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Casson's invariant and Seiberg-Witten gauge theory

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1. Introduction

Let Y be an oriented homology 3-sphere. Y bounds a smooth compact oriented spin 4-manifold W which induces the unique spin structure on Y. The Rohlin invariant of Y is defined by

$$\mu(Y) \equiv \frac{1}{8}\text{sign}(W) \pmod 2,$$

where $\text{sign}(W)$ is the signature of W. $\mu(Y)$ is well-defined by the famous theorem of Rohlin which asserts that the signature of a closed smooth spin 4-manifold is divisible by 16.

In 1985, A. Casson introduced an integer invariant of oriented homology 3-spheres via beautiful constructions on the representation spaces of the fundamental groups into $SU(2)$. Casson's invariant refines the Rohlin invariant and gives surprising corollaries in low dimensional topology. Later on Taubes [T] gave a gauge theory interpretation of Casson's invariant as one half of the Euler characteristic of the gradient of the Chern-Simons functional. Floer [F] then refined Taubes' construction and introduced an instanton homology theory—Floer homology.

Casson's invariant of an oriented homology 3-sphere Y, denoted by $\lambda(Y)$, is uniquely defined by the following Dehn surgery formula:

1. $\lambda(K_{n+1}) - \lambda(K_n) = \frac{1}{2}\Delta"_K(1)$
2. $\lambda(S^3) = 0$

where K_n is the manifold obtained from Y by $\frac{1}{n}$ surgery on a knot K in Y, and $\Delta"_K$ is the second derivative of the symmetrized Alexander polynomial of K. Casson's invariant also satisfies:

a) $\lambda(Y) + \lambda(-Y) = 0$
b) $\lambda(Y) \equiv \mu(Y) \pmod 2$
c) $\lambda(Y_1 \# Y_2) = \lambda(Y_1) + \lambda(Y_2)$

where $\mu(Y)$ is the Rohlin invariant of Y. See [AM] for details.

Beginning in 1994, many constructions in Donaldson theory have been carried out in Seiberg-Witten theory. It is generally believed that Seiberg-Witten theory and Donaldson theory are equivalent (see [D] [W]). However, no analogue of Floer homology has been constructed for Seiberg-Witten theory until recently Kronheimer and Mrowka [KM2] announced a reduced form of the Seiberg-Witten Floer homology and conjectured a relationship between Casson's invariant and the Euler characteristic of the unreduced Seiberg-Witten Floer homology. Set

$$\alpha(Y) = \chi_{SWF}(Y) - (indexD_W + \frac{1}{8}\text{sign}(W))$$

where $\chi_{SWF}(Y)$ is the Euler characteristic of the unreduced Seiberg-Witten Floer homology, and D_W is the Dirac operator on a smooth compact oriented spin 4-manifold W with boundary Y, satisfying the APS global boundary condition. $indexD_W + \frac{1}{8}\text{sign}(W)$ does not depend on W and can be expressed in terms of eta invariants (see [APS]).

Conjecture 1.1. *(Kronheimer-Mrowka)* $\lambda(Y) = \alpha(Y)$.

Remarks: This conjecture holds for Brieskorn 3-spheres [KM2].

In this paper, we give a rigorous definition of $\alpha(Y)$ and prove the following

Theorem 1.2. *Let Y be an oriented homology 3-sphere. Then*
1. *$\alpha(Y)$ is a topological invariant of Y, and $\alpha(Y) + \alpha(-Y) = 0$.*
2. *$\alpha(Y) \equiv \mu(Y) \pmod 2$, where $\mu(Y)$ is the Rohlin invariant of Y.*

On the other hand, Hitchin [H] constructed a family of metrics on S^3 which shows that $IndexD_W + \frac{1}{8}\text{sign}(W)$ varies with the metrics. So $\chi(S^3)$ is not an invariant.

Corollary 1.3. *The unreduced Seiberg-Witten Floer homology DOES depend on the Riemannian metric.*

Remarks: The proof of Theorem **1.2** works equally well for oriented rational homology 3-spheres, but the invariant α may depend on the choice of spin structures and may not be an integer. It would be interesting to know what this spin invariant is, and what the relationship with Casson-Walker's invariant [Wa] for oriented rational homology 3-spheres is. The Dehn surgery formula for invariant χ is obtained in [C].

In Section 2, we review some basic facts of the 3 dimensional Seiberg-Witten theory and introduce two types of perturbations of the Chern-Simons-Dirac functional, one of which is invariant with respect to a natural involution in Seiberg-Witten theory. This type of perturbations is used to prove the second assertion in Theorem **1.2**. Section 3 is devoted to the definition of the Euler characteristic for the gradient of the Chern-Simons-Dirac functional and the definition of α. In Section 4, we prove the topological invariance of α by carefully analyzing the Kuranishi model at a reducible critical point where a family of perturbations passes its singular point. In Section 5, we study the variation of Dirac operators with respect to the metrics. We show that certain perturbed Dirac operators are generically invertible and admit a chamber structure; they are still quaternionic and used in the construction of the involution-invariant perturbations of the Chern-Simons-Dirac functional introduced in Section 2. The involution-invariant perturbations are constructed in Section 6.

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2. Seiberg-Witten gauge theory in 3 dimensions

Let Y be an oriented homology 3-sphere equipped with a Riemannian metric g (many facts stated in this section hold for general 3-manifolds). There exists an unique $SU(2)$ vector bundle W_0 over Y as a Clifford module of the Clifford algebra bundle $Cl(TY)\otimes_{\mathbb{R}}\mathbb{C}$ such that the oriented volume form on Y acts as identity on W_0. Let $W = W_0 \otimes L$, where L is the trivial complex line bundle over Y. W is a $U(2)$ vector bundle.

Let (e^1, e^2, e^3) be an oriented local orthonormal basis of T^*Y. This gives rise to a local unitary basis of W_0 and W, within which the Clifford multiplication is given by the following matrices:

$$c(e^1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, c(e^2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c(e^3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $\psi = (z, w)$, $\varphi = (u, v)$, $\psi, \varphi \in W$, define

$$\tau(\psi, \varphi) = \frac{1}{2}\begin{pmatrix} Re(z\bar{u} - w\bar{v}) & z\bar{v} + \bar{w}u \\ \bar{z}v + w\bar{u} & -Re(z\bar{u} - w\bar{v}) \end{pmatrix}.$$

Lemma 2.1. *$i\tau(\psi, \varphi) = \frac{1}{2}(Re(z\bar{u} - w\bar{v})(e^1) + Im(z\bar{v} + \bar{w}u)(e^2) + Re(z\bar{v} + \bar{w}u)(e^3))$, so $\tau(\psi, \varphi) \in \Lambda^1(Y) \otimes i\mathbb{R}$. Moreover, we have*

$$\langle ie \cdot \psi, \varphi\rangle_{Re} = -2\langle e, i\tau(\psi, \varphi)\rangle$$

for any $e \in \Lambda^1(Y)$, and $|\tau(\psi, \psi)|^2 = \frac{1}{4}|\psi|^4$.

Proof: Direct computation. *QED*

The Levi-Civita connection of the Riemannian metric g lifts to a connection on W_0. Coupled with an $U(1)$ connection A on L, the Dirac operator D_A: $\Gamma(W) \longrightarrow \Gamma(W)$ is given in a local frame by

$$D_A = \sum_{j=1}^{3} e^j \cdot (\nabla_{e_j} + iA_j).$$

Let $\mathcal{A} = \mathcal{C} \times \Gamma(W)$ where $\mathcal{C}$ is the space of smooth $U(1)$ connections on L. The gauge group $\mathcal{G} = Map(Y, S^1)$ acts on $\mathcal{A}$ by $s \cdot (A, \psi) = (A - s^{-1}ds, s\psi)$, $s \in \mathcal{G}$, $(A, \psi) \in \mathcal{A}$. Note that $\pi_0(\mathcal{G}) = H^1(Y, \mathbb{Z}) = 0$. Each element in $\mathcal{G}$ can be written as e^f with $f \in \Gamma(\Lambda^0 Y \otimes i\mathbb{R})$

determined up to a constant $2\pi ik$, $k \in \mathbb{Z}$. So $\mathcal{G} = K(\mathbb{Z},1)$. Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$. The action of $\mathcal{G}$ is free on the subset $\mathcal{A}^* = \mathcal{A} \setminus \{\psi \equiv 0\}$, and with stabilizer S^1 on the rest. Hence $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ is homotopic to CP^∞.

We shall work within the context of Sobolev spaces and Banach manifolds. By fixing a trivialization of L, $\mathcal{C}$ can be identified with $\Omega^1(Y, i\mathbb{R})$. Define $\mathcal{A}_1^2 = L_1^2(\Lambda^1(Y, i\mathbb{R})) \times L_1^2(W_0)$, $\mathcal{G}_1^2 = \{L_2^2 \text{ maps from } Y \text{ to } S^1\}$. For simplicity, we still use the old symbols to denote the Sobolev objects.

Lemma 2.2. *$\mathcal{B}^*$ is a Banach manifold whose tangent space at (A,ψ) is*

$$T\mathcal{B}^*_{(A,\psi)} = \{(a,\varphi) \in \mathcal{A} | - d^*a + i\langle i\psi, \varphi\rangle_{Re} = 0\}.$$

Proof: Standard arguments. The key point is that the operator $d^*d + |\psi|^2$ is invertible if ψ is not identically zero. See [FU]. QED

Remarks: A neighborhood of $[A, 0]$ in $\mathcal{B}$ is given by U/S^1, where $U = \{(a,\varphi) \in \mathcal{A} | d^*a = 0, \|(a,\varphi)\| < \delta\}$.

There is an natural $\mathbb{Z}_4$ action σ on $\mathcal{A}$ given by $\sigma(A,\psi) = (-A, J\psi)$, where J is the quaternion structure on W_0. σ descends to an involution on $\mathcal{B}$ and acts freely on $\mathcal{B}^*$.

The Chern-Simons-Dirac functional on $\mathcal{A}$ is defined by

$$\mathcal{CSD}(A,\psi) = -\frac{1}{2}\int_Y A \wedge F_A + \frac{1}{2}\int_Y \langle \psi, D_A\psi\rangle_{gRe} Vol_g.$$

It is easy to see that $\mathcal{CSD}$ is gauge invariant and σ-invariant, so it descends to $\mathcal{B}$. The gradient of $\mathcal{CSD}$ at (A,ψ) is given by

$$s(A,\psi) = (*F_A + \tau(\psi,\psi), D_A\psi).$$

It can be regarded as a 'weak' tangent vector field on $\mathcal{B}^*$ in the sense that s is not in $T\mathcal{B}^*$ but in its L^2 completion $\mathcal{L}$, i.e., $\mathcal{L}_{(A,\psi)} = \{(a,\varphi) \in L^2 | - d^*a + i\langle i\psi, \varphi\rangle_{Re} = 0\}$.

The covariant derivative ∇s is given by

$$\nabla s_{(A,\psi)}(a,\varphi) = (*da + 2\tau(\psi,\varphi) - df(\varphi), D_A\varphi + a\psi + f(\varphi)\psi)$$

where $f(\varphi)$ is the unique solution to the equation $(d^*d + |\psi|^2)f = i\langle D_A\psi, i\varphi\rangle_{Re}$. As in [T], we have

Lemma 2.3. *$\nabla s_{(A,\psi)}$ defines a closed, essentially selfadjoint, Fredholm operator on $\mathcal{L}_{(A,\psi)}$, and its eigenvectors form an L^2-complete orthonormal basis for $\mathcal{L}_{(A,\psi)}$. The domain of $\nabla s_{(A,\psi)}$ is the L_1^2-Sobolev space completion of $\mathcal{L}_{(A,\psi)}$. The eigenvalues form a discrete subset of the real line which has no accumulation points, and which is unbounded in both directions. Each eigenvalue has finite multiplicity.*

The 3 dimensional Seiberg-Witten moduli space $\mathcal{M}$ is the set of critical points of $\mathcal{CSD}$ on $\mathcal{B}$, i.e., the equivalence classes of solutions to the Seiberg-Witten equations:

$$\begin{cases} *F_A + \tau(\psi,\psi) = 0 \\ D_A\psi = 0 \end{cases}$$

Let $[\theta]$ denote the unique reducible solution $[0,0]$. Then the moduli space of irreducible solutions is $\mathcal{M}^* = \mathcal{M} \setminus [\theta]$.

Lemma 2.4. *The moduli space $\mathcal{M}$ can be represented by smooth solutions and it is compact.*

Proof: Standard arguments of elliptic regularity and Maximum Principle. See [KM1].

QED

In order to define the Euler characteristic of $\mathcal{CSD}$, we need to perturb it suitably.

Definition 2.5. *A perturbation $\mathcal{CSD}'$ is admissible if:*

1. *The critical points in $\mathcal{B}^*$ are non-degenerate, i.e., $\nabla \mathrm{s}'_{[A,\psi]}$ is invertible at $[A,\psi] \in \mathcal{B}^* \bigcap \mathrm{s}'^{-1}(0)$.*
2. *The (perturbed) Dirac operator at the reducible $[\theta]$ is invertible so that $[\theta]$ is isolated.*

Here s' is the gradient of $\mathcal{CSD}'$ and $\nabla \mathrm{s}'$ is the covariant derivative of s'. The (perturbed) Dirac operator at $[\theta]$ will be clear when we specify the perturbation.

An admissible perturbation has only finitely many isolated critical points in $\mathcal{B}^*$, since we require $[\theta]$ to be isolated so that $\mathcal{M}^*$ is compact. Each irreducible critical point is assigned a sign by the mod 2 spectral flow of $\nabla \mathrm{s}'$. Since $\pi_1(\mathcal{B}^*) = 0$, the spectral flow does not depend on the path chosen. See [T].

We will consider two classes of admissible perturbations. The first class is σ-invariant. First we need to perturb the Dirac operator so that it is invertible and still quaternionic. These perturbations take the form of $D_g + f$ where g stands for the metric and f is a smooth real valued function on Y. The perturbed Chern-Simons-Dirac functional takes the form of

$$\mathcal{CSD}'_\mu(A,\psi) = \mathcal{CSD}(A,\psi) + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g + u,$$

where u is some functional on $\mathcal{B}$ which will be constructed in Section 6. For convenience, we set

$$\mathcal{CSD}_f(A,\psi) = \mathcal{CSD}(A,\psi) + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g.$$

The following proposition is proved in Section 5 in which Met stands for the space of metrics.

Proposition 2.6. *Let Y be a closed oriented 3-manifold. For a generic pair $(g,f) \in Met \times C^k(Y)$, the perturbed Dirac operator $D_g + f$ is invertible. Moreover, any two such regular pairs (g_0, f_0) and (g_1, f_1) can be connected by a path (g_t, f_t) such that the perturbed Dirac operators $D_{g_t} + f_t$ are invertible except for $t_i \in (0,1)$ with* Ker $(D_{g_{t_i}} + f_{t_i}) = \mathbb{H}$, *$i = 1, 2, \ldots, n$. Let λ_t, ψ_t be the eigenvalue and eigenvector near t_i, i.e., $(D_{g_t} + f_t)\psi_t = \lambda_t \psi_t$ with $\lambda_{t_i} = 0$ and $\|\psi_t\|_{L^2} = 1$, we have*

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{g_t} + f_t)(t_i)(\psi_{t_i}), \psi_{t_i}\rangle_{Re} Vol \neq 0.$$

As a corollary, the spectral flow of $D_{g_t} + f_t$ at t_i, $i = 1, 2, \ldots, n$ is ± 4.

The next proposition concerning the existence of σ-invariant admissible perturbations is proved in Section 6.

Proposition 2.7. *Fix a regular pair (g, f) so that the reducible $[\theta]$ is isolated. There exist σ-invariant admissible perturbations of $\mathcal{CSD}_f$ which are away from $[\theta]$ and the non-degenerate critical points of $\mathcal{CSD}_f$. Any two such admissible perturbations can be connected by a path which is away from $[\theta]$.*

The second class of admissible perturbations is given by

$$\mathcal{CSD}'_\mu(A, \psi) = \mathcal{CSD}(A, \psi) - \int_Y A \wedge *\mu.$$

for generic co-closed 1-forms μ with values in $i\mathbb{R}$. The gradient of $\mathcal{CSD}'_\mu$ at (A, ψ) is

$$\mathrm{s}'_\mu(A, \psi) = (*F_A + \tau(\psi, \psi) + \mu, D_A\psi).$$

The only reducible critical point is $[\theta_\mu] = [a_\mu, 0]$ where a_μ is the unique solution to the equations $*da_\mu + \mu = 0$ and $d^*a_\mu = 0$. The covariant derivative $\nabla \mathrm{s}'_\mu$ is given by

$$\nabla \mathrm{s}'_{\mu,(A,\psi)}(a, \varphi) = (*da + 2\tau(\psi, \varphi) - df(\varphi), D_A\varphi + a\psi + f(\varphi)\psi)$$

where $f(\varphi)$ is the unique solution to the equation

$$(d^*d + |\psi|^2)f = i\langle D_A\psi, i\varphi\rangle_{Re}.$$

The perturbed Dirac operator at the reducible $[\theta_\mu]$ is given by $D_\mu = D + a_\mu$ (see Definition 2.5).

Proposition 2.8. *For generic μ, $\mathcal{CSD}'_\mu$ is admissible. Moreover, any two such regular μ_0 and μ_1 can be connected by a path μ_t, $t \in [0, 1]$, such that*

1. *s'_{μ_t} is transversal to the zero section of the Hilbert bundle $\mathcal{L}$ over $\mathcal{B}^* \times [0, 1]$.*
2. *D_{μ_t} is invertible for all but finitely many points $t_i \in (0, 1)$ with* Ker $D_{\mu_{t_i}} = \mathbb{C}$. *Moreover, if λ_t and ψ_t are the eigenvalue and eigenvector of D_{μ_t} near t_i, i.e., $D_{\mu_t}\psi_t = \lambda_t\psi_t$ with $\|\psi_t\|_{L^2} = 1$ and $\lambda_{t_i} = 0$, then*

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{\mu_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re} Vol \neq 0.$$

In particular, the spectral flow of D_{μ_t} at t_i is equal to ± 1.

Proof: The "universal" gradient $s(\mu, A, \psi) = (*F_A + \tau(\psi, \psi) + \mu, D_A\psi)$ is a section of the Hilbert bundle $\mathcal{L}$ over Ker $d^* \times \mathcal{A}^*$ which is transversal to the zero section. So $s^{-1}(0)$ is a Banach manifold, and $s^{-1}(0)/\mathcal{G}$ is also a Banach manifold. The projection P: $s^{-1}(0)/\mathcal{G} \longrightarrow$ Ker d^* is a Fredholm map with index equal to 0. So for generic μ, $\nabla \mathrm{s}'_\mu$ is invertible at ${\mathrm{s}'}_\mu^{-1}(0)$, and any two such regular μ_0 and μ_1 can be connected by a path μ_t, $t \in [0, 1]$, such that s'_{μ_t} is transversal to the zero section of the Hilbert bundle $\mathcal{L}$ over $\mathcal{B}^* \times [0, 1]$.

Consider the real Hilbert bundle $\mathcal{E}$ over Ker $d^* \times (L^2_1(W_0) \setminus \{0\})$ given by $\mathcal{E}_{(a,\psi)} = \{\varphi \in L^2(W_0) | \varphi$ is orthogonal to $i\psi\}$. Then $L(a, \psi) = D\psi + a\psi$ is a section of $\mathcal{E}$ which is also transversal to the zero section. Therefore $L^{-1}(0)$ is a Banach manifold. The projection Π: $L^{-1}(0) \longrightarrow \text{Ker } d^*$ is Fredholm with index equal to 1. Since $D_a = D + a$ is complex linear, by Sard-Smale theorem, for generic $a \in \text{Ker } d^*$, $\Pi^{-1}(a)$ is empty, i.e., D_a is invertible. Two such regular a_0 and a_1 can be connected by a path a_t which is transversal to Π. We can take an analytic path a_t so that for all but finitely many points t_i, D_{a_t} is invertible and $\text{Ker } D_{a_{t_i}} = \mathbb{C}$ by index counting. If $D_{a_t}\psi_t = \lambda_t\psi_t$ with $\|\psi_t\|_{L^2} = 1$ and $\lambda_{t_i} = 0$, then

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(D_{a_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re} Vol.$$

Since a_t is transversal to Π, $\int_Y \langle \frac{d}{dt}(D_{a_t})(t_i)\psi_{t_i}, \psi_{t_i}\rangle_{Re} Vol \neq 0$. *QED*

Remarks:

1. The same conclusions hold if we also allow the metrics to change.
2. Throughout this paper, we also use μ to denote the first type perturbations of $\mathcal{CSD}$. In this case, the perturbed Dirac operator at $[\theta]$ is given by $D_\mu = D_g + f$ for the pair (g, f).

3. The definition of χ and α

Fix an admissible perturbation $\mathcal{CSD}'_\mu$ of $\mathcal{CSD}$ with gradient s'_μ. Let $\mathcal{M}^*_\mu = \{[A,\psi] \in \mathcal{B}^* | s'_\mu(A,\psi) = 0\}$. We define for $\beta_j \in \mathcal{M}^*_\mu$

$$\chi^j_\mu = \sum_{\beta_i \in \mathcal{M}^*_\mu} (-1)^{SF(\beta_j,\beta_i)}$$

where $SF(\beta_j, \beta_i)$ is the spectral flow between $\nabla s'_{\mu,\beta_j}$ and $\nabla s'_{\mu,\beta_i}$. As in [T], it is easy to show that $|\chi^j_\mu|$ is independent of the choice of β_j. In order to give a sign to $|\chi^j_\mu|$, we need to fix a sign near the reducible $[\theta_\mu]$. See [T].

At $(A,\psi) \in \mathcal{A}^*$, we have an exact complex

$$0 \longrightarrow T\mathcal{G}_1 \xrightarrow{d_{(A,\psi)}} T\mathcal{A}^* \longrightarrow \pi^*T\mathcal{B}^* \longrightarrow 0$$

where $d_{(A,\psi)}(f) = (-df, f\psi)$ and $\pi : \mathcal{A}^* \to \mathcal{B}^*$. This enables us to extend any endomorphism of $T\mathcal{B}^*$ to a $\mathcal{G}$-equivariant one of $T\mathcal{A} \oplus T\mathcal{G}_1$. An endomorphism L of $T\mathcal{B}^*$ is extended to

$$\mathcal{K}'_L = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & d_{(A,\psi)} \\ 0 & d^*_{(A,\psi)} & 0 \end{pmatrix},$$

an endomorphism of $T\mathcal{A} \oplus T\mathcal{G}_1 = T\mathcal{B}^* \oplus Im(d_{(A,\psi)}) \oplus T\mathcal{G}_1$. $\mathcal{K}'_L$ is self-adjoint if and only if L is. For $L = \nabla s'_\mu$, we use $\mathcal{K}'_\mu$ for $\mathcal{K}'_L$.

At $(A,\psi) \in \mathcal{A}$, we define a self-adjoint endomorphism $\mathcal{K}_{(A,\psi)}$ of $T\mathcal{A} \oplus T\mathcal{G}_1$ by

$$\mathcal{K}_{(A,\psi)}(a,\varphi,f) = (*da + 2\tau(\psi,\varphi) - df, D_A\varphi + a\psi + f\psi, -d^*a + i\langle i\psi,\varphi\rangle_{Re})$$

or

$$\mathcal{K}_{(A,\psi)} = \begin{pmatrix} D_A & \psi\cdot & \psi\cdot \\ 2\tau(\psi,\cdot) & *d & -d \\ i\langle i\psi,\cdot\rangle_{Re} & -d^* & 0 \end{pmatrix}.$$

Lemma 3.1. *For smooth $(A,\psi) \in \mathcal{A}$, $\mathcal{K}_{(A,\psi)}$ extends to $L^2(\Lambda^1(Y,i\mathbb{R}) \oplus W_0 \oplus \Lambda^0(Y,i\mathbb{R}))$ as a closed essentially selfadjoint, Fredholm operator. It has discrete spectrum with no accumulation points, and each eigenvalue has finite multiplicity. The spectrum is unbounded from above and below. The same holds for $\mathcal{K}'_{\mu,(A,\psi)}$ if $(A,\psi) \in \mathcal{A}^*$. Moreover, one can replace $\nabla s'_\mu$ by $\mathcal{K}$ for the purpose of computing the spectral flow.*

Proof: Similar arguments as in [T]. *QED*

For any $(a,\varphi) \in \mathcal{A}^*$, we need to study the small eigenvalues of $\mathcal{K}_{\mu,t}(a,\varphi) = \mathcal{K}_{\mu,0} + tC(a,\varphi)$ as $t \to 0$ where

$$\mathcal{K}_{\mu,0} = \begin{pmatrix} D_\mu & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix}, \text{ and } C(a,\varphi) = \begin{pmatrix} a & \varphi\cdot & \varphi\cdot \\ 2\tau(\varphi,\cdot) & 0 & 0 \\ i\langle i\varphi,\cdot\rangle_{Re} & 0 & 0 \end{pmatrix}.$$

We assume that D_μ is invertible. Then $\mathcal{K}_{\mu,0}$ has only one zero eigenvector which is the constant function i. $\mathcal{K}_{\mu,t}(a,\varphi)$ is expected to have exactly one small eigenvalue λ_t which is analytic in t as $t \to 0$. See [K].

Lemma 3.2. $\dot{\lambda}_t(0) = 0$, $\ddot{\lambda}_t(0) = -2\int_Y \langle D_\mu\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol$ *where* $\tilde{\varphi} = D_\mu^{-1}(i\varphi)$.

Proof: For simplicity let $K_t = \mathcal{K}_{\mu,t}(a,\varphi)$, $C = C(a,\varphi)$. Suppose $(K_t - \lambda_t)f_t = 0$ where $\|f_t\|_{L^2} = 1$, $f_0 = i$. By differentiating the equation, we have

$$(C - \dot{\lambda}_t)f_t + (K_t - \lambda_t)\dot{f}_t = 0.$$

So $\dot{\lambda}_t = (C(f_t), f_t)$, and $\dot{\lambda}_t(0) = (C(i), i) = (i\varphi, i) = 0$. $K_0(\dot{f}_t(0)) = -C(f_0) = -i\varphi$. Let $\tilde{\varphi} = D_\mu^{-1}(i\varphi)$, then $\ddot{\lambda}_t(0) = (C(\dot{f}_t(0)), f_0) + (C(f_0), \dot{f}_t(0)) = -2\int_Y \langle D_\mu\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol$. *QED*

Corollary 3.3. *For generic φ, $\ddot{\lambda}_t(0) \neq 0$. $\lambda_t \sim \lambda t^2$ where $\lambda = -\int_Y \langle D_\mu\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol$ and $\tilde{\varphi} = D_\mu^{-1}(i\varphi)$.*

For $\beta_j \in \mathcal{M}^*_\mu$, we define

$$sign(\beta_j) = -sign(\int_Y \langle D_\mu\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol) \cdot (-1)^{SF(\beta_j,\varphi)}$$

for generic φ, where $SF(\beta_j,\varphi)$ is the spectral flow between $\mathcal{K}_{\beta_j}$ and $\mathcal{K}_{\mu,t}(a,\varphi)$ for small t.

Definition 3.4. $\chi_\mu = sign(\beta_j) \cdot \chi^j_\mu$.

It is easy to see that sign(β_j) is independent of φ, and χ_μ is independent of β_j as in [T].

Lemma 3.5. *$\chi_\mu(Y) = -\chi_{-\mu}(-Y)$, and $\chi_\mu \equiv 0 \pmod 2$ if $\mathcal{CSD}'_\mu$ is a σ-invariant admissible perturbation.*

Proof: W_0 still can serve for $-Y$ if we change the Clifford multiplication by a factor of -1. Under this change, $\mathcal{CSD}'_\mu(Y) = -\mathcal{CSD}'_{-\mu}(-Y)$, $\nabla s'_\mu(Y) = -\nabla s'_{-\mu}(-Y)$, $\mathcal{M}_\mu(Y) = \mathcal{M}_{-\mu}(-Y)$, and $\int_Y \langle D_\mu\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol = -\int_{-Y}\langle D_{-\mu}\tilde{\varphi}, \tilde{\varphi}\rangle_{Re} Vol$. So $\chi_\mu(Y) = -\chi_{-\mu}(-Y)$. The other statement is obvious. *QED*

Let W be a smooth compact oriented spin 4-manifold with $\partial W = Y$. Equip W with a Riemannian metric so that a neighborhood of Y is isometric to $(-1, 0] \times Y$. Suppose D^μ_W is a perturbed Dirac operator on W which takes the form

$$c(dt)(\frac{d}{dt} + D^Y_\mu)$$

near the boundary Y where t is the outward normal coordinate. Here D^Y_μ is a perturbed Dirac operator on Y which takes the form of $D^Y_g + f + a$ where a is a co-closed imaginary valued 1-form, g stands for the metric and f is a smooth real valued function on Y. D^Y_μ is required to be invertible. $IndexD^\mu_W$ is the L^2 index if a semi-cylinder is attached to W, or the index of D^μ_W satisfying the APS global boundary condition.

Lemma 3.6. *([APS]) $IndexD^\mu_W + \frac{1}{8}\mathrm{sign}(W)$ is independent of W, and*

$$(IndexD^{\mu_1}_W + \frac{1}{8}\mathrm{sign}(W)) - (IndexD^{\mu_2}_W + \frac{1}{8}\mathrm{sign}(W)) = -SF(D^Y_{\mu_1}, D^Y_{\mu_2}).$$

In the case that $a = 0$ and (g, f) is a regular pair, $IndexD^\mu_W + \frac{1}{8}\mathrm{sign}(W) \equiv \mu(Y) \pmod 2$ where $\mu(Y)$ is the Rohlin invariant. $IndexD^\mu_W + \frac{1}{8}\mathrm{sign}(W)$ changes by a factor of -1 if the orientation of Y is changed.

Definition 3.7. *$\alpha_\mu = \chi_\mu - (IndexD^\mu_W + \frac{1}{8}\mathrm{sign}(W))$ where μ is an admissible perturbation. Here D^μ_W takes the form of $c(dt)(\frac{d}{dt} + D^Y_\mu)$ near the boundary of W, where D^Y_μ is the perturbed Dirac operator at the reducible $[\theta_\mu]$ associated to the perturbation μ.*

4. Topological invariance of α_μ

In this section, we shall prove that α_μ is independent of the choice of the Riemannian metric and the admissible perturbation involved in the definition.

Given any two such choices, we can connect them by a path μ_t for which Proposition 2.8 holds. So we only need to consider the following two situations:

1. D_{μ_t} is invertible for all t.
2. D_{μ_t} is invertible for all t but $t = 0$.

In the first case, $IndexD^\mu_W + \frac{1}{8}\mathrm{sign}(W)$ does not change, and χ_μ also does not change. In fact, we have

Lemma 4.1. *Suppose two admissible perturbations μ_0 and μ_1 are connected by a path μ_t which provides a cobordism X between part of $\mathcal{M}^*_0$ and part of $\mathcal{M}^*_1$. If $\beta_0 \in \mathcal{M}^*_0$ is cobordant to $\beta_1 \in \mathcal{M}^*_1$ via X, then $SF(\beta_0, \beta_1)$ is even. If $\beta_0 \in \mathcal{M}^*_0$ is cobordant to*

$\beta_1 \in \mathcal{M}_0^*$ via X, then $SF(\beta_0, \beta_1)$ is odd. Here $SF(\beta_0, \beta_1)$ stands for the spectral flow between $\nabla s'_{\beta_0}$ and $\nabla s'_{\beta_1}$.

Proof: The lemma follows from the fact that the cobordism X can be arranged so that the projection from X to $[0,1]$ is a Morse function. See [DK], p.143. *QED*

In the second case, $IndexD_W^\mu + \frac{1}{8}\text{sign}(W)$ is changed by ± 1. We shall prove that χ_μ is also changed by ± 1 which is compatible to the change of $IndexD_W^\mu + \frac{1}{8}\text{sign}(W)$ so that α_μ remains unchanged. This is done by analyzing the Kuranishi model near the reducible at $t = 0$.

Nonlinear Fredholm maps between Hilbert spaces admit local reductions to finite dimensional maps. Suppose Ψ: $X \to Y$ is a nonlinear Fredholm map satisfying $\Psi(0) = 0$. Let $T = (d\Psi)_0$. Then there are splittings $X = \text{Ker } T \oplus (\text{Ker } T)^\perp$, $Y = ImT \oplus cokT$ and a map $\psi : X \to cokT$ so that Ψ is equivalent to $T + \psi$ near 0 by a diffeomorphism of X, and $\psi(0) = 0$, $(d\psi)_0 = 0$. Moreover, $\Psi^{-1}(0)$ is diffeomorphic to $\{\psi|_{\text{Ker}T} = 0\}$ near 0. If there is a group action, the above can be made equivariant.

The detailed construction goes as follows. Let $\pi_k : X \to \text{Ker}T$, $\pi_c : Y \to cokT$ be the orthogonal projections. Then $\chi : X \to X$ given by $\chi : x \to \pi_k(x) + T^{-1}(1 - \pi_c)(\Psi(x))$ is a local diffeomorphism at 0. Define $\psi(y) = \pi_c(\Psi(\chi^{-1}(y)))$. Then $\Psi \circ \chi^{-1} = T + \psi$, and $\Psi^{-1}(0) = \{\psi|_{\text{Ker}T} = 0\}$. See [FU].

Suppose two admissible perturbations μ_{-1} and μ_1 are connected by a path μ_t, $t \in [-1,1]$ in the sense of Proposition 2.8 and D_{μ_t} is invertible except for $t = 0$. We will study the Kuranishi model near the reducible at $t = 0$ of the following family of Seiberg-Witten equations:

$$\begin{cases} *_t F_A + \tau_t(\psi, \psi) = 0 \\ (D_{\mu_t} + A)\psi = 0 \end{cases}$$

where $A \in \text{Ker}d^*$. Here d^* stands for d^{*_t} at $t = 0$.

Consider map $\Psi : \mathbb{R} \oplus L_1^2(\text{Ker}d^* \oplus W_0) \to L^2(\text{Ker}d^* \oplus W_0)$ given by

$$\Psi(t, A, \psi) = (\pi(*_t F_A + \tau_t(\psi, \psi)), (D_{\mu_t} + A)\psi)$$

where $\pi : \Omega^1(Y, i\mathbb{R}) \to \text{Ker}d^*$ is the L^2 orthogonal projection. Then $\text{Ker}(d\Psi)_0 = \mathbb{R} \oplus \text{Ker } D_0$, $cok(d\Psi)_0 = \text{Ker } D_0$. Here D_0 stands for D_{μ_0}. Write $\psi = \psi_0 + \psi_1$ where $\psi_0 \in \text{Ker } D_0$ and $\psi_1 \in (\text{Ker } D_0)^\perp$, then we have a local diffeomorphism $\chi : \mathbb{R} \oplus L_1^2(\text{Ker}d^* \oplus W_0) \to \mathbb{R} \oplus L_1^2(\text{Ker}d^* \oplus W_0)$

$$\chi : (t, A, \psi_0 + \psi_1) \to (t, (*d)^{-1}(\pi(*_t F_A + \tau_t(\psi_0 + \psi_1))), \psi_0 + D_0^{-1}(1 - \pi_k)((D_{\mu_t} + A)(\psi_0 + \psi_1)))$$

and $\chi^{-1}(t, 0, \psi_0) = (t, A, \psi_0 + \psi_1)$ where $A = A(t, \psi_0)$, $\psi_1 = \psi_1(t, \psi_0)$ satisfy

$$\begin{cases} A + (\pi *_t d)^{-1}(\pi\tau_t(\psi_0 + \psi_1)) = 0 \\ \psi_1 + D_0^{-1}(1 - \pi_k)(D_{\mu_t} - D_0 + A)(\psi_0 + \psi_1) = 0. \end{cases}$$

Lemma 4.2. *$(D_{\mu_t} + A(t, \psi_0))(\psi_0 + \psi_1(t, \psi_0)) \in \text{Ker } D_0$ and if we write*

$$(D_{\mu_t} + A(t, \psi_0))(\psi_0 + \psi_1(t, \psi_0)) = a\psi_0 + ib\psi_0$$

where a, b are real numbers, then $b = 0$.

Proof: For simplicity, denote $D_{\mu_t} + A(t,\psi_0)$ by D. Then $b\|\psi_0\|^2 = \int_Y \langle ib\psi_0, i\psi_0\rangle_{Re} = \int_Y \langle D(\psi_0+\psi_1) - a\psi_0, i\psi_0\rangle_{Re} = \int_Y \langle D\psi_1, i\psi_0\rangle_{Re} = -\int_Y \langle i\psi_1, D\psi_0\rangle_{Re} = -\int_Y \langle i\psi_1, a\psi_0 + ib\psi_0 - D\psi_1\rangle_{Re} = 0$. *QED*

Lemma 4.3. *There exists a constant C so that for small s, if $\|\psi_0\|_{L^2_1} \leq s, t \leq s$, then*

$$\|\psi_1(t,\psi_0)\|_{L^2_1} \leq Cs^2, \text{ and } \|A(t,\psi_0)\|_{L^2_1} \leq Cs^2.$$

Proof: First of all, we have continuous maps $L^2_1 \times L^2_1 \to L^2$ and $(*d)^{-1}, D_0^{-1} : L^2 \to L^2_1$. Then apply Banach lemma to the map

$$B(A,\psi_1) = ((\pi *_t d)^{-1}(\pi\tau_t(\psi_0+\psi_1)), D_0^{-1}(1-\pi_k)(D_{\mu_t} - D_0 + A)(\psi_0+\psi_1)).$$

B maps $\{\|A\|_{L^2_1} \leq Cs^2, \|\psi_1\|_{L^2_1} \leq Cs^2\}$ into itself, if $t \leq s$ and $\|\psi_0\|_{L^2_1} \leq s$ for small s. *QED*

Next we examine the finite dimensional reduction $\varphi|_{\mathrm{Ker}(d\Psi)_0} : \mathbb{R} \oplus \mathrm{Ker}\, D_0 \to \mathrm{Ker} D_0$. Let $\psi_0 \in \mathrm{Ker} D_0$, $\|\psi_0\|_{L^2} = 1$. We have

$$\varphi|_{\mathrm{Ker}(d\Psi)_0}(t, s\psi_0) = (D_{\mu_t} + A(t,s\psi_0))(s\psi_0 + \psi_1(t,s\psi_0)).$$

Without loss of generality, we can assume that s is real and positive. By lemma 4.2, $\varphi|_{\mathrm{Ker}(d\Psi)_0}(t,s\psi_0) = 0$ if and only if

$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t,s\psi_0)), \psi_0\rangle_{Re} + \int_Y \langle A(t,s\psi_0)(s\psi_0+\psi_1(t,s\psi_0)), \psi_0\rangle_{Re} = 0.$$

Lemma 4.4. *Let $D_{\mu_t}\psi_t = \lambda_t\psi_t$, $\lambda_t(0) = 0$, $\psi_t(0) = \psi_0$ as in Proposition 2.8. Then*

1.
$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t,s\psi_0)), \psi_0\rangle_{Re} = s(\lambda_t + O(st+t^2))$$
as $t, s \to 0$.

2.
$$\int_Y \langle A(t,s\psi_0)(s\psi_0+\psi_1(t,s\psi_0)), \psi_0\rangle_{Re} = s(-2s^2\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle + O(s+t)s^2)$$
as $t, s \to 0$.

Proof: Let $D_{\mu_t}\psi_t = \lambda_t\psi_t$, and $\psi_t = a_t\psi_0 + b_t\psi_t^\perp$ where $\psi_t^\perp \in (\mathrm{Ker}D_0)^\perp$, $\|\psi_t^\perp\|_{L^2} = 1$, $a_t \to 1$, $b_t = O(t)$. Then

$$\lambda_t = |a_t|^2(D_{\mu_t}\psi_0, \psi_0) + 2|b_t|^2\lambda_t - |b_t|^2(D_{\mu_t}\psi_t^\perp, \psi_t^\perp).$$

Since $a_t \to 1$, $b_t = O(t)$, we have $(D_{\mu_t}\psi_0,\psi_0) = \lambda_t + O(t^2)$.

On the other hand, for any $\psi_2 \in (\mathrm{Ker}D_0)^\perp$, we have

$$(D_{\mu_t}\psi_2, \psi_0) = a_t^{-1}b_t(\lambda_t(\psi_t^\perp,\psi_2) - (D_{\mu_t}\psi_t^\perp, \psi_2)) = O(\|\psi_2\|\cdot t).$$

So

$$\int_Y \langle D_{\mu_t}(s\psi_0 + \psi_1(t,s\psi_0)), \psi_0\rangle_{Re} = s(\lambda_t + O(st+t^2))$$

as $t, s \to 0$.

For the second assertion, we have

$$A(t, s\psi_0) = -(\pi *_t d)^{-1}(\pi\tau_t(s\psi_0 + \psi_1(t, s\psi_0))) = -(*d)^{-1}(\tau(\psi_0))s^2 + O(ts^2 + s^3).$$

So

$$\int_Y \langle A(t, s\psi_0)(s\psi_0 + \psi_1(t, s\psi_0)), \psi_0\rangle_{Re} = s(-2s^2 \int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle + O(s+t)s^2)$$

as $t, s \to 0$. *QED*

Corollary 4.5. *Suppose* $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \neq 0$. *Then* $\varphi|_{\mathrm{Ker}(d\Psi)_0}(t, s\psi_0) = 0$ *has exactly one solution s for and only for those t such that* λ_t *and* $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle$ *have the same sign, and* $t \sim cs^2$ *as* $t, s \to 0$.

Remarks: $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle \neq 0$ is generically true by slightly perturbing μ_t near $t = 0$, observing that $\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0, \varphi)\rangle = 0$ for any φ implies that $\psi_0 = 0$, and also observing that μ_t is transversal to the projection Π (see Proposition 2.8).

Lemma 4.6. *Let* (A, ψ) *be the solution to*

$$\begin{cases} *_t F_A + \tau_t(\psi, \psi) = 0 \\ (D_{\mu_t} + A)\psi = 0 \end{cases}$$

near the reducible and $t = 0$, *then* $SF(\mathcal{K}_{(A,\psi)}, \mathcal{K}_{\mu_t,s}(0, \psi_0))$ *is odd as* $t, s \to 0$.

Proof: $\mathcal{K}_{(A,\psi)}$ is an analytic perturbation in $s = (\psi, \psi_0)$ of

$$\mathcal{K}_0 = \begin{pmatrix} D_0 & 0 & 0 \\ 0 & *d & -d \\ 0 & -d^* & 0 \end{pmatrix}.$$

$\mathcal{K}_0$ has three zero eigenvectors $E^1 = \psi_0$, $E^2 = \frac{1}{\sqrt{2}}(i\psi_0 + i)$, $E^3 = \frac{1}{\sqrt{2}}(i\psi_0 - i)$. Let $\mathcal{K}_{(A,\psi)} E^i_s = \lambda^i_s E^i_s$ where $E^i_s(0) = E^i$, $\lambda^i_s(0) = 0$. Then

$$\dot{\lambda}^1_s(0) = 0,\ \ddot{\lambda}^1_s(0) = -8\int_Y \langle (*d)^{-1}(\tau(\psi_0)), \tau(\psi_0)\rangle,\ \dot{\lambda}^2_s(0) = 1,\ \dot{\lambda}^3_s(0) = -1.$$

So $\lambda^1_s \sim \lambda s^2$, $\lambda^2_s \sim s$ and $\lambda^3_s \sim -s$ where λ has the same sign with $-\lambda_t$ (see corollary 4.5).

On the other hand, $\mathcal{K}_{\mu_t,s}(0, \psi_0)$ has three small eigenvalues $\lambda_t, \lambda_t, \lambda_1 s^2$ as $t \to 0$ and $s = o(t)$ where $\lambda_1 = -(D_{\mu_t}\tilde{\psi_0}, \tilde{\psi_0})$ and $\tilde{\psi_0} = D^{-1}_{\mu_t}(i\psi_0)$. It is easy to see that λ_1 has the same sign with $-(D_{\mu_t}\psi_0, \psi_0) \sim -\lambda_t$ as $t \to 0$. So $SF(\mathcal{K}_{(A,\psi)}, \mathcal{K}_{\mu_t,s}(0, \psi_0))$ is odd as $t, s \to 0$. *QED*

The Proof of Theorem 1.2:

There will be a family of irreducible critical points disappearing or being created when t passes 0. Call it β_t. Then it is easy to see from lemma 4.6 that $\mathrm{sign}(\beta_t) = \mathrm{sign}\lambda_t$. The rest of $\mathcal{M}^*_{\mu_t}$ provides a cobordism between the rest of $\mathcal{M}^*_{\mu_{-1}}$ and $\mathcal{M}^*_{\mu_1}$. The sign convention fixed near the reducibles does not change since $\mathcal{K}_{\mu_t,s}(0, \psi_0)$ has a spectral flow equal to ± 1 when t passes 0 (the point is that D_{μ_t} is complex linear). So we have $\chi_{\mu_{-1}} - \chi_{\mu_1} = -SF(D_{\mu_{-1}}, D_{\mu_1})$ and α_μ remains unchanged.

The second assertion is an easy consequence of the existence of σ-invariant admissible perturbations. We will construct them in the next two sections. *QED*

5. Perturbations of Dirac operators

In this section, we show that the perturbed Dirac operators $D_g + f$ are invertible for generic pairs of (g, f) and they admit a chamber structure.

Throughout this section, we assume that Y is a closed oriented 3-manifold. Given a metric g on Y, let P_{SO} be the orthonormal tangent frame bundle of Y. Let $H \subset GL(3,\mathbb{R})$ be the subset of symmetric matrices with positive eigenvalues, then $C^k(P_{SO} \times_{Ad} H)$ which is the set of C^k sections of the associated fiber bundle $P_{SO} \times_{Ad} H$ parameterizes all of the C^k–smooth Riemannian metrics on Y. We use the C^k–norm of $C^k(P_{SO} \times_{Ad} H)$ to topologize it. Let h be a section of $P_{SO} \times_{Ad} H$, g^h be the correspondent metric, and P^h_{SO} be the orthonormal tangent frame bundle associated to g^h. Let ξ be a given spin structure on Y, $\pi : P_{\mathrm{Spin}(\xi)} \to P_{SO}$, $\pi : P^h_{\mathrm{Spin}(\xi)} \to P^h_{SO}$ be the Spin(3) bundles correspondent to the metrics g and g^h, then we have a lifting $\tilde{h}$

$$\begin{array}{ccc} P_{\mathrm{Spin}(\xi)} & \xrightarrow{\tilde{h}} & P^h_{\mathrm{Spin}(\xi)} \\ \downarrow \pi & & \downarrow \pi \\ P_{SO} & \xrightarrow{h} & P^h_{SO} \end{array}$$

Note that if h is not symmetric, we may not remain in the same spin structure. Let $V = P_{Spin(\xi)} \times_\rho \mathbb{C}^2$, $V^h = P^h_{Spin(\xi)} \times_\rho \mathbb{C}^2$ be the spinor bundles where $\rho : \mathrm{Spin}(3) \longrightarrow SU(2)$ is the standard representation. We have an isometry $\tilde{h} : V \longrightarrow V^h$ given by $\tilde{h}(\sigma, \theta) = (\tilde{h}(\sigma), \theta)$.

Let $\mathbb{D} : \Gamma(V) \times C^k(P_{SO} \times_{Ad} H) \longrightarrow \Gamma(V)$ be the map defined by $\mathbb{D}(\psi, h) = \tilde{h}^{-1} \cdot D_{g^h} \cdot \tilde{h}(\psi)$ where $\psi \in \Gamma(V)$ and $h \in C^k(P_{SO} \times_{Ad} H)$. Let σ be a local frame of $P_{Spin(\xi)}$, $\pi(\sigma) = (e_1, e_2, e_3)$, and $(f_1, f_2, f_3) = (e_1, e_2, e_3)h$ which is the local orthonormal frame with respect to the metric g^h. Write $\psi = (\sigma, \theta)$,$h = (\pi(\sigma), (h_{ij}))$, then

$$\begin{aligned} \mathbb{D}(\psi, h) &= \tilde{h}^{-1} \cdot D_{g^h} \cdot (\tilde{h}(\sigma), \theta) \\ &= \tilde{h}^{-1} \cdot (\tilde{h}(\sigma), \sum_{i=1}^{3} (c_i f_i(\theta) - \frac{1}{2} \sum_{k<j} \omega^i_{kj}(h) c_i c_k c_j \theta)) \\ &= (\sigma, \sum_{i=1}^{3} (c_i h_{si} e_s(\theta) - \frac{1}{2} \sum_{k<j} \omega^i_{kj}(h) c_i c_k c_j \theta)) \end{aligned}$$

where $\omega^i_{kj}(h)$ is the Levi-Civita connection 1-forms of the metric g^h with respect to (f_1, f_2, f_3), i.e., $\nabla^h_{f_i} f_j = f_k \omega^i_{kj}(h)$, and

$$c_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, c_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

See [LM]. Direct computation shows that

$$\begin{aligned} \omega^i_{kj}(h) &= \frac{1}{2}(h^{-1}_{kr} h_{li} h_{sj} + h^{-1}_{jr} h_{lk} h_{si} - h^{-1}_{ir} h_{lj} h_{sk})(\omega^l_{rs} - \omega^s_{rl}) \\ &\quad + \frac{1}{2} h^{-1}_{ks} h_{li} e_l(h_{sj}) - \frac{1}{2} h^{-1}_{kl} h_{sj} e_s(h_{li}) + \frac{1}{2} h^{-1}_{js} h_{lk} e_l(h_{si}) \\ &\quad - \frac{1}{2} h^{-1}_{jl} h_{si} e_s(h_{lk}) - \frac{1}{2} h^{-1}_{is} h_{lj} e_l(h_{sk}) + \frac{1}{2} h^{-1}_{il} h_{sk} e_s(h_{lj}) \end{aligned}$$

where $\nabla_{e_i} e_j = e_k \omega^i_{kj}$, $h^{-1}_{ij} h_{jk} = \delta_{ik}$ (see [KN]).

Lemma 5.1. $\mathbb{D}(\cdot, h)$*:* $\Gamma(V) \longrightarrow \Gamma(V)$ *is smooth in* h*. Moreover,* $\mathbb{D}(\cdot, h)$ *is self-adjoint if* $\det(h) = 1$ *pointwise on* Y*.*

Proof: That $\mathbb{D}(\cdot, h)$ is smooth in h follows from the local expressions of $\mathbb{D}(\cdot, h)$ and $\omega^i_{kj}(h)$. For the self-adjointness of $\mathbb{D}(\cdot, h)$, we have

$$\begin{aligned} \int_Y \langle \mathbb{D}(\psi, h), \varphi \rangle_g Vol_g &= \int_Y \langle \tilde{h}^{-1} \cdot D_{g^h} \cdot \tilde{h}(\psi), \varphi \rangle_g Vol_g \\ &= \int_Y \langle D_{g^h} \cdot \tilde{h}(\psi), \tilde{h}(\varphi) \rangle_{g^h} Vol_{g^h} \\ &= \int_Y \langle \tilde{h}(\psi), D_{g^h}(\tilde{h}(\varphi)) \rangle_{g^h} Vol_{g^h} \\ &= \int_Y \langle \psi, \tilde{h}^{-1} \cdot D_{g^h} \cdot \tilde{h}(\varphi) \rangle_g Vol_g \\ &= \int_Y \langle \psi, \mathbb{D}(\varphi, h) \rangle_g Vol_g \end{aligned}$$

where $Vol_g = Vol_{g^h}$ since $\det(h) = 1$ pointwise on Y. *QED*

Lemma 5.2. *Given any metric* g *on* Y*, let* (e_1, e_2, e_3) *be an oriented local orthonormal frame in an open subset* A *of* Y*. Let* f *be a smooth real valued function on* Y*. Suppose* $\psi, \varphi \in \mathrm{Ker}\,(D_g + f)$*. If*

$$\frac{d}{dt}\left(\int_Y \langle \mathbb{D}(\psi, e^{tX}), \varphi \rangle_g Vol_g\right) = 0$$

at $t = 0$ *for any symmetric matrix function* X *compactly supported in* A *satisfying* $tr(X) = 0$*, then in* A *we have*

$$\langle e_j \nabla_{e_j} \psi, \varphi \rangle_g + \langle \psi, e_j \nabla_{e_j} \varphi \rangle_g = -\frac{2}{3} \langle f\psi, \varphi \rangle_g$$

for $j = 1, 2, 3$, *and*

$$\langle e_j \nabla_{e_i}\psi, \varphi\rangle_g + \langle \psi, e_j \nabla_{e_i}\varphi\rangle_g = -\frac{1}{2} e_k(\langle \psi, \varphi\rangle_g)$$

for any i, j, k *such that* $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$.

The proof of this lemma is a lengthy computation which is given at the end of this section.

Let Met_0 be the subspace of $Met = C^k(P_{SO} \times_{Ad} H)$ given by

$$Met_0 = \{h \in Met | \det(h) = 1\}.$$

Then every metric in Met is conformal to a metric in Met_0.

The Proof of Proposition 2.6:

Consider the real Hilbert bundle E over the Banach manifold $B = Met_0 \times C^k(Y) \times (L_1^2(V) \setminus \{0\})$. At $(h, f, \psi) \in B$, $E_{(h,f,\psi)} = \{\varphi \in L^2(V) | \varphi$ is orthogonal to $i\psi, j\psi, k\psi\}$. Here $i, j, k \in \mathbb{H}$ satisfying

$$ij = k, \quad jk = i, \quad ki = j, \quad \textit{and} \quad i^2 = j^2 = k^2 = -1.$$

The map $L : (h, f, \psi) \longrightarrow \mathbb{D}(\psi, h) + f\psi$ defines a section of the bundle E over the Banach manifold B. Suppose that $(h, f, \psi) \in L^{-1}(0)$, then the differential of L at (h, f, ψ) is

$$\delta L_{(h,f,\psi)}(H, F, \Psi) = \mathbb{D}(\Psi, h) + f\Psi + \delta\mathbb{D}(\psi, \cdot)(h)(H) + F\psi,$$

from which it is easy to see that if $\varphi \in (Im\delta L)^{\perp}$, then $\varphi \in$ Ker $(\mathbb{D}(\cdot, h) + f)$ and $\varphi = a_1(i\psi) + a_2(j\psi) + a_3(k\psi)$ for some real functions a_1, a_2, a_3. Moreover, by lemma 5.2,

$$\int_Y \langle \delta\mathbb{D}(\psi, \cdot)(h)(H), \varphi\rangle_{Re} Vol = 0$$

for any H implies that

$$\langle e_i \nabla_{e_i}\psi, \varphi\rangle_{Re} + \langle \psi, e_i \nabla_{e_i}\varphi\rangle_{Re} = -\frac{2}{3}\langle f\psi, \varphi\rangle_{Re}$$

for $i = 1, 2, 3$, and

$$\langle e_j \nabla_{e_i}\psi, \varphi\rangle_{Re} + \langle \psi, e_j \nabla_{e_i}\varphi\rangle_{Re} = -\frac{1}{2} e_k(\langle \psi, \varphi\rangle_{Re})$$

for i, j, k such that $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$. From this we obtain that

$$\langle \psi, e_s \cdot (e_l(a_1)(i\psi) + e_l(a_2)(j\psi) + e_l(a_3)(k\psi))\rangle_{Re} = 0$$

for any $s, l = 1, 2, 3$. Since ψ is not identically zero, we have $e_l(a_i) = 0$ for any $l, i = 1, 2, 3$. Hence a_1, a_2, a_3 are constant. So L is transversal to the zero section of E and $L^{-1}(0)$ is a Banach submanifold in B. The projection

$$P : L^{-1}(0) \longrightarrow Met_0 \times C^k(Y)$$

is Fredholm with index equal to 3. Note that $L(h, f, \cdot) = \mathbb{D}(\cdot, h) + f$ is quaternionic, so by Sard-Smale theorem, for a generic pair $(h, f) \in Met_0 \times C^k(Y)$, $P^{-1}(h, f)$ is empty, i.e., $\mathbb{D}(\cdot, h) + f$ is invertible. Any two such regular pairs (h_0, f_0) and (h_1, f_1) can be connected by an analytic path (h_t, f_t) which is transversal to the projection P. The operators $\mathbb{D}(\cdot, h_t) + f_t$ are invertible except for finitely many points $t_i \in (0,1)$, $i = 1, 2, \ldots, n$. The fact that Ker $(\mathbb{D}(\cdot, h_{t_i}) + f_{t_i}) = \mathbb{H}$ follows from index counting. Suppose that $\mathbb{D}(\psi_t, h_t) + f_t\psi_t = \lambda_t\psi_t$ near t_i with $\lambda_{t_i} = 0$ and $\|\psi_t\|_{L^2} = 1$, then

$$\frac{d\lambda_t}{dt}(t_i) = \int_Y \langle \frac{d}{dt}(\mathbb{D}(\psi_{t_i}, h_t) + f_t\psi_{t_i})(t_i), \psi_{t_i}\rangle_{Re} Vol.$$

Since the path (h_t, f_t) is transversal to the projection P, we have

$$\int_Y \langle \frac{d}{dt}(\mathbb{D}(\psi_{t_i}, h_t) + f_t\psi_{t_i})(t_i), \psi_{t_i}\rangle_{Re} Vol \neq 0.$$

Suppose $h_1 \in Met$ is conformal to $h \in Met_0$ and $g^{h_1} = e^{2u}g^h$. Let $m : V^{h_1} \to V^h$ be the isometry. The Dirac operators are related in the following way (see [H] or [LM]):

$$D_{g^h} = e^{2u} m D_{g^{h_1}} m^{-1} e^{-u}.$$

It is easy to see from this that $D_{g^{h_1}} + f$ is invertible if and only if $D_{g^h} + e^u f$ is. Similar arguments justify the chamber structure. *QED*

The Proof of Lemma 5.2:

Let $\psi = (\sigma, \theta)$, $\pi(\sigma) = (e_1, e_2, e_3)$, then

$$\begin{aligned}\mathbb{D}(\psi, h) \;=\; & (\sigma, c_1e_1(\theta) + c_2e_2(\theta) + c_3e_3(\theta) - \frac{1}{2}((\omega_{12}^2(h) + \omega_{13}^3(h))c_1\theta \\ & + (\omega_{23}^3(h) - \omega_{12}^1(h))c_2\theta - (\omega_{13}^1(h) + \omega_{23}^2(h))c_3\theta \\ & + (\omega_{12}^3(h) - \omega_{13}^2(h) + \omega_{23}^1(h))c_1c_2c_3\theta)).\end{aligned}$$

For $h = e^{tX}$, where $X = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$\begin{aligned}\omega_{12}^2(h) + \omega_{13}^3(h) &= (\omega_{12}^2 + \omega_{13}^3)(1+tx) + (1+tx)^2 e_1(1-tx) + O(t^2),\\ \omega_{23}^3(h) - \omega_{12}^1(h) &= (\omega_{23}^3 - \omega_{12}^1)(1-tx) + (1-tx)^2 e_2(1+tx) + O(t^2),\\ \omega_{13}^1(h) + \omega_{23}^2(h) &= -(1-tx)e_3(1+tx) - (1+tx)e_3(1-tx) + O(t^2),\\ \omega_{12}^3(h) - \omega_{13}^2(h) + \omega_{23}^1(h) &= \frac{1}{2}((1+tx)^2(\omega_{23}^1 + \omega_{12}^3) + (1-tx)^2(\omega_{12}^3 - \omega_{13}^2)) + O(t^2).\end{aligned}$$

So we have

$$\begin{aligned}\frac{d}{dt}(\mathbb{D}(\psi, h))(0) \;=\; & (\sigma, xc_1e_1(\theta) - xc_2e_2(\theta) - \frac{1}{2}((x(\omega_{12}^2 + \omega_{13}^3) - e_1(x))c_1\theta \\ & - (x(\omega_{23}^3 - \omega_{12}^1) - e_2(x))c_2\theta + x(\omega_{23}^1 + \omega_{13}^2)c_1c_2c_3\theta)).\end{aligned}$$

If we write $\psi = (\sigma, \theta)$, $\varphi = (\sigma, \xi)$, then

$$\begin{aligned}\int_Y \langle \frac{d}{dt}(\mathbb{D}(\psi, h))(0), \varphi\rangle Vol &= \int_Y (\langle x c_1 e_1(\theta), \xi\rangle - \langle x c_2 e_2(\theta), \xi\rangle \\ &- \frac{1}{2}(x\omega_{12}^2 + x\omega_{13}^3 - e_1(x))\langle c_1\theta, \xi\rangle \\ &- \frac{1}{2}(x\omega_{23}^3 - x\omega_{12}^1 - e_2(x))\langle c_2\theta, \xi\rangle \\ &+ \frac{1}{2}x(\omega_{23}^1 + \omega_{13}^2)\langle \theta, \xi\rangle) Vol.\end{aligned}$$

Let (e^1, e^2, e^3) be the dual to (e_1, e_2, e_3), then

$$\begin{aligned}d(x\langle c_1\theta, \xi\rangle * e^1) &= e_1(x)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3 \\ &+ x(\langle c_1 e_1(\theta), \xi\rangle + \langle c_1\theta, e_1(\xi)\rangle) e^1 \wedge e^2 \wedge e^3 \\ &- x(\omega_{12}^2 + \omega_{13}^3)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

Integration by parts, we have

$$\begin{aligned}\int_Y e_1(x)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3 &= -\int_Y x(\langle c_1 e_1(\theta), \xi\rangle + \langle c_1\theta, e_1(\xi)\rangle) e^1 \wedge e^2 \wedge e^3 \\ &+ \int_Y x(\omega_{12}^2 + \omega_{13}^3)\langle c_1\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_Y e_2(x)\langle c_2\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3 &= -\int_Y x(\langle c_2 e_2(\theta), \xi\rangle + \langle c_2\theta, e_2(\xi)\rangle) e^1 \wedge e^2 \wedge e^3 \\ &+ \int_Y x(\omega_{23}^3 - \omega_{12}^1)\langle c_2\theta, \xi\rangle e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

These give us

$$\begin{aligned}\int_Y \langle \frac{d}{dt}(\mathbb{D}(\psi, h))(0), \varphi\rangle Vol &= \frac{1}{2}\int_Y x(\langle e_1\nabla_{e_1}\psi, \varphi\rangle + \langle \psi, e_1\nabla_{e_1}\varphi\rangle) e^1 \wedge e^2 \wedge e^3 \\ &- \frac{1}{2}\int_Y x(\langle e_2\nabla_{e_2}\psi, \varphi\rangle + \langle \psi, e_2\nabla_{e_2}\varphi\rangle) e^1 \wedge e^2 \wedge e^3.\end{aligned}$$

Therefore, if $\int_Y \langle \frac{d}{dt}(\mathbb{D}(\psi, h))(0), \varphi\rangle Vol = 0$ for all $h = e^{tX}$ where $X = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$\langle e_1\nabla_{e_1}\psi, \varphi\rangle + \langle \psi, e_1\nabla_{e_1}\varphi\rangle = \langle e_2\nabla_{e_2}\psi, \varphi\rangle + \langle \psi, e_2\nabla_{e_2}\varphi\rangle.$$

Similarly, we have

$$\langle e_1\nabla_{e_1}\psi, \varphi\rangle + \langle \psi, e_1\nabla_{e_1}\varphi\rangle = \langle e_3\nabla_{e_3}\psi, \varphi\rangle + \langle \psi, e_3\nabla_{e_3}\varphi\rangle.$$

But $\psi, \varphi \in \text{Ker}\,(D_g + f)$, we have

$$\begin{aligned}\sum_{i=1}^{3}(\langle e_i\nabla_{e_i}\psi,\varphi\rangle + \langle\psi, e_i\nabla_{e_i}\varphi\rangle) &= \langle D_g\psi,\varphi\rangle + \langle\psi, D_g\varphi\rangle \\ &= -2\langle f\psi,\varphi\rangle.\end{aligned}$$

So we have

$$\langle e_i\nabla_{e_i}\psi,\varphi\rangle + \langle\psi, e_i\nabla_{e_i}\varphi\rangle = -\frac{2}{3}\langle f\psi,\varphi\rangle$$

for $i = 1, 2, 3$. Similar computation with $X = \begin{pmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ yields

$$\langle e_2\nabla_{e_1}\psi,\varphi\rangle + \langle\psi, e_2\nabla_{e_1}\varphi\rangle + \langle e_1\nabla_{e_2}\psi,\varphi\rangle + \langle\psi, e_1\nabla_{e_2}\varphi\rangle = 0.$$

Combined with

$$(\langle e_2\nabla_{e_1}\psi,\varphi\rangle + \langle\psi, e_2\nabla_{e_1}\varphi\rangle) - (\langle e_1\nabla_{e_2}\psi,\varphi\rangle + \langle\psi, e_1\nabla_{e_2}\varphi\rangle) = -e_3(\langle\psi,\varphi\rangle),$$

we have

$$\langle e_2\nabla_{e_1}\psi,\varphi\rangle + \langle\psi, e_2\nabla_{e_1}\varphi\rangle = -\frac{1}{2}e_3(\langle\psi,\varphi\rangle).$$

In general, we have

$$\langle e_j\nabla_{e_i}\psi,\varphi\rangle + \langle\psi, e_j\nabla_{e_i}\varphi\rangle = -\frac{1}{2}e_k(\langle\psi,\varphi\rangle)$$

for i, j, k such that $e_i \wedge e_j \wedge e_k = e_1 \wedge e_2 \wedge e_3$. *QED*

6. The σ-invariant perturbations

In this section, we give the construction of the σ-invariant admissible perturbations by using holonomy along embedded loops. Assume that (g, f) is regular. Let s^f denote the gradient of $\mathcal{CSD}_f$ and $\mathcal{M}_f$ denote the set of critical points where

$$\mathcal{CSD}_f = \mathcal{CSD} + \frac{1}{2}\int_Y f|\psi|_g^2 Vol_g, \quad \textit{and} \quad s^f(A,\psi) = (*F_A + \tau(\psi,\psi), D_A\psi + f\psi).$$

The moduli space $\mathcal{M}_f$ is compact and can be represented by smooth sections.

Definition 6.1. *A thickened loop is an embedding $\gamma : S^1 \times D^2 \to Y$, together with a bump function $\eta(y)$ on D^2 centered at $0 \in D^2$, with $\int_{D^2}\eta(y)dy = 1$.*

Given a thickened loop $\lambda = (\gamma, \eta)$, one can define a pair of σ-invariant functions $(p, q)_\lambda : \mathcal{B} \to [-1, 1] \times \mathbb{R}^+$ given by

$$p_\lambda(A,\psi) = \int_{D^2}\cos(\theta_y)\eta(y)dy,$$

where $e^{i\theta_y}$ is the holonomy of A along the loop $\gamma_y = S^1 \times \{y\}$, and

$$q_\lambda(A,\psi) = \int_{D^2\times S^1}|\psi|^2\eta(y)dydt.$$

Lemma 6.2. *The function (p,q) is smooth on $\mathcal{A}$.*

Proof: The same arguments as in [T]. It is useful to know that

$$dp_\lambda|_{(A,\psi)}(a,\varphi) = \int_{D^2} i\sin(\theta_y)\eta(y)(\int_{S^1\times\{y\}} \gamma_y^* a)dy$$

and

$$dq_\lambda|_{(A,\psi)}(a,\varphi) = 2\int_{D^2\times S^1} \langle\psi,\varphi\rangle\eta(y)dydt.$$

QED

For any set Λ of finitely many thickened loops, we have a smooth map $\Phi_\Lambda : \mathcal{B}^* \to \prod_{\lambda\in\Lambda}([-1,1]\times\mathbb{R}^+)_\lambda$ given by

$$\Phi_\Lambda([A,\psi]) = ((p,q)_\lambda(A,\psi), \lambda\in\Lambda).$$

The map Φ_Λ is σ-invariant and continuous on $\mathcal{B}$.

Lemma 6.3. *There is a set Λ of finitely many thickened loops such that*

1. $\mathrm{Ker}\nabla s^f \bigcap \bigcap_{\lambda\in\Lambda} \mathrm{Ker}(d(p,q)_\lambda) = \{0\}$ *at any* $[A,\psi]\in{\mathcal{M}_f}^*$.
2. Φ_Λ *is injective up to the σ action on $\mathcal{M}_f$. Therefore we can identify $\mathcal{M}_f/\sigma$ with a compact subset of* $\prod_{\lambda\in\Lambda}([-1,1]\times\mathbb{R}^+)_\lambda$.

Proof: Suppose $[A,\psi]\in{\mathcal{M}_f}^*$, and $(a,\varphi)\in\mathrm{Ker}\nabla s^f_{(A,\psi)}$, i.e., (a,φ) satisfies

$$\begin{cases} D_A\varphi + f\varphi + a\psi = 0 \\ *da + 2\tau(\psi,\varphi) = 0 \\ -d^*a + i\langle i\psi,\varphi\rangle_{Re} = 0. \end{cases}$$

Since A is not flat, if $(a,\varphi)\in\mathrm{Ker}(d(p,q)_\lambda)$ for all thickened loops, then $\int_{S^1\times\{y\}}\gamma_y^*a = 0$ for all γ. So $da = 0$. $da = 0$ implies $\tau(\psi,\varphi) = 0$. So $\varphi = v\psi$ for some function $v\in\Omega^0(Y,i\mathbb{R})$ wherever $\psi\neq 0$. This implies $dv + a = 0$ and $\int_Y(|dv|^2 + |v|^2|\psi|^2)Vol = 0$ by plugging into the equations. Since ψ is not identically zero, we have $(a,\varphi) = 0$.

So for each $[A,\psi]\in{\mathcal{M}_f}^*$, there is a set of finitely many thickened loops such that the first assertion holds for $[A,\psi]$. Then the first assertion follows by the compactness of $\mathcal{M}_f^*$ and the smoothness of the function (p,q).

For the second assertion, suppose $[A_1,\psi_1],[A_2,\psi_2]\in{\mathcal{M}_f}^*$ such that $(p,q)_\lambda(A_1,\psi_1) = (p,q)_\lambda(A_2,\psi_2)$ for all loops. Then $dA_1 = \pm dA_2$, and $|\psi_1|^2 = |\psi_2|^2$. Assume $dA_1 = dA_2$, then $\tau(\psi_1) = \tau(\psi_2)$. By writing in a local frame, it is easy to see that $\psi_1 = s\psi_2$ for some $s\in Map(Y,S^1)$. Then it is easy to see that $[A_1,\psi_1] = [A_2,\psi_2]$. In the case of $dA_1 = -dA_2$, apply σ.

Now suppose $[A_1,\psi_1]\neq[A_2,\psi_2]$ in ${\mathcal{M}_f}^*/\sigma$. Then there is a thickened loop λ separating them. By the compactness of ${\mathcal{M}_f}^*/\sigma$ and the smoothness of (p,q), there exists a set of finitely many loops separating any two points in ${\mathcal{M}_f}^*/\sigma$ with distance greater then a fixed number. Combining with the first assertion, since each point in ${\mathcal{M}_f}^*/\sigma$ has a neighborhood described by a Kuranishi model, the second assertion follows. *QED*

For any smooth function h on $\prod_{\lambda\in\Lambda}([-1,1]\times\mathbb{R}^+)_\lambda$, the composition $u = h\circ\Phi_\Lambda$ is a smooth function on $\mathcal{A}$. We will perturb $\mathcal{CSD}_f$ by adding u, i.e., $\mathcal{CSD}'_\mu = \mathcal{CSD}_f + u$. Denote the gradient of $\mathcal{CSD}'_\mu$ by s'_μ. The following lemma is standard (see [T]).

Lemma 6.4. 1. *∇s^f and $\nabla \mathrm{s}'_\mu$ are continuous family of Fredholm operators from bundle $T\mathcal{B}^*$ to $\mathcal{L}$ over $\mathcal{B}^*$, and $\nabla s^f - \nabla \mathrm{s}'_\mu$ are compact.*

2. *$\mathcal{M}_\mu$ can be represented by smooth sections.*
3. *There exists a constant $\varepsilon > 0$ such that when $|dh| < \varepsilon$, $\mathcal{M}_\mu$ is compact.*
4. *When $|dh| \to 0$, the distance between $\mathcal{M}_f$ and $\mathcal{M}_\mu$ goes to zero.*

Next we define a section G of the bundle $\mathcal{L}$ over $\mathcal{B}^*\times V$ where V is the dual of the vector space $\prod_{\lambda\in\Lambda}(\mathbb{R}\times\mathbb{R})_\lambda$:

$$G((A,\psi),(v,w)_\lambda) = s^f(A,\psi) + grad(\rho(\Phi_\Lambda)(\sum_{\lambda\in\Lambda}(v_\lambda p_\lambda + w_\lambda q_\lambda)))(A,\psi).$$

Here the set Λ of thickened loops satisfies the conditions in lemma 6.3, and ρ is a cutoff function on $\prod_{\lambda\in\Lambda}(\mathbb{R}\times\mathbb{R})_\lambda$ satisfying that $\rho\equiv 0$ in a neighborhood of $\prod_{\lambda\in\Lambda}([-1,1]\times\{0\})_\lambda$ and $\Phi_\Lambda([A,\psi])$ where $[A,\psi]\in\mathcal{B}^*$ are non-degenerate critical points of $\mathcal{CSD}_f$, and $\rho\equiv 1$ in a neighborhood of the rest of $\mathcal{M}_f{}^*$.

Lemma 6.5. *There exists $\varepsilon > 0$ (depending on ρ) such that G is transversal to the zero section of $\mathcal{L}$ when restricted to [illegible] $B(\varepsilon)$, where $B(\varepsilon)$ is a ball of radius ε centered at 0 in $V = (\prod_{\lambda\in\Lambda}(\mathbb{R}\times\mathbb{R})_\lambda)^*$.*

Proof: G is transversal to the zero section of $\mathcal{L}$ over $\mathcal{M}_f{}^*\times\{0\}$ by the choice of the set Λ. By continuity and lemma 6.4 (4), this lemma is proved. *QED*

The Proof of Proposition 2.7:

Apply Sard-Smale theorem to the projection $\Pi : G^{-1}(0)\to B(\varepsilon)$. For generic $(v,w)_\lambda \in B(\varepsilon)$, the perturbation $\mathcal{CSD}'_\mu = \mathcal{CSD}_f + u$ is admissible where

$$u = \rho(\Phi_\Lambda)(\sum_{\lambda\in\Lambda}(v_\lambda p_\lambda + w_\lambda q_\lambda)).$$

QED

References

[AM] Akbulut,S. and McCarthy,J. *Casson's invariant for oriented homology 3-spheres.* Princeton University Press, 1990.

[APS] Atiyah,M.F. Patodi,V.K. and Singer,I.M. *Spectral asymmetry in Riemannian geometry.* **I**, Math. Proc. Cambridge Philos. Soc.**77** (1975a) 43-69.

[C] Chen, W. *Dehn surgery formula for Seiberg-Witten invariants of homology 3-spheres*, preprint, 1997.

[D] Donaldson,S.K. *The Seiberg-Witten equations and 4-manifold topology*, Bull. Amer. Math. Soc.**33**(1996),45-70.

[DK] Donaldson,S.K. and Kronheimer,P.B. *The Geometry of Four-Manifolds*, Clarendon, Oxford, 1990.

[F] Floer,A *An instanton invariant for 3-manifolds*, Comm. Math. Phys.**118**(1988) 215-240.

[FU] Freed,D.S. and Uhlenbeck,K.K. *Instantons and four-manifolds*, M.S.R.I. Publications, Vol.**1** Springer, New York, 1984.

[H] Hitchin,N.J. *Harmonic spinors*, Advances in Mathematics, **14**(1974),1-54.

[K] Kato,T. *Perturbation theory for linear operators*, 2nd ed., Springer Verlag, Berlin and New York, 1980.

[KM1] Kronheimer,P.B. and Mrowka,T.S. *The genus of embedded surfaces in the projective plane*, Math.Research Letters **1**(1994),797-808.

[KM2] Kronheimer,P.B. and Mrowka,T.S. The 1st International Press Lectures, UC Irvine, March 1996.

[KN] Kobayashi,S. and Nomizu,K. *Foundations of Differential Geometry* Vols.**I,II**. Wiley, New York,1969.

[LM] Lawson, H.B. and Michelsohn, M *Spin Geometry*, Princeton University Press, 1989.

[T] Taubes,C.H. *Casson's invariant and gauge theory*, Journal of Differential Geometry,**31**(1990), 547-99.

[W] Witten,E. *Monopoles and four-manifolds*, Math. Research Letters **1** (1994), 769-796.

[Wa] Walker, K. *An Extension of Casson's Invariant*, Annals of Math. Studies, **No.126**, Princeton University Press, 1992.

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