K_1 \times K_2)^* \Theta. \end{aligned} \tag{7.32}$$

Here the second step is given by Fubini's theorem, and the last step follows by letting u_1 and u_2 tend to the distributions γ_{K_1} and γ_{K_2} respectively.

This argument then exemplifies the proper homological interpretation of the quadratic form $\int_M \varphi \, d\varphi$ on any $(4q-1)$-dimensional manifold. It describes the linking of two submanifolds A, B in dimension $2q-1$ in the following sense. Given u_1 and u_2 in $\Omega^{2q}(M)$ and α_i in Ω^{2q-1} such that

$$d\alpha_1 = u_1, \qquad d\alpha_2 = u_2,$$

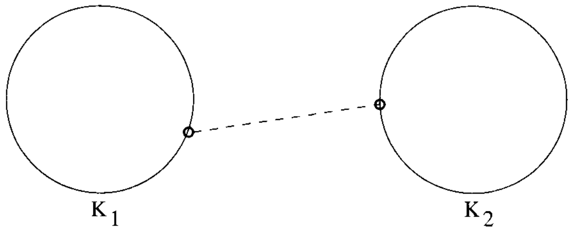

FIGURE 6

then by Stokes' theorem applied to $d(\alpha_1\alpha_2) = u_1\alpha_2 - u_2\alpha_1$, we obtain the identity:

$$\int u_1\alpha_2 = \int u_2\alpha_1, \tag{7.33}$$

and if u_1 and u_2 are the Poincaré duals of disjoint null-homologous submanifolds A, B of dimension $2q-1$, then this linking number is given by

$$l(A,B) = \int u_1\alpha_2 = \int u_2\alpha_1. \tag{7.34}$$

In fact, the bilinear form $\int_M \varphi\, d\psi$ describes the linking phenomenon in all dimensions.

Returning to our physics-inspired discussion, we have now seen how linking appears in this context as the coefficient of $1/\lambda$ in the asymptotic expansion of the quadratic moment $\langle \gamma_{K_1}, \gamma_{K_2}\rangle$. In the physics literature, this term has the suggestive graphical description indicated in Figure 6. The dotted line represents the "propagator" applied between a point on K_1 and K_2, and the whole diagram stands for the sum of all these applications, in short, for our integral

$$\int_{S^1\times S^1} (K_1 \times K_2)^*\Theta. \tag{7.35}$$

What about knots that do intersect, or indeed what about $\langle \gamma_{K_1}, \gamma_{K_1}\rangle$? These questions force one to deal with the behavior of the propagator near the diagonal, where it is *a priori* not defined, and so forces one to consider configuration spaces and their compactifications, etc. — precisely the topics treated in last year's lecture. There our choice for dealing with the diagonal in the present context was to exploit the fact that the propagator extended to a form Θ on $M\widehat{\times}M$. Hence, the natural integral to propose for $\langle \gamma_{K_1}, \gamma_{K_1}\rangle$ is

$$\int_{S^1\widehat{\times}S^1} (K_1 \times K_1)^*\Theta, \tag{7.36}$$

arising from the map

$$K\widehat{\times}K : S^1\widehat{\times}S^1 \to M\widehat{\times}M.$$

This does work, and it produces a finite number for the "self-linking" of K, but a price must be paid. The space $S^1\widehat{\times}S^1$ is a manifold with nontrivial boundary. As a consequence, the integral (7.36) varies as K moves in its isotopy class. In short, this self-linking number of K is not a knot invariant in the usual sense since it varies under isotopies and under changes in the underlying Riemann structure of M.

8. The Chern-Simons Action

In Section 6, we discussed the action $S(\varphi) = \frac{1}{2}\int \varphi \wedge d\varphi$ and saw how it related to the linking of knots in homology spheres. From the point of view of physics, this is the abelian, or $U(1)$, theory, and goes back to the early quantum mechanics of the electromagnetic field. The "new" knot invariants emerged in the physics context only after Witten's remarkable paper [W] presenting the "Chern-Simons action" as the natural nonabelian generalization of the abelian action $\frac{1}{2}\int \varphi \wedge d\varphi$.

In this last section, I will very briefly describe this extension, and indicate how it leads to the sort of integrals we encountered in last year's lecture [B].

In the nonabelian extension of $\Omega^1(M;\mathbb{R})$, one steps up to the space $\Omega^1(M;\mathfrak{g})$ of "matrix-valued" 1-forms. More precisely, one starts with a compact Lie group G with Lie-algebra $\mathfrak{g}$, and if we think of G as imbedded as a subgroup of some large orthogonal group, then $\mathfrak{g}$ becomes realized as a Lie subalgebra of the skew-symmetric matrices, so that $\Omega^1(M;\mathfrak{g})$ can be thought of as matrix-valued 1-forms, or equivalently as matrices of 1-forms on M. The second and much more profound step, is to find a proper nonabelian generalization of the symmetry group.

Here the physics literature had an answer long ago. Namely, they decreed that in so-called "gauge theories", the group of symmetries should be the group of smooth maps of M to G:

$$\mathcal{G} = \mathrm{Map}(M, G) \tag{8.1}$$

and that this group of "gauge transformations" should act on $\Omega^1(M;\mathfrak{g})$ by the rule:

$$g^*\omega = g^{-1}\omega g + g^{-1}\,dg. \tag{8.2}$$

Remark. Here I have written things in terms of matrix-valued 1-forms and functions, so matrix multiplication is to be understood on the right-hand side of (8.2). Note that when $G = S^1$, so that $\mathfrak{g} = \mathbb{R}$, then (8.2) simplifies to

$$g^*\omega = \omega + g^{-1}\,dg, \tag{8.3}$$

and that if M is a homology sphere, then:

$$g^{-1}\,dg = d\log g. \tag{8.4}$$

Thus, in this case, (8.2) reduces to the abelian situation with Ω^0 acting on Ω^1 by $\omega \to \omega + df$.

A natural first try for an action on $\Omega^1(M;\mathfrak{g})$ invariant under $\mathcal{G}$ is

$$S(\varphi) = \mathrm{trace}\int_M \varphi\,d\varphi, \qquad \varphi \in \Omega^1(M;\mathfrak{g}), \tag{8.5}$$

but this ansatz fails the test. However, the action

$$S(\varphi) = \frac{1}{4\pi}\int_M \operatorname{trace}\left(\varphi\, d\varphi + \frac{2}{3}\varphi^3\right) \tag{8.6}$$

turns out to be invariant under the action of $\mathcal{G}_0$, the identity component of $\mathcal{G}$, and changes by multiples of 2π under the group of components of $\mathcal{G}$.

This is the "Chern-Simons action", Chern and Simons having understood this behavior of (8.6) long ago, and having put it to good use in differential geometry in a very different context from the present one [CS]. The extrema of the Chern-Simons action are quite easily determined and are found to be given by those φ for which the matrix-valued 2-form:

$$F = d\varphi + \varphi^2 \tag{8.7}$$

vanishes.

In the mathematics literature, the transformation rule (8.2) is associated with *connections* on principal G-bundles over M, and the F of (8.7) then corresponds to the *curvature* of that connection. From this point of view it is then clear that the $\mathcal{G}$-orbits of the extrema of the Chern-Simons action are in one-to-one correspondence with isomorphism classes of flat bundles and hence also with isomorphism classes of representations of $\pi_1(M)$ in G.

In Witten's work, he considers the asymptotics of

$$T_\lambda\{S\} = \int e^{i\lambda S}\mathcal{D}\varphi \tag{8.9}$$

but remarks that here λ should be restricted to the integers because it is only then that $e^{i\lambda S}$ is a well-defined $\mathcal{G}$-invariant function on $\Omega^1(M;\mathfrak{g})$. Moreover he argues that, for $k \in \mathbb{Z}$,

$$W_k(M) = \int e^{ikS(\varphi)}\mathcal{D}\varphi \tag{8.10}$$

should be a well-defined numerical invariant of M and prescribes its behavior under surgery on M.

He also incorporates knots into this framework. Recall that in the abelian theory, the function $\gamma_k : \theta \to \int_K \theta$ was gauge-invariant, and in terms of it, $e^{-\gamma_k}$ gives the holonomy of K relative to the connection defined by θ. In the nonabelian theory, the natural gauge-invariant functions on $\Omega^1(M;\mathfrak{g})$ are given by the value of a class function χ for G on the holonomy along K relative to the connection determined by θ. Thus one now deals with the path-ordered integral of $h_K = e^{-\int_K \theta}$ to get at the holonomy, and one can take for χ, the trace χ_V of any representation of G in V. In [W], Witten goes on to argue that the expectation value of $\chi_V(h_K)$ is related to the Jones polynomial of K, etc.

In our framework, it is more natural to consider the "Taylor expansion" of $\chi_V(h_K)$, which is provided by the path-ordered integral.

Precisely, suppose we parameterize K by the map

$$K : [0,1] \longrightarrow M. \tag{8.11}$$

Then the path-ordered integral of θ along K is given by the series

$$\text{p.o. } e^{-\int_K K^*\theta\,dt} = \sum_{n=0}^{\infty} \pi_*^{\Delta_n} e_K^* \underbrace{\theta_K \otimes \cdots \otimes \theta_K}_{n}, \tag{8.12}$$

where Δ_n is the simplex $0 \leq t_0 \leq t_1 \ldots \leq t_n \leq 1$, $\pi_*^{\Delta_n}$ denotes the integral over Δ_n, and

$$e_K(t_0, t_1, \ldots, t_n) = K(t_1) \times \cdots \times K(t_n) \in M^n.$$

Applying the representation V to (8.12) yields a power series in $\operatorname{End}(V)$ starting with the identity and whose next term is

$$\int_0^1 K^*\theta_V. \tag{8.13}$$

After taking the trace, this integral is therefore the natural extension of the γ_K in the abelian theory.

Consider now the asymptotics of

$$\langle \chi_V h_K \rangle = \int_{\Omega^1/\mathcal{G}} e^{i\lambda S} \chi_V(h_K) \Big/ \int_{\Omega^1/\mathcal{G}} e^{i\lambda S}, \tag{8.14}$$

where S is the Chern-Simons action

$$S = \frac{1}{4\pi} \int_M \operatorname{trace}_V \left(d\theta \wedge \theta + \frac{2}{3}\theta^3 \right) \tag{8.15}$$

in the vicinity of the *trivial representation* of $\pi_1(M)$ in C. Once we introduce a metric g on M and an invariant inner product on $\mathfrak{g}$, the quadratic term in (8.15) is given by $\theta \to (\theta, *d\theta)$, with $*d$ acting on $\mathfrak{g}$-valued forms componentwise. Hence its inverse on its image is again made up of the same Green's form Θ encountered in the abelian theory. Thus in these asymptotics of (8.14) we would expect integrals of the type schematically given in Figure 7 to arise from the expansion of h_V. (In the figure, the number of dotted lines keeps track of the power of $\frac{1}{\lambda}$ at which the term occurs.) On the other hand, the cubic term in the Chern-Simons action also gets into the act, as we saw when we dealt with the function $S = \frac{x}{2} + x^3$ in Section 3. For instance, diagrams of the type shown in Figure 8 will now also occur due to the interaction of the h_V term with the θ^3 terms, and also due to the interactions of the θ^3 terms with themselves.

It is in this context then that, using the procedures explained in the earlier sections, the new self-linking invariant for knots in $\mathbb{R}^3$ was discovered by Dror Bar-Natan [BN] and independently by E. Guadagnini, M. Martellini, and M. Mintchev [GMM] at about the same time.

Diagrammatically, this new invariant is given by Figure 9, and, as is explained in my lecture last year, a precise definition of the relevant integrals involves compactification of configuration spaces along their diagonals. The reader is therefore referred to [B] and [BT] or to the original papers for more details.

We have now at long last reached the goal of this account. Although the "physics route" to this invariant and its generalizations has the undisputible advantage that it

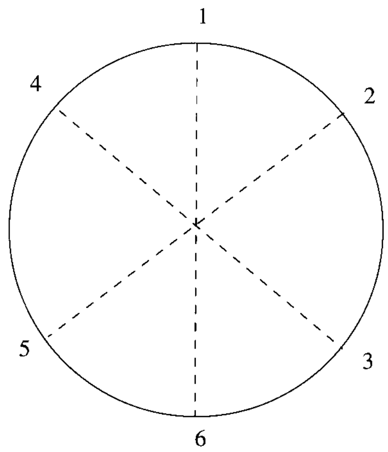

FIGURE 7

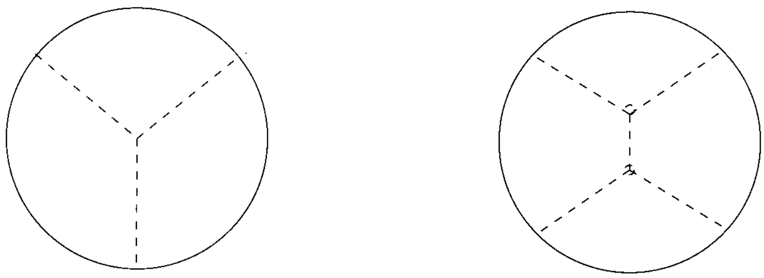

FIGURE 8

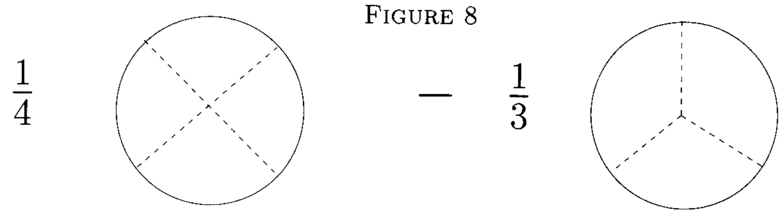

FIGURE 9

got there first, the validity of the Feynman expansion in the present context remains in doubt. One is plagued by the question of "hidden faces" which the Feynman procedures do not "see", and which, to my knowledge, can be dealt with in $\mathbb{R}^3$ only for spherically symmetric metrics.

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