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LOCALLY TOPOLOGICAL GROUPOIDS

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Abstract

The notion of locally topological groupoid was introduced by Aof and Brown in [2]. On the other hand in [6] by Mackenzie a topological groupoid MG, called monodromy groupoid, is constructed. In this paper we prove that this groupoid MG gives a locally topological groupoid.

Introduction

A groupoid whose explicit definition is given in Definition 1.1 is a category such that each morphism has an inverse.

For example a group is a groupoid with only one object. If X is a topological space, the homotopy classes of the paths in X form a groupoid on X. The composition of the paths in X gives a composition of the homotopy classes. This groupoid is called *fundamental groupoid* of X and denoted by $\pi_1 X$.

A topological groupoid defined in Definition 1.3, is a groupoid having topology such that all maps are continuous.

A locally topological groupoid is a pair (G, W) of a groupoid G and a topological space W such that $W \subseteq G$ and the conditions given in Definition 2.2 are satisfied.

Let G be a topological groupoid such that each fibre $G_x = \alpha^{-1}(x)$ has a universal covering. Let $(\overline{G}_x)_{1_x}$ be the universal covering of G_x at the base point 1_x. On the other hand it is well known that if X is a topological space which has a iniversal cavering, then

$$\beta_x : (\pi_1 X)_x \longrightarrow X,$$

the restriction of the final point map β, is the universal covering of X at the base point x. Here $\pi_1 X$ is the fundamental groupoid of X. Hence we can take $(\overline{G}_x)_{1_x}$ as $(\pi_1 G_x)_{1_x}$. So the elements of $(\pi_1 G_x)_{1_x}$ are the homotopy classes of the pats $a : [0,1] \longrightarrow G_x$ such tahat $a(0) = 1_x$. Let

$$MG = \bigcup_{x \in O_G} (\overline{G}_x)_{1_x}.$$

In [6] on MG a groupoid is defined as follows:

$$\begin{array}{c} MG(y,z)\times MG(x,y)\longrightarrow MG(x,z)\\ ([b],[a])\longmapsto [b\,\mathbf{0}a]\end{array}$$

where $b\,\mathbf{0}a$ is defined to be

$$b\,\mathbf{0}\,a=\begin{cases} a(2t), & 0\le t\le 1/2\\ b(2t-1)g, & 1/2\le t\le 1\end{cases}$$

for $g=a(1)$. This multiplication is well defined and MG is a groupoid on O_G. This groupoid is called monodromy groupoid. Monodromy groupoid of a topological groupoid is also the main object of [7] (see also [3]).

The main object of this paper is to prove that this groupoid MG gives rise to a locally topological groupoid.

1 Groupoids

Definition 1.1 *A groupoid consists of two sets G and O_G called respectively the set of elements or morphisms and the set of objects of the groupoid, together with two maps $\alpha,\beta: G\longrightarrow O_G$, called respectively the source and target maps, a map $1_{()}: O_G\longrightarrow G, x\mapsto 1_x$ called the object map and a partial multiplication*

$$G*G\longrightarrow G,(h,g)\mapsto hg$$

defined on the fibre product set

$$G*G=\{(h,g)\in G\times G:\alpha(h)=\beta(g)\}.$$

These maps are subject to the following conditions

i) $\alpha(hg)=\alpha(g)$ and $\beta(hg)=\beta(h)$ for all $(h,g)\in G*G$;

ii) $k(hg)=(kh)g$ for all $g,h,k\in G$ such that $\alpha(h)=\beta(g)$ and $\alpha(k)=\beta(h)$;

iii) $\alpha(1_x)=\beta(1_x)=x$ for all $x\in O_G$, where 1_x is the identity at x;

iv) $g1_{\alpha(g)}=g$ and $1_{\beta(g)}g=g$ for all $g\in G$; and

v) each $g\in G$ has an inverse g^{-1} such that $\alpha(g^{-1})=\beta(g)$, $\beta(g^{-1})=\alpha(g)$ and $g^{-1}g=1_{\alpha(g)}, gg^{-1}=1_{\beta(g)}$.

If the pair (G,O_G) is a groupoid we say G is a groupoid on O_G. If G is a groupoid and W is a subset of G containing all the identities we write $O_G\subseteq W$.

For a groupoid G we write G_x for $\alpha^{-1}(x)$ and $G(x,y)$ for $\alpha^{-1}(x) \cap \beta^{-1}(y)$ where$x, y \in O_G$. In a groupoid $G, \delta : G \underset{\alpha}{\times} G \longrightarrow G, (h,g) \rightarrow hg^{-1}$ is called groupoid difference map, where

$$G \underset{\alpha}{\times} G = \{(h,g) \in G \times G : \alpha(h) = \alpha(g)\}.$$

Definition 1.2 *Let G and H be groupoids. A local morphism of groupoids is a map $f : W \longrightarrow H$ from a subset of G containing all the identities in G such that for $u \in W, \alpha_H(fu) = f(\alpha_G u), \beta_H(fu) = f(\beta_G u)$ and $f(vu) = f(v)f(u)$ whenever $v, u \in W$ and vu is defined and belongs to W.*

A morphism from G to H is a pair of maps

$$f : G \longrightarrow H \text{ and } O_f : O_G \longrightarrow O_H$$

such that

$$\alpha_H \mathbf{0} f = O_f \mathbf{0} \alpha_G, \quad \beta_H \mathbf{0} f = O_f \mathbf{0} \beta_G$$

and $f(vu) = f(v)f(u)$ for all $(v,u) \in G * G$, where

$$G * G = \{(v,u) \in G \times G : \alpha(v) = \beta(u)\}.$$

For such a morphism we simply write $f : G \longrightarrow H$.

Definition 1.3 *A topological groupoid is a groupoid G in which the sets G and O_G are topological spaces and the following maps are continuous.*

i) partial multiplication $G * G \longrightarrow G, (h,g) \mapsto hg$, where $G * G$ has the relative topology;

ii) inverse map $G \longrightarrow G, g \mapsto g^{-1}$;

iii) source and target maps $\alpha, \beta : G \longrightarrow O_G$;

iv) object map $1_{()} : O_G \longrightarrow G, x \mapsto 1_x$.

2 Review of holonomy groupoids

We recall the following definition due to Ehresmann[5].

Definition 2.1. *Let G be a groupoid and O_G a topological space. An admissible lacal section of G is a function $s : U \longrightarrow G$ from an open neighbourhood in O_G such that*

i) $\alpha s(x) = x$ for all $x \in U$;

ii) $\beta s(U)$ is open in O_G; and

iii) βs maps U homeomophically to $\beta s(U)$.

Let W be a subset of G such that $O_G \subseteq W$, that is W contains all the identities and let W have the structure of a topological space. We give O_G the subspace topology. We say that (α, β, W) *has enough continuous admissible local sections* if for each $w \in W$ there is an admissible local section $s : U \longrightarrow G$ of G such that

i) $s\alpha(w) = w$;

ii) $s(U) \subseteq W$; and

iii) s is continuous from U to W.

Such an s is called a *continuous admissible local section*

Let G be a groupoid and W a subset of G. We say that W generates G, if each element of G is written as a multiplication of some elements of W.

The following definition is taken from [2].

Definition 2.2. *A locally topological groupoid is a pair (G, W) consisting of a groupoid G and a topological space W such that*

i) $O_G \subseteq W \subseteq G$ (that is , W is a subset of G including all the identities)

ii) $W = W^{-1}$;

iii) W generates G as a groupoid;

iv) the set $W_\delta = W \underset{\alpha}{\times} W \cap \delta^{-1}(W)$ is open in $W \underset{\alpha}{\times} W$ and the restriction to W_δ of the difference map $\delta : G \underset{\alpha}{\times} G \longrightarrow G, (g, h) \mapsto gh^{-1}$ is continuous, where

$$W \underset{\alpha}{\times} W = \{(v, u) \in W \times W : \alpha(u) = \alpha(v)\}.$$

and

v) the restriction to W of the source and target maps α and β are continuous and the triple (α, β, W) has enough continuous admissible local sections.

In this definition, G is a groupoid but not necessarily a topological groupoid. The locally topological groupoid (G, W) is said to be *extendible* if a topology can be found on

G making it a topological groupoid such taht W is an open subspace of G. See [2] for a locally topological groupoid which is not extendible.

From a locally topological groupoid (G,W) a topological groupoid, called *Holonomy groupoid,* is obtained in the following theorem. This theorem was first stated by Pradines in [8] and then completely proved in [1] (see also [2]).

Theorem 2.3. *Let (G,W) be a locally topological groupoid. Then there is a topological groupoid H, A morphism $\phi : H \longrightarrow G$ of groupoids and an embedding $i : W \longrightarrow H$ of W to an open neighbourhood of O_H such that the following conditions are satisfied.*

i) ϕ is the identity on objects, $\phi i = id_W, \phi^{-1}(W)$ is open in H, and the restriction $\phi_W : \phi^{-1}(W) \longrightarrow W$ of ϕ is continuous;

ii) if A is a topological groupoid and $\zeta : A \longrightarrow G$ is a morphism of groupoids such that

a) ζ is the identity on objects;

b) the restriction $\zeta_W : \zeta^{-1}(W) \longrightarrow W$ of ζ is continuous and $\zeta^{-1}(W)$ is open in A and generates A;

c) the triple (α_A, β_A, A) has enough continuous admissible local sections;
then there is a unique morphism $\zeta' : A \longrightarrow H$ of topological groupoids such that $\phi\zeta' = \zeta$ and $\zeta' a = i\zeta a$ for $a \in \zeta^{-1}(W)$.

The groupoid H is called *holonomy groupoid* of the locally topological groupoid (G,W) and denoted by Hol (G,W). See [7] for some applications of Theorem 2.3

3 Main Theorem

Definition 3.1. *Let X be a topological space which has a simply connected covering. A subset W of X is called canonical if it is open, path connected and for each $x \in W$, the fundamental group $\pi_1(W,x)$ is singleton, that is, has just only one element.*

Let G be a topological groupoid and W a supspace of G. Then W is *star connected* if each $W_x = W \cap G_x$ is connected and W is *star canonical* if each W_x is canonical. Thus G is *star connected* if for each $x \in O_G, G_x$ is connected.

It is well known that if G is a topological group and V is an open neighbourhood of the identity e in G then there exists an open neighbourhood W of e in G such that $W = W^{-1}$ and $W^2 \subseteq V$. Because in a topological group G the group difference map

$$\delta : G \times G \longrightarrow G, (g,h) \mapsto gh^{-1}$$

is continuous, and so there is an open neighbourhood N of e in G such that $N \times N \subseteq \delta^{-1}(V)$. If we take $W = N \cap N^{-1}$ then $W = W^{-1}$ and $W^2 \subseteq V$. Note that if V is canonical then W can be chosen as canonical.

In topological groupoid case in [1] Aof first proved that if G is a paracompact topological groupoid (that is the topologies of G and O_G are paracompact) and V is an

open subset of G, such that $O_G \subseteq V$, then there exists an open subset W of G, with $O_G \subseteq W$, satisfying the following conditions.
i) $W = W^{-1}$
ii) $W^2 \subseteq V$.
Then by Paradines it was pointed out in a letter, an appendix to [1], that for such a neighbourhood W to exist the paracompactness of O_G is sufficient. Similarly if V is star canonical, then W can be chosen star canonical.

Theorem 3.2. *Let G be a star connected topological groupoid such that each fibre G_x has a universal covering. Let V be an open neighbourhood of O_G in G such that V is star canonical in G and (α, β, V) has enough continuous admissible local sections. Suppose that there exists an open neighbourhood W of O_G in G such that $W = W^{-1}, W^2 \subseteq V$ and W_x is star canonical. Then MG may be given a locally topological groupoid structure.*

Proof. First of all we note that by the above remark by choosing O_G paracompact it is possible to have such a neighbourhood W from V. Construct the groupoid MG as above. Define a map $f : W \longrightarrow MG$ as follows: Let $u \in W(x,y)$, where $W(x,y) = W \cap G(x,y)$. Then $u \in W_x$. Since W_x is path connected, there is a path a from 1_x to u. Note that $1_x \in W_x$. Define $f(u)$ to be the unique homotopy class of the path a in W_x. Since W_x is canonical, f is well defined. Then we prove the following lemmas □

Lemma 3.3. *The map $f : W \longrightarrow MG$ is injective*

Proof. Consider the composition of the maps $W \xrightarrow{f} MG \xrightarrow{p} G$, where $p : MG \longrightarrow G$ is defined by $p([a]) = a(1)$. Then $pf = i$, and i is injective. Hence $f : W \longrightarrow MG$ is injective. □ □

Lemma 3.4. *The map $f : W \longrightarrow MG$ is a local morphism*

Proof. Let $u \in W(x,y), v \in W(y,z)$ and $vu \in W$. Since $W^2 \subseteq V$ and V is star canonical we have □

$$f(vu) = f(v)f(u).$$

Hence the map $f : W \longrightarrow MG$ is a local morphism. □

Let $\overline{W}$ denote the image of W under the map $f : W \longrightarrow MG$. Hence $\overline{W}$ has a topology such that $f : W \longrightarrow \overline{W}$ is a homeomorphism. We now prove that the pair $(MG, \overline{W})$ satisfies the conditions of Definition 2.2.

i) Since W is isomorphic to $\overline{W}$, $O_{MG} = O_G$ and $O_G \subseteq W \subseteq G$, we have that $O_{MG} \subseteq \overline{W} \subseteq MG$

ii) Since $W = W^{-1}$ and W is isomorphic to $\overline{W}$, obviously $\overline{W} = (\overline{W})^{-1}$.
The main part of the proofs is to prove that $\overline{W}$ generates MG as a groupoid, that is, each element of MG can be written as a multiplication of some elements of $\overline{W}$.

iii) W generates G as a groupoid. To prove this we use a technical method.

Let $[a] \in MG(x,y)$, so that by the construction of MG, a is a path such that $a(0) = 1_x$ and $a(1) = g \in G(x,y)$. Let $S \subseteq [0,1]$ be the set of $s \in [0,1]$ such that $a^s = a|[0,s]$ can be written $a^s = a_n\mathbf{0}\cdots\mathbf{0}a_1$ for some n and $Ima_i \subseteq W$. Since $S \subseteq [0,1], S$ is bounded above by 1, and so $u = supS$ exists. Then we have to prove that
A) $u \in S$;
B) u = 1

Proof bf of A. Let $a(u) \in G(x, x_u)$, where $x_u = \beta a(u)$. Then the map $f : [0,1] \longrightarrow G_{x_u}$ defined by $t \mapsto a(t)a(u)^{-1}$ is continuous and $f(u) = 1_{x_u} \in W$. Hence there is an $\varepsilon > 0$ such that $f([u-\varepsilon, u+\varepsilon]) \subseteq W$. Hence the composition

$$\delta_W\mathbf{0}(f \times f) : [u-\varepsilon, u+\varepsilon] \times [u-\varepsilon, u+\varepsilon] \longrightarrow W \underset{\alpha}{\times} W \longrightarrow G$$

$$(t_1, t_2) \mapsto (a(t_1)a(u)^{-1}, a(t_2)a(u)^{-1}) \mapsto a(t_1)a(t_2)^{-1}$$

is continuous, where δ_W is the restriction to $W \underset{\alpha}{\times} W \longrightarrow G$ of the difference map

$G \underset{\alpha}{\times} G, (g,h) \rightarrow gh^{-1}$. Hence there is an $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon$ and

$$\delta_W(f \times f)([u-\varepsilon', u+\varepsilon'] \times [u-\varepsilon', u+\varepsilon']) \subseteq W \quad (*)$$

Since $u = supS$, there in an element $s \in S$ such that $u - \varepsilon' < S$. Hence a^s can be written $a_n \cdots a_1$ for n with $Ima_i \subseteq W$ and so we have

$$a_u = a_{n+1}\mathbf{0}(a_n\mathbf{0}\cdots\mathbf{0}a_1)$$

where $a_{n+1}(t) = a(t)a(s)^{-1}$ for $t \in [s,u]$. By $(*)$ we have that $Ima_{n+1} \subseteq W$. Hence $u \in S$.

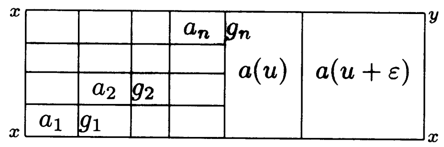
Proof of B. To prove this suppose that $u < 1$. Since $u \in S$, we have that

$$a^u = a_n\mathbf{0}\cdots\mathbf{0}a_1$$

for some n such that $Ima_i \subseteq W$. Let $a_i(1) = gi \in G(x_{i-1}, x)$ for $1 \leq i \leq n$ with $x_0 = x$ and $x_{n=y}$. Hence we have

$$a(u) = g_n\mathbf{0}\cdots\mathbf{0}g_1$$

and the path a can be divided into small paths as follows:



where $Ima_i \subseteq W$. Since the map

$$[u,1] \longrightarrow G_{x_n}, t \mapsto a(t)a(u)^{-1}$$

is continuous, there is an $\varepsilon > 0$ such that $a(t)a(u)^{-1} \in W$ for $t \in [u, u+\varepsilon]$. Hence

$$a^{u+\varepsilon} = a_{n+1}\mathbf{0}(a_n\mathbf{0}\cdots 0a_1),$$

with $a_{n+1}(t) = a(t)a(u)^{-1}$ for $t \in [u, u+\varepsilon]$.
Hence we have that $a^{u+e} \in S$, which is a contradiction. This proves that $u = 1$. This completes the proof of (B).

iv) Since G is a topological groupoid the groupoid difference map

$$G \underset{\alpha}{\times} G, (g,h) \mapsto gh^{-1}$$

is continuous, so also the restriction map $\delta_W : W \underset{\alpha}{\times} W \longrightarrow G$ is. So $W_\delta = (W \underset{\alpha}{\times} W) \cap \delta^{-1}(W)$ is open in $W \underset{\alpha}{\times} W$. Hence $\overline{W} \underset{\alpha}{\times} \overline{W}) \cap \delta^{-1}(\overline{W})$ is open in $\overline{W} \underset{\alpha}{\times} \overline{W}$ and $\overline{W} \underset{\alpha}{\times} \overline{W} \longrightarrow MG$ is continuous

v) Since $\alpha, \beta : W \longrightarrow O_G$ are continuous, so also $\alpha, \beta : \overline{W} \longrightarrow O_G$ are. Further since (α, β, W) has enough continuous admissible local sections, so also $(\alpha, \beta, \overline{W})$ is.

So $(MG, \overline{W})$ (becomes a locally topological groupoid. □

By theorem 2.3 this locally topological groupoid $(MG, \overline{W})$ gives a holonomy groupoid. In [4] it is also obtained a locally topological groupoid from a foliation.

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Yerel Topolojik Groupoidler

Özet

Referanslardan [2] de Aof and Brown tarafından yerel topolojik groupoid kavramı tanıtıldı. Diğer yandan [6] da Mackenzie tarafından monodromy groupoidi olarak adlandırılan bir MG groupoidi inşa ediliyor. Bu makalede MG groupoidinden bir yerel topolojik groupoidinin elde edildiğini ispat ediyoruz.

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