Seiberg-Witten Invariants when Reversing Orientation

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Let X be a closed, oriented 4-manifold and let $\bar{X}$ denote the manifold X with the reversed orientation. Denote by $\chi(X)$ the Euler characteristic and by $\sigma(X)$ the signature of X. An interesting conjecture formulated in the framework of Donaldson's invariants, but easily translated to the Seiberg-Witten theory states:

Conjecture: *For each compact, orientable, simply-connected 4-manifold X, all the invariants vanish either on X or on $\bar{X}$.*

Although some work has been done (see [6]), within the frame of Donaldson's theory the conjecture remained wide open. Translated to Seiberg-Witten invariants, an affirmative answer to the conjecture has been recently given for a large class of complex 4-manifolds. The following theorem is due to Kotschick, but particular cases have been earlier obtained by N. Leung in [8] and the author in [3], independently.

Theorem: (Kotschick, [5]) *Let X be a complex surface of general type and assume that $\bar{X}$ admits a non-zero Seiberg-Witten invariant (of any degree). Then X has ample canonical bundle, $c_1^2(X)$ is even and the signature $\sigma(X)$ is non-negative. Moreover, X has zero signature if and only if it is uniformized by the polydisk.*

This result implies that the above conjecture is true for any complex surface X of general type satisfying one of the following conditions:
(i) $c_1^2(X)$ is odd;
(ii) canonical bundle is not ample;
(iii) $\sigma(X) < 0$.

It is worth remarking that the result does not use simply connectedness. However, the conjecture is not true for **all** complex surfaces of general type. Signature zero examples are easy to obtain, as there exist Kähler surfaces with orientation reversing diffeomorphisms. We will show that positive signature examples also exist. First we start by giving a sketch of proof for the theorem above, to emphasize the role played by the signature (for details see [5]).

Sketch of the Proof: The conclusion that $c_1^2(X)$ must be even comes from the fact that the dimension of the moduli space for the *spin$_c$* structure with non-zero invariant on $\bar{X}$ is even. Then Kotschick argues that the canonical bundle is ample, by showing that there are no embedded holomorphic spheres of self-intersection -1 or -2 in X. Indeed, if X contains an embedded sphere of negative self-intersection, non-trivial in homology, in $\bar{X}$

it becomes sphere of positive self-intersection and this would imply that all invariants of $\bar{X}$ vanish.

When the canonical bundle is ample, by Yau's solution to the Calabi conjecture [11], it follows that X admits a Kähler-Einstein metric g (this is the case treated in [8] and [3]). Rescaling this metric, we may assume that $Vol_g(X) = Vol_g(\bar{X}) = 1$.

Because g is a Kähler-Einstein metric on X, we have

$$c_1^2(X) = \frac{s^2}{32\pi^2}. \tag{1}$$

Denote by L the determinant line bundle of the $spin_c$ structure on $\bar{X}$ with non-zero Seiberg-Witten invariant. A theorem of LeBrun [7] imply

$$c_1(L)^2(\bar{X}) \le \frac{s^2}{32\pi^2}. \tag{2}$$

On the other hand, from the dimension formula of the Seiberg-Witten moduli space,

$$c_1(L)^2(\bar{X}) \ge 3\sigma(\bar{X}) + 2\chi(\bar{X}) = -3\sigma(X) + 2\chi(X) \tag{3}$$

$$= -6\sigma(X) + 3\sigma(X) + 2\chi(X) = -6\sigma(X) + c_1^2(X) = -6\sigma(X) + \frac{s^2}{32\pi^2},$$

where we used (1) in the last equality.

Relations (2) and (3) imply that $\sigma(X) \ge 0$. For the equality case, the reader is sent to [8]. We will just say that $\sigma(X) = 0$ implies equality in (2) and this equality holds if and only if there exists a Kähler-Einstein structure $(g, \bar{J}, \bar{\omega})$ on $\bar{X}$ as well. But then a holonomy argument implies that X is covered by the product of two disks.

Next we show that the conclusion about the signature in the theorem of Kotschick is sharp. We achieve this by giving examples of bi-symplectic 4-manifolds X (i.e. both X and $\bar{X}$ are symplectic) and invoking the following important result of Taubes:

Theorem: (Taubes, [9]) *Let (X, ω) be a closed, symplectic 4-manifold with $b_+ \ge 2$. Then the Seiberg-Witten invariant of the canonical class is equal to ± 1.*

For zero signature, the simplest examples are products of two Riemann surfaces, $X = \Sigma_1 \times \Sigma_2$. If ω_i is a volume form on Σ_i, $i = 1, 2$, then $\omega = \omega_1 + \omega_2$ and $\bar{\omega} = \omega_1 - \omega_2$ are symplectic forms on X inducing opposite orientations. If we take Σ_1, Σ_2 with the genus of each at least 2, then $X = \Sigma_1 \times \Sigma_2$ is a complex surface of general type. If ω_1, ω_2 are volume forms corresponding to hyperbolic metrics on each surface, then the product metric on X is Kähler-Einstein metric compatible with both ω and $\bar{\omega}$.

Let us remark that the above examples of bi-symplectic 4-manifolds are very particular cases in a much larger class. "Almost" all locally trivial fibre bundles $F \to X^4 \to \Sigma$, where F and Σ are closed Riemann surfaces, admit bi-symplectic structure. To see this we just have to repeat Thurston's construction of symplectic forms [10].

The only restriction is $[F] \neq 0$ in $\mathbf{H}_2(X, \mathbf{R})$. If this is satisfied, Thurston shows that there exists a closed 2-form α on X, which restricts to a symplectic form on each fibre, $F_x, x \in X$. Taking σ a symplectic form on the base Σ, for $\epsilon > 0$ small enough, $\omega = \pi^\star \sigma + \epsilon \alpha$ is a symplectic form on X. The induced volume form is

$$\omega \wedge \omega = \epsilon \pi^\star \sigma \wedge \alpha + \epsilon^2 \alpha \wedge \alpha.$$

But then, for ϵ eventually smaller, $\bar{\omega} = \pi^\star \sigma - \epsilon \alpha$ is also a symplectic form on X and

$$\bar{\omega} \wedge \bar{\omega} = -\epsilon \pi^\star \sigma \wedge \alpha + \epsilon^2 \alpha \wedge \alpha$$

gives the opposite orientation.

Many other examples of bi-symplectic 4-manifolds with signature zero can be obtained in this way. For instance, if we take $F \to X^4 \to \Sigma$ to be a holomorphic fibre bundle, then it is shown easily that the signature of the total space must be zero.

However, the signature of the total space of a fibre bundle is not always zero. Independently, Kodaira [4] and Atiyah [1] constructed a class of examples of non-zero signature. In fact, with one of the orientations, the total space X is a complex surface of general type, and with this orientation the signature is positive. Here is a short description of the examples. Take R to be a Riemann surface which has a fixed point free holomorphic involution denoted by τ (any surface of odd genus has fixed point free holomorphic involutions). Let C be the cover of R corresponding to the homomorphism

$$\pi_1(R) \longrightarrow \mathbf{H}_1(R; \mathbf{Z}) \longrightarrow \mathbf{H}_1(R; \mathbf{Z}_2),$$

and let $f : C \longrightarrow R$ be the covering map. In $C \times R$ consider the divisor $\Gamma = \Delta \cup \Delta'$, where $\Delta = graph(f)$, $\Delta' = graph(\tau \circ f)$. From the way the covering f was chosen, Γ induces an even class in $\mathbf{H}_2(C \times R, \mathbf{Z})$. Denote by X the 2-fold cover of $C \times R$ branched over Γ. Remark that X fibers over both C and R, but X is not a holomorphic fibre bundle. As for the signature of X, using the general formula for the signature of branched covers, we get

$$\sigma(X) = 2\sigma(C \times R) - \frac{1}{2}\Gamma \cdot \Gamma,$$

where $\Gamma \cdot \Gamma$ is the self-intersection of the branch locus in $C \times R$. Since $\sigma(C \times R) = 0$ and

$$\Gamma \cdot \Gamma = \Delta \cdot \Delta + \Delta' \cdot \Delta' = 2\Delta \cdot \Delta = 2\chi(C) < 0,$$

it follows that $\sigma(X) > 0$.

It is worth remarking that the existence of symplectic forms inducing both orientations may be used in this case to show that the canonical bundle of X is ample, therefore X admits a Kähler-Einstein metric.

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