Tr. J. of Mathematics
21 (1997) , 87 – 92.

Proceedings of 5^{th} Gökova
Geometry-Topology Conference

Seiberg-Witten Equations on $\mathbf{R}^8$

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1. Introduction

The Seiberg-Witten equations are meaningful on any even-dimensional manifold. To state them, let us recall the general set-up, adopting the terminology of the forthcoming book by D.Salamon ([1]).

A $spin^c$-structure on a $2n$-dimensional real inner-product space V is a pair (W, Γ), where W is a 2^n-dimensional complex Hermitian space and $\Gamma : V \to End(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \qquad \Gamma(v)^2 = -\|v\|^2$$

for $v \in V$.Globalizing this defines the notion of a $spin^c$-structure $\Gamma : TX \to End(W)$ on a $2n$-dimensional (oriented) manifold X, W being a 2^n-dimensional complex Hermitian vector bundle on X. Such a structure exists iff $w_2(X)$ has an integral lift. Γ extends to an isomorphism between the complex Clifford algebra bundle $C^c(TX)$ and $End(W)$). There is a natural splitting $W = W^+ \oplus W^-$ into the $\pm i^n$ eigenspaces of $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ where $e_1, e_2, \cdots, e_{2n}$ is any positively oriented local orthonormal frame of TX.

The extension of Γ to $C_2(X)$ gives via the identification of $\Lambda^2(T^*X)$ with $C_2(X)$ a map

$$\rho : \Lambda^2(T^*X) \to End(W)$$

given by

$$\rho(\sum_{i<j} \eta_{ij} e_i^* \wedge e_j^*) = \sum_{i<j} \eta_{ij}\Gamma(e_i)\Gamma(e_j).$$

The bundles $W^\pm$ are invariant under $\rho(\eta)$ for $\eta \in \Lambda^2(T^*X)$. Denote $\rho^\pm(\eta) = \rho(\eta)|_{W^\pm}$. The map ρ (and $\rho^\pm$) extends to

$$\rho : \Lambda^2(T^*X) \otimes \mathbf{C} \to End(W).$$

(If $\eta \in \Lambda^2(T^*X) \otimes \mathbf{C}$ is real-valued then $\rho(\eta)$ is skew-Hermitian and if η is imaginary-valued then $\rho(\eta)$ is Hermitian.)

A Hermitian connection ∇ on W is called a $spin^c$ connection (compatible with the Levi-Civita connection) if

$$\nabla_v(\Gamma(w)\Phi) = \Gamma(w)\nabla_v\Phi + \Gamma(\nabla_v w)\Phi$$

where Φ is a spinor (section of W),v and w are vector fields on X and $\nabla_v w$ is the Levi-Civita connection on X. ∇ preserves the subbundles $W^\pm$.

There is a principal $Spin^c(2n) = \{e^{i\theta}x | \theta \in \mathbf{R}, x \in Spin(2n)\} \subset C^c(\mathbf{R}^{2n})$ bundle P on X such that W and TX can be recovered as the associated bundles

$$W = P \times_{Spin^c(2n)} \mathbf{C}^{2^n}, \qquad TX = P \times_{Ad} \mathbf{R}^{2n},$$

Ad being the adjoint action of $Spin^c(2n)$ on $\mathbf{R}^{2n}$.We get then a complex line bundle $L_\Gamma = P \times_\delta \mathbf{C}$ using the map $\delta : Spin^c(2n) \to S^1$ given by $\delta(e^{i\theta}x) = e^{2i\theta}$.

There is a one-to-one correspondence between $spin^c$ connections on W and $spin^c(2n) = Lie(Spin^c(2n) = spin(2n) \oplus i\mathbf{R}$ -valued connection-1-forms $\hat{A} \in \mathbf{A}(P) \subset \Omega^1(P, spin^c(2n))$ on P.

Now consider the trace-part A of $\hat{A}$: $A = \frac{1}{2^n} trace(\hat{A})$. This is an imaginary valued 1-form $A \in \Omega^1(P, i\mathbf{R})$ which is equivariant and satisfies

$$A_p(p \cdot \xi) = \frac{1}{2^n} trace(\xi)$$

for $v \in T_pP, g \in Spin^c(2n), \xi \in spin^c(2n)$(where $p \cdot \xi$ is the infinitesimal action). Denote the set of imaginary valued 1-forms on P satisfying these two properties by $\mathbf{A}(\Gamma)$. There is a one-to-one correspondence between these 1-forms and $spin^c$ connections on W. Denote the connection corresponding to A by ∇_A. $\mathbf{A}(\Gamma)$ is an affine space with parallel vector space $\Omega^1(X, i\mathbf{R})$. For $A \in \mathbf{A}(\Gamma)$ the 1-form $2A \in \Omega^1(P, i\mathbf{R})$ represents a connection on the line bundle L_Γ. Because of this reason A is called a *virtual connection* on the *virtual line bundle* $L_\Gamma^{1/2}$. Let $F_A \in \Omega^2(X, i\mathbf{R})$ denote the curvature of the 1-form A. Finally, let D_A denote the Dirac operator corresponding to $A \in \mathbf{A}(\Gamma)$,

$$C^\infty(X, W^+) \to C^\infty(X, W^-)$$

defined by

$$D_A(\Phi) = \sum_{i=1}^{2n} \Gamma(e_i) \nabla_{A,e_i}(\Phi)$$

where $\Phi \in C^\infty(X, W^+)$ and $e_1, e_2, \cdots, e_{2n}$ is any local orthonormal frame.

The Seiberg-Witten equations can now be expressed as follows. Fix a $spin^c$ structure $\Gamma : TX \to End(W)$ on X and consider the pairs $(A, \Phi) \in \mathbf{A}(\Gamma) \times C^\infty(X, W^+)$. The SW-equations read

$$D_A(\Phi) = 0 \quad , \qquad \rho^+(F_A) = (\Phi\Phi^*)_0$$

where $(\Phi\Phi^*)_0 \in C^\infty(X, End(W^+))$ is defined by $(\Phi\Phi^*)(\tau) =< \Phi, \tau > \Phi$ for $\tau \in C^\infty(X, W^+)$ and $(\Phi\Phi^*)_0$ is the traceless part of $(\Phi\Phi^*)$.

In dimension $2n = 4$, $\rho^+(F_A) = \rho^+(F_A^+) = \rho(F_A^+)$ (where F^+ is the self-dual part of F and the second equality understood in the obvious sense), and therefore self-duality comes intimately into play. The first problem in dimensions $2n > 4$ is that there is not a generally accepted notion of self-duality. Although there are some meaningful definitions ([2],[3],[4],[5],[6]) (Equivalence of self-duality notions in [2],[3],[5],[6] has been shown in [7], making them more relevant as they separately are), they do not assign a well-defined self-dual part to a given 2-form. Even though $\rho^+(F_A)$ is still meaningful, it is apparently less important due to the lack of an intrinsic self-duality of 2-forms in higher dimensions.

The other serious problem in dimensions $2n > 4$ is that the SW-equations as they are given above are overdetermined. So it is improbable from the outset to hope for any solutions. We verify below for $2n = 8$ that there aren't indeed any solutions.

In dimension $2n = 4$ it is well-known that there are no finite-energy solutions ([1]), but otherwise whole classes of solutions are found which are related to vortex equations ([8]). It seems to us that it would be desirable to have higher dimensional modifications of Seiberg-Witten equations (at least in the physically important dimension $2n = 8$) having nontrivial solutions (possibly including 4-dimensional Seiberg-Witten solutions as special cases) and related to generalized self-duality referred to above.

2. Seiberg-Witten Equations on $\mathbf{R}^8$

We fix the constant *spin*^{c} structure $\Gamma : \mathbf{R}^8 \longrightarrow \mathbf{C}^{16\times 16}$ given by

$$\Gamma(e_i) = \begin{bmatrix} 0 & \gamma(e_i) \\ -\gamma(e_i)^* & 0 \end{bmatrix}$$

($e_i, i = 1, 2, ..., 8$ being the standard basis for $\mathbf{R}^8$),where

$$\gamma(e_1) = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \gamma(e_2) = \begin{bmatrix} i&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0\\ 0&0&0&0&i&0&0&0\\ 0&0&0&0&0&i&0&0\\ 0&0&0&0&0&0&i&0\\ 0&0&0&0&0&0&0&-i \end{bmatrix}$$

$$\gamma(e_3) = \begin{bmatrix} 0&1&0&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0 \end{bmatrix}, \quad \gamma(e_4) = \begin{bmatrix} 0&i&0&0&0&0&0&0\\ i&0&0&0&0&0&0&0\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&-i&0&0\\ 0&0&-i&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0\\ 0&0&0&0&0&0&0&i\\ 0&0&0&0&0&0&i&0 \end{bmatrix}$$

$$\gamma(e_5)=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},\quad \gamma(e_6)=\begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix}$$

$$\gamma(e_7)=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},\quad \gamma(e_8)=\begin{bmatrix} 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \end{bmatrix}.$$

(We obtain this $spin^c$ structure from the well-known isomorphism of the complex Clifford algebra $C^c(\mathbf{R}^{2n})$ with $End(\Lambda^*\mathbf{C}^n)$.)

In our case $X = \mathbf{R}^8, W = \mathbf{R}^8 \times \mathbf{C}^{16}, W^{\pm} = \mathbf{R}^8 \times \mathbf{C}^8$ and $L_\Gamma = {L_\Gamma}^{1/2} = \mathbf{R}^8 \times \mathbf{C}$. Consider the connection 1-form

$$A = \sum_{i=1}^{8} A_i dx_i \in \Omega^1(\mathbf{R}^8, i\mathbf{R})$$

on the line bundle $\mathbf{R}^8 \times \mathbf{C}$. Its curvature is given by

$$F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j \in \Omega^2(\mathbf{R}^8, i\mathbf{R})$$

where $F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}$. The $spin^c$ connection $\nabla = \nabla_A$ on W^+ is given by

$$\nabla_i \Phi = \frac{\partial \Phi}{\partial x_i} + A_i \Phi$$

$(i = 1, ..., 8)$ where $\Phi : \mathbf{R}^8 \to \mathbf{C}^8$.

$$\rho^+ : \Lambda^2(T^*X) \otimes \mathbf{C} \to End(W^+)$$

is given by

$$\rho^+(F_A) = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & G_{16} & G_{17} & 0 \\ \bar{G}_{12} & G_{22} & G_{23} & G_{24} & G_{25} & G_{26} & 0 & G_{17} \\ \bar{G}_{13} & \bar{G}_{23} & G_{33} & G_{34} & G_{35} & 0 & G_{26} & -G_{16} \\ \bar{G}_{14} & \bar{G}_{24} & \bar{G}_{34} & G_{44} & 0 & G_{35} & -G_{25} & G_{15} \\ \bar{G}_{15} & \bar{G}_{25} & \bar{G}_{35} & 0 & -G_{44} & G_{34} & -G_{24} & G_{14} \\ \bar{G}_{16} & \bar{G}_{26} & 0 & \bar{G}_{35} & \bar{G}_{34} & -G_{33} & G_{23} & -G_{13} \\ \bar{G}_{17} & 0 & \bar{G}_{26} & -\bar{G}_{25} & -\bar{G}_{24} & \bar{G}_{23} & -G_{22} & G_{12} \\ 0 & \bar{G}_{17} & -\bar{G}_{16} & \bar{G}_{15} & \bar{G}_{14} & -\bar{G}_{13} & \bar{G}_{12} & -G_{11} \end{bmatrix},$$

where

$$\begin{aligned} G_{11} &= iF_{12} + iF_{34} + iF_{56} + iF_{78}, & G_{12} &= F_{13} + iF_{14} + iF_{23} - F_{24}, \\ G_{13} &= F_{15} + iF_{16} + iF_{25} - F_{26}, & G_{14} &= F_{17} + iF_{18} + iF_{27} - F_{28}, \\ G_{15} &= F_{35} + iF_{36} + iF_{45} - F_{46}, & G_{16} &= F_{37} + iF_{38} + iF_{47} - F_{48}, \\ G_{17} &= F_{57} + iF_{58} + iF_{67} - F_{68}, & G_{22} &= -iF_{12} - iF_{34} + iF_{56} + iF_{78}, \\ G_{23} &= -F_{35} - iF_{36} + iF_{45} - F_{46}, & G_{24} &= -F_{37} - iF_{38} + iF_{47} - F_{48}, \\ G_{25} &= F_{15} + iF_{16} - iF_{25} + F_{26}, & G_{26} &= F_{17} + iF_{18} - iF_{27} + F_{28}, \\ G_{33} &= -iF_{12} + iF_{34} - iF_{56} + iF_{78}, & G_{34} &= -F_{57} - iF_{58} + iF_{67} - F_{68}, \\ G_{35} &= -F_{13} - iF_{14} + iF_{23} - F_{24}, & G_{44} &= -iF_{12} + iF_{34} + iF_{56} - iF_{78}. \end{aligned}$$

For $\Phi = (\phi_1, \phi_2, ..., \phi_8) \in C^\infty(X, W^+) = C^\infty(\mathbf{R}^8, \mathbf{R}^8 \times \mathbf{C}^8)$,

$$(\Phi\Phi^*)_0 = \begin{bmatrix} \phi_1\bar{\phi}_1 - 1/8\sum\phi_i\bar{\phi}_i & \phi_1\bar{\phi}_2 & \cdots & \phi_1\bar{\phi}_8 \\ \phi_2\bar{\phi}_1 & \phi_2\bar{\phi}_2 - 1/8\sum\phi_i\bar{\phi}_i & \cdots & \phi_2\bar{\phi}_8 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \phi_8\bar{\phi}_1 & \phi_8\bar{\phi}_2 & \cdots & \phi_8\bar{\phi}_8 - 1/8\sum\phi_i\bar{\phi}_i \end{bmatrix}.$$

It was remarked by Salamon([1],p.187) that $\rho^+(F_A) = 0$ implies $F_A = 0$. (i.e.reducible solutions of 8-dim. SW-equations are flat.)
It can be explicitly verified that all solutions are reducible and flat:

Proposition 2.1. *There are no nontrivial solutions of the Seiberg-Witten equations on $\mathbf{R}^8$ with constant standard spin^c structure,i.e.*
$\rho^+(F_A) = (\Phi\Phi^*)_0$ *(alone) implies* $F_A = 0$ *and* $\Phi = 0$.

Proof. Trivial but tedious manipulation with the linear system. □

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