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A lower bound of the first eigenvalue of certain self-adjoint elliptic operators on manifolds containing long necks

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Let X be an oriented Riemannian manifold with a cylindrical end modeled on Y, i.e., there exists a compact subset K such that $X \setminus K$ is isometric to $(-1, \infty) \times Y$.

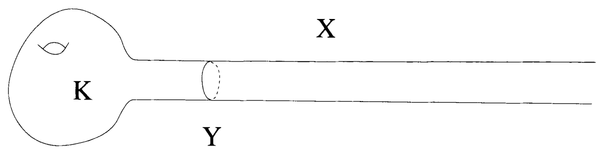

FIGURE 1.

Let E be a cylindrical Riemannian vector bundle over X. This means that there is a Riemannian vector bundle E_0 over Y such that E is isometric to $\pi^* E_0$ on the cylindrical end $(-1, \infty) \times Y$, where $\pi : (-1, \infty) \times Y \to Y$ is the natural projection. Assume that $D : \Gamma(E) \to \Gamma(E)$ is a first order formally self-adjoint elliptic operator on X, which takes the following form on the cylindrical end $(-1, \infty) \times Y$:

$$D = I(\frac{\partial}{\partial t} + A),$$

where I is a bundle automorphism of E_0 which preserves the inner product, and $A : \Gamma(E_0) \to \Gamma(E_0)$ is a self-adjoint elliptic operator on Y independent of t. The self-adjointness of D implies that I and A satisfy the following conditions:

$$I^2 = -1, \ \ I^* = -I, \ \ A^* = A, \ \ IA + AI = 0.$$

Note that the spectrum of A is symmetric about the origin and the automorphism I maps E_λ to $E_{-\lambda}$ where E_λ is the eigenspace correspondent to eigenvalue λ (see [M]). **Throughout this note, we assume that** Ker $A \neq 0$. Then the automorphism I defines a complex structure on Ker A which induces a symplectic structure on it. In particular, the dimension of Ker A is even. The operator D as described will be said cylindrical compatible in this note.

Definition 1. *An exponentially small perturbation of a cylindrical compatible operator D is a first order formally self-adjoint elliptic operator D' satisfying the following conditions:*

a) *D' is a zero order perturbation of D,*

b) *on the cylindrical end* $(-1,\infty)\times Y$, $D' = D+P(t)$ *where* $P(t) : \Gamma(E_0) \to \Gamma(E_0)$ *is a smooth family of zero order self-adjoint operators satisfying the following exponential decay conditions: there exist a small* $\delta > 0$, *some* $T_0 > 0$ *and a constant* C *such that when* $t > T_0$,

$$\|P(t)\psi\|_{L^2(Y)} \le Ce^{-\delta(t-T_0)}\|\psi\|_{L^2(Y)} \text{ and } \|\frac{\partial P}{\partial t}\psi\|_{L^2(Y)} \le Ce^{-\delta(t-T_0)}\|\psi\|_{L^2(Y)}$$

for $\psi \in L^2(E_0)$.

Let D' be an exponentially small perturbation of a cylindrical compatible operator. The space of "bounded" harmonic sections of D' is denoted by $H_B(D')$, i.e.,

$$H_B(D') = \{\psi \in \Gamma(E) | D'\psi = 0, \|\psi\|_{C^0(X)} < \infty\}.$$

The space of L^2 harmonic sections of D' is denoted by $H_{L^2}(D')$, i.e.,

$$H_{L^2}(D') = \{\psi \in L^2(E) | D'\psi = 0\}.$$

Let β be a fixed cut-off function which is equal to one at ∞, and $\pi : (-1,\infty)\times Y \to Y$ be the natural projection.

Lemma 2. *There exists a small* $\delta_1 > 0$ *such that for any* $\psi \in H_B(D')$, *there exists an unique limiting value* $r(\psi) \in \mathrm{Ker}\, A$ *such that*

$$\|\psi - \beta\pi^* r(\psi)\|_{L^2_{\delta_1}(E)} < \infty.$$

In particular, $\psi \in H_{L^2}(D')$ *if and only if* $r(\psi) = 0$. *Moreover,*

$$dim H_B(D') - dim H_{L^2}(D') = \frac{1}{2} dim \mathrm{Ker}\, A.$$

Now consider a pair of triples (X_i, E_i, D'_i) for $i = 1, 2$. Suppose that there is an orientation reversing isometry $h : Y_1 \to Y_2$ which is covered by correspondent bundle maps which identify A_1 with A_2 in a suitable way so that for any $L > 0$, we can form a triple (X_L, E_L, D'_L) where $X_L = X_1 \setminus [L+1,\infty)\times Y_1 \bigcup_h X_2 \setminus [L+1,\infty)\times Y_2$ with $h : (L, L+1)\times Y_1 \to (L+1, L)\times Y_2$ given by $h(L+t, y) = (L+1-t, h(y))$, $E_L = E_1 \bigcup_h E_2$, $D_L = D_1 \bigcup_h D_2$ and $P_L = \beta_L P_1 + (1-\beta_L)h^* P_2$ for some cut-off function β_L supported in $(L, L+1)\times Y_1$ with $|\nabla\beta_L| \le 2$, and $D'_L = D_L + P_L$ (see Figure 2). Set

$$\lambda_L = \inf_{\psi\ne 0} \frac{\int_{X_L} |D'_L\psi|^2}{\int_{X_L} |\psi|^2}.$$

The purpose of this note is to investigate the behavior of λ_L as $L \to \infty$.

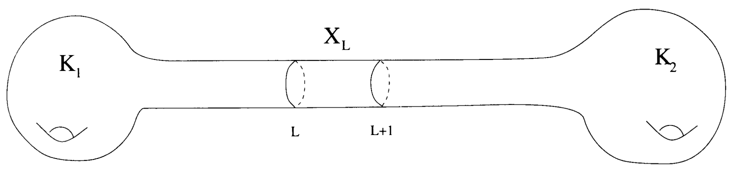

FIGURE 2.

Definition 3. *Suppose D', D'_1 and D'_2 are exponentially small perturbations of cylindrical compatible operators.*

a) *D' is said to be regular if $H_{L^2}(D') = 0$.*
b) *(D'_1, D'_2) is said to be a transversal pair if*

$$r(H_B(D'_1)) \bigcap h^*(r(H_B(D'_2))) = \{0\}.$$

Here is the main result.

Theorem 4. 1) *$\lambda_L = O(\frac{1}{L^2})$ as $L \to \infty$,*
2) *if (D'_1, D'_2) is a regular transversal pair, then for any function $\gamma(L) = o(\frac{1}{L^2})$ as $L \to \infty$, there exists $L_0 > 0$ such that when $L > L_0$, we have*

$$\lambda_L > \gamma(L).$$

In particular, D'_L is invertible for large L.

Remarks:

1. More general results are obtained in [CLM] for the unperturbed case.
2. Theorem 4 in this note is used in the gluing of Seiberg-Witten moduli spaces of 3-manifolds along T^2. See [C1] and [C2].

We first introduce some notations. Let λ_i, $i \in \mathbb{Z}$ denote the eigenvalues of the operator A, and u_i denote the correspondent eigensections. Set $\mu = \inf_{\lambda_i \neq 0} |\lambda_i|$. For simplicity, we omit the subscript L if no confusion is caused.

Lemma 5. *There exist $L_0 > 0$ and $M > 1$ with the following significance. Assume that ψ and c satisfy $D'\psi = c\psi$ with $\psi \neq 0$ and $|c| < \delta(\mu)$ for some small $\delta(\mu)$, then ψ can be rescaled so that $\|\psi\|_{C^0(X_L)} < M$ and one of the following conditions holds:*

- *either $\int_{X_1(L_0)} |\psi|^2$ or $\int_{X_2(L_0)} |\psi|^2$ is equal to one,*
- *either $\|\psi\|_{L^2(Y_1)}(L_0)$ or $\|\psi\|_{L^2(Y_2)}(L_0)$ is greater than or equal to one.*

Here $X_i(L_0) = X_i \setminus (L_0, \infty) \times Y_i$, $i = 1, 2$.

Proof: Let Π_1, Π_2 be the L^2-orthogonal projection onto Ker A and $(\text{Ker } A)^\perp$. On the cylindrical neck of X_L, write $\psi = f_1 + f_2$ where $f_1 \in \text{Ker } A$ and $f_2 \in (\text{Ker } A)^\perp$. Set $\xi(t) = \int_Y |f_2|^2$.

Direct computation shows that

$$\begin{aligned}
\frac{\partial f_1}{\partial t} &= I\Pi_1 P\psi - cI(f_1) \\
\frac{\partial f_2}{\partial t} &= -Af_2 + I\Pi_2 P\psi - cI(f_2) \\
\frac{\partial^2 f_2}{\partial t^2} &= (A^2 - c^2)f_2 + IA\Pi_2 P\psi + I\Pi_2 \frac{\partial P}{\partial t}\psi + I\Pi_2 P\frac{\partial \psi}{\partial t} + c\Pi_2 P\psi.
\end{aligned}$$

For any $\varepsilon > 0$, there exists $L_0 > 0$ such that on the neck $[L_0, 2L+1-L_0] \times Y_1$ we have

$$\begin{aligned} \frac{\partial^2 \xi}{\partial t^2} &\geq 2\int_Y (\frac{\partial^2 f_2}{\partial t^2}, f_2) \\ &\geq K(\mu^2 \|f_2\|^2_{L^2_1(Y)} - \varepsilon \|f_2\|_{L^2_1(Y)}(\|f_1\|_{L^2(Y)} + \|f_2\|_{L^2(Y)})) \end{aligned}$$

for some constant K. Here $|c| < \delta(\mu)$ for some small $\delta(\mu)$. If $\xi(t)$ reaches its maximum in an interior point $t_0 \in (L_0, 2L+1-L_0)$, then on the neck, we have

$$\max \|f_1\|_{L^2(Y)} \geq \|f_1\|_{L^2(Y)}(t_0) \geq \frac{\mu^2 - \varepsilon}{\varepsilon} \max \|f_2\|_{L^2(Y)}.$$

Otherwise, $\xi(t) = \|f_2\|^2_{L^2(Y)}$ reaches its maximum at the end points.

On the other hand, we have on the neck that

$$\begin{aligned} \tfrac{\partial f_1}{\partial t} + cI(f_1) &= I\Pi_1 P\psi \\ \tfrac{\partial (If_1)}{\partial t} - c(f_1) &= -\Pi_1 P\psi. \end{aligned}$$

Set $C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, then

$$\begin{pmatrix} f_1 \\ If_1 \end{pmatrix}(t) = e^{-Ct}\int_{L_0}^t e^{Cs}\begin{pmatrix} I\Pi_1 P\psi \\ -\Pi_1 P\psi \end{pmatrix} ds + e^{-C(t-L_0)}\begin{pmatrix} f_1(L_0) \\ If_1(L_0) \end{pmatrix}.$$

This implies that on the interval $[L_0, 2L+1-L_0]$

$$\|f_1\|_{L^2(Y)}(t) \leq c_1 e^{-\delta(L_0 - T_0)}(\max \|f_1\|_{L^2(Y)} + \max \|f_2\|_{L^2(Y)}) + \|f_1(L_0)\|_{L^2(Y)}.$$

If $\|f_2\|_{L^2(Y)}$ reaches its maximum in the interior, then

$$\max \|f_1\|_{L^2(Y)} \leq 2\|f_1(L_0)\|_{L^2(Y)}$$

for large enough L_0. If $\|f_2\|_{L^2(Y)}$ reaches its maximum at the end points, assuming that it is the left end point without loss of generality, we have

$$\max(\|f_1\|_{L^2(Y)} + \|f_2\|_{L^2(Y)}) \leq 2(\|f_1(L_0)\|_{L^2(Y)} + \|f_2(L_0)\|_{L^2(Y)})$$

for large enough L_0. Lemma 5 follows easily from these estimates. *QED*

The Proof of Theorem 4: 1). Pick $\varphi \in \operatorname{Ker} A$ with $\|\varphi\|_{L^2(Y)} = 1$. Let ρ_L be a cut-off function which equals to one on $[\frac{3L}{4}, \frac{3L}{4} + \frac{L}{2} + 1] \times Y_1$ and equals to zero outside $[\frac{L}{2}, \frac{L}{2} + L + 1] \times Y_1$ with $|\nabla \rho_L| = O(\frac{1}{L})$. Then

$$\int_{X_L} |D'_L(\rho_L \varphi)|^2 \leq \int_{X_L} |\nabla \rho_L|^2 |\varphi|^2 + \int_{X_L} |P_L(\rho_L \varphi)|^2 = O(\frac{1}{L}), \text{and} \int_{X_L} |\rho_L \varphi|^2 \geq \frac{L}{10}.$$

So $\lambda_L = O(\frac{1}{L^2})$ as $L \to \infty$.

2). Suppose that there exists a sequence of $L_n \to \infty$ such that $\lambda_{L_n} \leq \gamma(L_n)$. Then there exist ψ_n, c_n such that $D'_{L_n}\psi_n = c_n \psi_n$ with $c_n^2 = \lambda_{L_n}$. By lemma 5, there exist $\psi_1 \in H_B(D'_1)$, $\psi_2 \in H_B(D'_2)$ such that a subsequence of ψ_n converges to ψ_1 over X_1 and ψ_2 over X_2 in C^k norm on any compact subset. Note that one of ψ_1 and ψ_2 is nonzero.

The second assertion of Theorem 4 follows if we show that $r(\psi_1) = h^* r(\psi_2)$. But this follows from the fact that if we write $\psi = f_1 + f_2$ as in lemma 5,

$$\|f_1(t) - f_1(2L+1-t)\|_{L^2(Y)} \leq C(e^{-\delta t} + |\cos(c(2L+1-2t)) - 1| + |\sin(c(2L+1-2t))|),$$

for large enough t and L. C is some constant independent of t and L.

The Proof of Lemma 2:

Suppose $\psi \in \Gamma(E)$ and $D'\psi = 0$. On the cylindrical end $(T_0, \infty) \times Y$, write $\psi = \sum_i f_i u_i$ where u_i are the eigensections of the operator A correspondent to eigenvalues λ_i, and f_i are smooth functions in t. Then we have

$$\frac{\partial f_i}{\partial t} + \lambda_i f_i = (IP(t)\psi, u_i).$$

Set $g_i = (IP(t)\psi, u_i)$, then $\sum_i g_i^2 = \|P\psi\|^2_{L^2(Y)}$ and

$$f_i(t) = \int_{T_0}^{t} e^{-\lambda_i(t-s)} g_i(s) ds + f_i(T_0) e^{-\lambda_i(t-T_0)}.$$

Now assume that $\psi \in L^2_{-\gamma}$ for any small enough $\gamma > 0$. Assume that $\delta_1 < \min(\frac{\delta}{2}, \frac{\mu}{4})$ where $\mu = \inf_{\lambda_i \neq 0} |\lambda_i|$.

- For $\lambda_i = 0$, we have for any $t' > t$,

$$e^{\delta_1 t} |f_i(t') - f_i(t)| \leq C\left(\int_t^{t'} \int_Y e^{-\frac{\delta}{10}s} |\psi|^2 Vol_Y ds\right)^{\frac{1}{2}},$$

  so $f_i(\infty) = \lim_{t \to \infty} f_i(t)$ exists and $f_i - f_i(\infty) \in L^2_{\delta_1}$.
- For $\lambda_i > 0$, we have for some constant $C(\mu)$ that

$$e^{2\delta_1 t}\left(\sum_i f_i^2(t)\right) \leq C(\mu) \int_{T_0}^{\infty} e^{2\delta_1 s}\left(\sum_i g_i^2(s)\right) ds + \left(\sum_i f_i^2(T_0)\right) e^{2\delta_1 T_0}.$$

- For $\lambda_i < 0$. First of all, we have

$$f_i(t) = -e^{-\lambda_i t} \int_t^{\infty} e^{\lambda_i s} g_i(s) ds$$

  since $\psi \in L^2_{-\gamma}$ for any small enough $\gamma > 0$. On the other hand, for some constant $C(\mu)$, we have

$$e^{2\delta_1 t}\left(\sum_i f_i^2(t)\right) \leq C(\mu) \int_t^{\infty} e^{2\delta_1 s}\left(\sum_i g_i^2(s)\right) ds.$$

Take $r(\psi) = \sum_i f_i(\infty) u_i$ where $u_i \in \text{Ker } A$, then

$$\|\psi - \beta \pi^* r(\psi)\|_{L^2_{\delta_1}(E)} < \infty$$

where β is a fixed cut-off function which is equal to one at ∞, and $\pi : (-1, \infty) \times Y \to Y$ is the natural projection. As for $dimH_B(D') - dimH_{L^2}(D') = \frac{1}{2} dim\text{Ker } A$, it follows from Theorem 7.4 in [LM].

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