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DIFFERENTIABLE FUNCTIONS AND THE GENERATORS ON A HILBERT-LIE GROUP

Erdal Coşkun

Abstract

A convolution semigroup plays an important role in the theory of probability measure on Lie groups. The basic problem is that one wants to express a semigroup as a Lévy-Khinckine formula. If $(\mu_t)_{t\in\mathbf{R}_+^*}$ is a continuous semigroup of probability measures on a Hilbert-Lie group G, then we define

$$T_{\mu_t}f := \int f_a\mu_t(da)\ (f \in C_u(G), t > 0).$$

It is apparent that $(T_{\mu_t})_{t\in\mathbf{R}_+^*}$ is a continuous operator semigroup on the space $C_u(G)$ with the infinitesimal generator N. The generating functional A of this semigroup is defined by $Af := \lim_{t\downarrow 0} \frac{1}{t}(T_{\mu_t}f(e) - f(e))$. We have the problem of construction of a subspace $C_{(2)}(G)$ of $C_u(G)$ such that the generating functional A on $C_{(2)}(G)$ exists. This result will be used later to show that the Lévy-Khinchine formula holds for Hilbert-Lie groups.

Key words: Continuous convolution semigroup, operator semigroup, Hilbert-Lie group, Lévy measure, infinitesimal generator, generating functional

Introduction

Let $(\mu_t)_{t\in\mathbf{R}_+^*}$ be a continuous convolution semigroup of probability measures on a Hilbert-Lie group G and $C_u(G)$ the Banach space of all bounded left uniformly continuous real-valued functions on G. Then there is associated a strongly continuous semigroup $(T_{\mu_t})_{t\in\mathbf{R}_+^*}$ of contraction operators on $C_u(G)$ with the infinitesimal generator $(N, D(N))$. The generating functional $(A, D(A))$ of the convolution semigroup $(\mu_t)_{t\in\mathbf{R}_+^*}$ is defined by

$$Af := \lim_{t\downarrow 0} \frac{1}{t}(T_{\mu_t}f(e) - f(e))$$

for all f in its domain $D(A)$. For finite dimensional Lie groups, infinite dimensional Hilbert spaces and Banach spaces of cotype 2, we have

$$C_{(2)}(G) \subset D(A)$$

(cf. [4], [6] and [8] resp.). In this paper we shall prove that the above result is also true for a class of infinite dimensional Hilbert-Lie groups. At several points we shall use ideas and techniques used in [4]. We first obtain the Taylor expansion for the functions $f \in C_{(2)}(G)$. In Lemma 2.1 we prove that, for every neighborhood of e in any Hilbert-Lie group G, the supremum $\sup_{t>0} \frac{1}{t}\mu_t(U^c)$ is finite. Using this result and Banach-Steinhaus Theorem, we prove Theorem 2.9.

1. Preliminaries

$\mathbf{N}$ and $\mathbf{R}$ denote the sets of positive integers and real numbers, respectively. Moreover let $\mathbf{R}_+ := \{r : r \geq 0\}, \mathbf{R}_+^* := \{r : r > 0\}$.

Let A be a set and B a subset of A. Then by 1_B we denote the indicator function of B. Let I be a nonvoid set. δ_{ij} is the Kronecker delta $(i, j \in I)$.

By G we denote a topological Hausdorff group with identity e. G is called Polish group, if G is a topological group with a countable basis of its topology and with a complete left invariant metric d which induces the topology.

For every function $f : G \to \mathbf{R}$ and $a \in G$ the functions $f^*, R_a f = f_a$ and $L_a f = {}_a f$ are defined by $f^*(b) = f(b^{-1})$, $f_a(b) = f(ba)$ and ${}_a f(b) = f(ab)$ for all $b \in G$, respectively. Moreover let $supp(f) = \overline{\{a \in G : f(a) \neq 0\}}$ denote the support of f. By $C_u(G)$ we denote the Banach space of all real-valued bounded left uniformly (or d-uniformly) continuous functions on G furnished with the supremum norm $\|\cdot\|$. A Hilbert-Lie group is a separable analytic manifold modeled on a separable Hilbert space, whose group operations are analytic. It is we known that the Hilbert-Lie groups are Polish (cf. [2]).

For the exponential mapping $\mathcal{E}xp : T_c \to G$ there exists an inverse mapping log from a neighborhood U_e of e onto a neighborhood N_0 of zero in T_e, where T_e is the tangential space in $e \in G$ ([5]).

By $\mathcal{B}(G)$ we denote the σ-field of Borel subsets of G. Moreover, $\mathcal{V}(e)$ denotes the system of neighborhoods of the identity e of G which are in $\mathcal{B}(G)$.

$\mathcal{M}(G)$ denotes the vector space of real-valued (signed) measures on $\mathcal{B}(G)$. As it is well known, $\mathcal{M}(G)$ is a Banach algebra with respect to convolution $*$ and the norm $\|\cdot\|$ of total variation. $M_+(G)$ is the set of positive measures in $\mathcal{M}(G)$ and $\mathcal{M}^1(G) = \{\mu \in \mathcal{M}_+(G) : \mu(G) = 1\}$ is the set of probability measures on G.

Now let $\gamma_X(t) := \mathcal{E}xp(tX)$ for $X \in H$ and $t \in \mathbf{R}^* := \mathbf{R} \setminus \{0\}$.

Definition 1.1 *Let $f \in C_u(G), X \in H$ and $a \in G$.*

f is called left differentiable at $a \in G$ with respect to X ("$Xf(a)$ exists" for short), if

$$Xf(a) := \lim_{t\to 0} \frac{1}{t}[L_{\gamma_X(t)}f(a) - f(a)]$$

exists. f *is called continuously left differentiable, if* $Xf(a)$ *exists for all* $a \in G$ *and* $X \in H$, *and if the mappings* $a \longmapsto Xf(a), X \longmapsto Xf(a)$ *are continuous.*

Derivatives of higher orders are defined inductively. Differentiability from the right is defined in replacing $L_{\gamma_X(t)}$ by $R_{\gamma_X(t)}$.

The following properties of the derivatives are well known for continuously left differentiable functions (cf. [1]).

Remark 1.2 Let $f, g \in C_u(G), X \in H$ and $a \in G$.

(i) If $Xf(a)$ exists, then the mapping $X \longmapsto Xf(a)$ is linear.

(ii) If $Xf(a)$ and $Xg(a)$ exists, then also $X(f\cdot g)(a)$ exists and $X(f\cdot g)(a) = Xf(a)\cdot g(a) + f(a)\cdot Xg(a)$.

Now let $f \in C_u(G)$ be twice continuously left differentiable function. Then the mapping

$$Df(a) : X \longmapsto Xf(a) \quad (D^2f(a) : (X,Y) \longmapsto XYf(a))$$

is continuous and linear (resp. symmetric, continuous and bilinear) functional on H (resp. $H \times H$) for all $a \in G$. There hold

$$< Df(a), X >= Xf(a) \text{ and } < D^2f(a)(X), Y >= XYf(a)$$

for all $a \in G$ and $X, Y \in H$.

We define by $C_2(G)$ the space of all twice continuously left differentiable functions $f \in C_u(G)$ such that the mapping $a \longmapsto D^2f(a)$ is d-uniformly continuous and $\|Df\| := \sup_{a\in G}\|Df(a)\| < \infty$, $\|D^2f\| := \sup_{a\in G}\|D^2f(a)\| < \infty$. It is easy to see that the space $C_2(G)$ is a Banach space with respect to the norm

$$\|f\|_2 := \|f\| + \|Df\| + \|D^2f\|, \quad f \in C_2(G)$$

and

$$R_aC_2(G) \subset C_2(G)$$

is satisfied for all $a \in G$. However $C_2(G)$ is not dense in $C_u(G)$ (cf. [6].) By $a_i(a) :=< \log(a), X_i > (i \in \mathbf{N})$ we define maps a_i from the canonical neighborhood U_e in $\mathbf{R}$. Now we call the system $(a_i)_{i\in\mathbf{N}}$ of maps from U_e in $\mathbf{R}$ a system of canonical coordinates of G with respect to the orthonormal basis $(X_i)_{i\in\mathbf{N}}$, if for all $a \in U_e$ the property $a = \mathcal{E}xp(\sum_{i=1}^{\infty} a_i(a)X_i)$ is satisfied.

Lemma 1.3 *Let $f \in C_2(G)$. Then*

(i) $(\sum_{i=1}^{\infty} a_i(a)X_i)f = \sum_{i=1}^{\infty} a_i(a)X_i f$ *for all* $a \in U_e$.

(ii) $(\sum_{i=1}^{\infty} a_i(a)X_i)((\sum_{j=1}^{\infty} a_j(c)X_i)f) = \sum_{i=1,j=1}^{\infty} a_i(a)a_j(c)X_iX_jf$ *for all* $a, c \in U_e$.

Proof. (i) For any $a \in U_e$ there exists an $X \in H$ with $X = \log(a)$. Then we have $X = \sum_{i=1}^{\infty} < X, X_i > X_i = \sum_{i=1}^{\infty} a_i(a)X_i$. Thus

$$\begin{aligned} Xf(e) &= \frac{d}{dt}|_{t=0} f(\gamma_X(t)) =< Df(e), X > \\ &= \sum_{i=1}^{\infty} a_i(a) < Df(e), X_i >= \sum_{i=1}^{\infty} a_i(a)X_i f(e). \end{aligned}$$

Now let $b \in G$ be an arbitrary point. Then $R_b f \in C_2(G)$, whence the assertion. The proof of (ii) can be carried out similarly. □

In the following we give the Taylor expansion for the functions $f \in C_2(G)$.

Proposition 1.4 *Let $f \in C_2(G)$. Then the Taylor-expanison of the second order for f at $e \in G$ is given by*

$$f(a) = f(e) + \sum_{i=1}^{\infty} a_i(a)X_i f(e) + \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i(a)a_j(a)X_iX_jf(\bar{a})$$

for all $a \in U_e$, where $\bar{a}$ is a point of U_e.

Proof. Let $f \in C_2(G)$ and $X \in H$. Then the function $\xi : t \longmapsto f(\gamma_X(t))$ is twice differentiable on $\mathbf{R}$ and therefore admits a Taylor-expansion valid up to the second order:

$$\xi(t) = \xi(0) + \xi'(0) \cdot t + \frac{1}{2}\xi''(\bar{t}) \cdot t^2$$

for some $\bar{t} \in [-|t|, |t|]$. Since $\xi'(0) = Xf(e)$ and $\xi''(\bar{t}) = XXf(\gamma_X(\bar{t}))$, it follows from Lemma 1.3 that

$$\begin{aligned} f(\gamma_X(t)) &= f(e) + \sum_{i=1}^{\infty} < tX, X_i > X_i f(e) \\ &+ \frac{1}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} < tX, X_i >< tX, X_i > X_iX_jf(\gamma_X(\bar{t})) \end{aligned}$$

for some $\bar{t} \in [-|t|, |t|]$. This yields the assertion. □

Remark 1.5 The Taylor-expansion of $f \in C_2(G)$ can be written in a closed form, i.e.

$$f(a) = f(e) + < Df(e), \log(a) > + \frac{1}{2} < D^2 f(\bar{a})(\log(a)), \log(a) >$$

for all $a \in U_e$ and for some $\bar{a}$ in the canonical neighborhood U_e.

2. Convolution Semigroups of Probability Measures and the Generators

For any probability measure μ on G, we define the operator T_μ on $C_u(G)$ by

$$T_\mu f := \int f_a \mu(da) \qquad \text{(Bochner-Integral)}.$$

It is easy to see that $T_\mu C_u(G) \subset C_u(G)$ and $T_{\mu * \nu} = T_\mu \circ T_\nu$.

A *convolution semigroup* is a family $(\mu_t)_{t \in \mathbf{R}_+^*}$ in $\mathcal{M}^1(G)$ such that $\mu_0 = \varepsilon_e$ and $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \in \mathbf{R}_+^*$.

$(\mu_t)_{t \in \mathbf{R}_+^*}$ is called *continuous* if $\lim_{t \longmapsto 0} \mu_t = \varepsilon_e$ (weakly). It is well known that the convolution semigroup $(\mu_t)_{t \in \mathbf{R}_+^*}$ is continuous iff the corresponding operator semigroup $(T_{\mu_t})_{t \in \mathbf{R}_+^*}$ is (strongly) continuous. The Hille-Yosida theory establishes a bijection between (strongly) continuous operator semigroups $(T_{\mu_t})_{t \in \mathbf{R}_+^*}$ and their infinitesimal generators. N is defined on its domain $D(N)$ which is dense in $C_u(G)$. It is clear that N commutes with the left translations, i.e.

$$L_a D(N) \subset D(N) \text{ and } L_a \circ N = N \circ L_a \text{ for all } a \in G.$$

A continuous convolution semigroup $(\mu_t)_{t \in \mathbf{R}_+^*}$ in $\mathcal{M}^1(G)$ admits a Lévy measure η, i.e η is a σ-finite positive measure on $\mathcal{B}(G)$ such that $\eta(\{e\}) = 0$ and such that

$$\lim_{t \downarrow 0} \frac{1}{t} \int f d\mu_t = \int f d\eta,$$

for all $f \in C_u(G)$ with $e \notin supp(f)$ (cf. [7]).

Lemma 2.1 *Let* $(\mu_t)_{t \in I\!\!R_+^*}$ *be a continuous convolution semigroup in* $\mathcal{M}^1(G)$. *Then for every* $U \in \mathcal{V}(e)$

$$\sup_{t \in I\!\!R_+^*} \frac{1}{t} \mu_t(U^c) < \infty.$$

Proof. Let U and V be two neighborhoods of $e \in G$ with $\overline{V} \subset U$. Since G is a normal group, there exists a function $f \in C_u(G)$ such that

$$0 \leq f \leq 1, \;\; f(V) = \{0\} \;\; \text{and} \;\; f(U^c) = \{1\}.$$

Then we have $\frac{1}{t}\mu_t(U^c) \leq \frac{1}{t}\int f d\mu_t$ for all $t \in \mathbf{R}_+^*$. $f \in C_u(G)$ with $e \notin supp(f)$ implies that

$$\lim_{t \downarrow 0} \frac{1}{t} \int f d\mu_t = \int f d\eta < \infty.$$

Hence the assertion. □

Let H be a separable Hilbert space with a complete orthonormal system $(X_i)_{i \in \mathbf{N}}$ and G a Hilbert-Lie group on H. Moreover, let

$$H_n := < \{x_1, X_2, \ldots, X_n\} >$$

be the space of all linear combinations of $X_1, X_2, \ldots, X_n$ and $H_n^\perp$ the orthogonal complement of H_n in H (for all $n \in \mathbf{N}$). Then $H/H_n^\perp$ and H_n are isomorphic. Clearly

$$G_n := \mathcal{E}xp(H_n^\perp)$$

is a closed subgroup of G for all $n \in \mathbf{N}$. The quotient spaces G/G_n are finite-dimensional Hilbert-Lie groups. Now let p_n be the canonical projection from G onto G/G_n and $\{b_i^n : i = 1, 2, \ldots, n\}$ a system of canonical coordinates with respect to $\{X_1, X_2, \ldots, X_n\}$. We now define the functions $d_i^n := b_i^n \circ p_n \in C_2(G)$; then $X_j d_i^n$ exist and

$$X_j d_i^n = X_j(b_i^n \circ p_n) = X_j b_i^n \circ p_n = 0$$

hold for all $j > n$ and $i = 1, 2, \ldots, n$.

Definition 2.2 *Let G be a Hilbert-Lie group on H, and $(X_i)_{i \in I\!R}$ an orthonormal basis in H. For any $n \in I\!N$ we define*

$$\begin{aligned} C_{(2),n}(G) := \{f \in C_2(G) \; &: \; X_i f = 0 \textit{ for all } i > n \textit{ and} \\ &: \; X_i X_j f = 0 \textit{ for all } i > n \textit{ or } j > n\}. \end{aligned}$$

Remark 2.3 Let $f \in C_u(G)$ be a left uniformly differentiable function with respect to X which satisfies $X_i f = 0$ for all $i > n \, (n \in \mathbf{N})$. Let π_n be the orthogonal projection from H onto H_n. Then we have

$$Xf = \pi_n(X) f \text{ for all } X \in H.$$

Hence f is continuously left differentiable and clearly $(C_{(2),n}(G))_{n\in\mathbf{N}}$ is a strictly increasing sequence of Banach subalgebra of Banach algebra $C_2(G)$.
Further properties of $C_{(2),n}(G)(n \in \mathbf{N})$
(i) $C_{(2),n}(G)$ are $\|\cdot\|_2$-closed in $C_2(G)$
and
(ii) For any probability measure $\mu \in \mathcal{M}^1(G)$, we have

$$T_\mu C_{(2),n}(G) \subset C_{(2),n}(G) \text{ for all } n \in \mathbf{N}.$$

Thus $\overline{C_{(2),n}(G) \cap D(N)}^{\|\cdot\|_2} = C_{(2),n}(G)$. Now consider the subspace

$$C_{(2)}(G) := \bigcup_{n\in\mathbf{N}} C_{(2),n}(G).$$

$C_{(2)}(G)$ is obviously an linear subspace of $C_2(G)$ with $T_\mu C_{(2)}(G) \subset C_{(2)}(G)$ for probability measures $\mu \in \mathcal{M}^1(G)$. Especially $\overline{C_{(2)}(G)}^{\|\cdot\|_2}$ is a Banach space with $\overline{T_\mu C_{(2)}(G)}^{\|\cdot\|_2} \subset \overline{C_{(2)}(G)}^{\|\cdot\|_2}$.

Definition 2.4 *For $n \in I\!N$ let $\{b_i^n : i = 1, 2, \ldots, n\}$ be a system of extended cononical coordinates with respect to $\{X_1, X_2, \ldots, X_n\}$. Then we say that the Hilbert-Lie group G has the property (K), if*

$$b_i^n \in C_{(2)}(G) \textit{ for all } i = 1, 2, \ldots, n,\ n \geq n_0$$

and for any $n_0 \in I\!N$.

Every commutative Hilbert-Lie group and every finite dimensional Lie group have clearly the property (K). In the finite dimensional case we have $n_0 = dim(G)$. Since $C_{(2),n}(G) \subset C_{(2),n+1}(G)$, a system $\{b_i^n, b_{n+1}^{n+1} : i = 1, 2, \ldots, n\} \subset C_{(2),n+1}(G)$ of canonical coordinates exists with respect to $\{X_1, X_2, \ldots, X_{n+1}\}$. We also have the following Proposition:

Proposition 2.5 *Let G be a Hilbert-Lie group with the property (K). Then a system $(d_n)_{n\in I\!N}$ of functions in $C_{(2)}(G)$ exists with*

$$d_i = b_i^{n_0} \textit{ for all } i = 1, 2, \ldots, n_0$$

and

$$d_n = b_n^n \textit{ for all } n > n_0.$$

This system $(d_n)_{n\in\mathbf{N}}$ is called a system of local canonical coordinates with respect to $(X_i)_{i\in\mathbf{N}}$.

Now let G be a Hilbert-Lie group with the property (K). We define for any $n \in \mathbf{N}$ the functions

$$\Phi_n(a) := \sum_{i=1}^{n} d_i(a)^2, \quad a \in G,$$

where $(d_i)_{i=1,2,\ldots,n}$ is a system of local canonical coordinates with respect to $\{X_1, X_2, \ldots, X_n$. Then $\Phi_n \in C_{(2),n}(G)$ and $\Phi_n(a) > 0$ for all $a \in G \setminus \{\Phi_n = 0\}$. Therefore

$$X_i\Phi_n(e) = 0, \quad X_iX_j\Phi_n(e) = 2\delta_{ij}, \quad i,j = 1, 2 \ldots, n$$

(cf. [3], Lemma 4.1.9 and 4.1.10).

Remark 2.6 (a) For $f \in C_{(2),n}(G), n \in \mathbf{N}$ and $i,j = 1, 2, \ldots, n$ we denote the numbers $X_if(e)$ and $X_iX_jf(e)$ by A_if and $A_{ij}f$, resp. Obviously $f \mapsto A_if$ and $f \mapsto A_{ij}f$ are continuous linear functionals on $C_{(2),n}(G)$ for $i,j = 1, 2, \ldots, n$.

(b) Let E be a locally convex vector space and E_1 a dense subspace of E. Moreover, let F be a subspace of E of finite condimension, $y \in E$ and $M := y + F$. Then $M_1 := M \cap E_1$ is dense in M ([3], Lemma 4.1.11).

Lemma 2.7 *For every $f \in C_{(2),n}(G)$ and every $\varepsilon > 0$ there exists a $g := g_e \in C_{(2),n}(G) \cap D(N)$ such that $\|f - g\|_2 < \varepsilon$, $f(e) = g(e), X_if(e) = X_ig(e)$ and $X_iX_j(e) = X_iX_jg(e)$ for $i,j = 1, 2, \ldots, n$.*

Proof. Let K_n be a map from $C_{(2),n}(G)$ to $\ell^2(n^2)$ with

$$f \longmapsto K_n(f) := (X_iX_jf(e))_{i,j=1,2,\ldots,n} = (A_{ij}f)_{i,j,=1,2,\ldots,n}, \; n \in \mathbf{N}.$$

Then K_n is linear and continuous, where $\ell^2(n)$ is a finite-dimensional subspace of the Hilbert space ℓ^2.

Similary, let L_n be a continuous linear map from $C_{(2),n}(G)$ to $\ell^2(n+1)$ with

$$f \longmapsto L_n(f) := (f(e), X_1f(e), \ldots, X_nf(e)) = (f(e), A_1f, \ldots, A_nf).$$

Moreover, let

$$F := Kern(L_n) \cap Kern(K_n),$$

then F is a closed subspace of $C_{(2),n}(G)$ of finite condimension. From Remark 2.6 b)

$$\overline{[f+F] \cap [C_{(2),n}(G) \cap D(N)]}^{\|\cdot\|_2} = f + F$$

the assertions follow. □

Proposition 2.8 *Let G be a Hilbert-Lie group with the property (K), $(\mu_t)_{t\in \mathbb{R}_+^*}$ a convolution semigroup in $\mathcal{M}^1(G)$ and $\Phi_n (n \in \mathbb{N})$ be as above. Then the suprema*

$$\sup_{t\in \mathbb{R}_+^*} \frac{1}{t} \int \Phi_n d\mu_t$$

are finite for every $n \in \mathbb{N}$.

Proof. Application of Lemma 2.7 to the function $\Phi_n \in C_{(2),n}(G)$ yields the existence of a function $\Psi_n \in C_{(2),n}(G) \cap D(N)$ with the property

$$\begin{aligned} \|\Phi_n - \Psi_n\|_2 < \varepsilon, \Psi_n(e) &= \Phi_n(e) = 0,\ X_i\Psi_n(e) = X_i\Phi_n(e) = 0 \\ \text{and } X_iX_j\Psi_n(e) &= X_iX_j\Phi_n(e) = 2\delta_{ij},\ i,j = 1,2,\dots,n. \end{aligned}$$

Taylor expansion of $\Psi_n \in C_{(2),n}(G) \cap D(N)$ in a neighborhood W_1 of e with $W_1 \subset U_e$ gives

$$\Psi_n(a) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_i(a)d_j(a)X_iX_j\Psi_n(\bar{a}),$$

for all $a \in W_1$ and for some $\bar{a} \in W_1$. Since $\|\Phi_n - \Psi_n\|_2 < \varepsilon$ and $X_iX_j\Psi_n(e) = 2\delta_{ij}, i, j = 1, 2, \dots, n$ there exists a neighborhood W_2 of e with the properties

$$-\varepsilon \le X_iX_j\Psi_n(a) \le \varepsilon \text{ for all } i,j = 1,2,\dots,n,\ i \ne j,$$

$$2 - \varepsilon \le X_iX_j\Psi_n(a) \le 2 + \varepsilon \text{ for all } i = 1,2,\dots,n,$$

whenever $a \in W_2$. Putting $\delta_n := \delta_n(e) := \frac{1}{2}(2 - \varepsilon - \varepsilon(n-1))$ and $W := W_1 \cap W_2$, we obtain

$$\Psi_n(a) \ge \delta_n \cdot \sum_{i=1}^{n} d_i(a)^2 \ \text{ for all } \ a \in W.$$

Since $\Psi_n \in C_{(2),n}(G) \cap D(N)$, we obtain $\sup_{t\in \mathbf{R}_+^*} \frac{1}{t}|\int_W \Psi_n d\mu_t| < \infty$ from Lemma 2.1. Thus $\sup_{t\in \mathbf{R}_+^*} \frac{1}{t}\int_W \Phi_n d\mu_t < \infty$, and since Φ_n is bounded, the assertion follows from Lemma 2.1. □

Now let G be a Hilbert-Lie group with the property (K) and $(d_i)_{i\in \mathbf{N}}$ a system of local canonical coordinates with respect to $(X_i)_{i\in \mathbf{N}}$. By Lemma 2.7 there exist functions $z_i \in C_{(2),n}(G) \cap D(N), (n \in \mathbf{N})$ with the property

$$z_i(e) = d_i(e) = 0,\ X_jz_i(e) = X_jd_i(e) = \delta_{ij},\ i,j = 1,2\dots,n.$$

Theorem 2.9 *Let G be a Hilbert-Lie group with the property (K) and $(\mu_t)_{t\in \mathbb{R}_+^*}$ a convolution semigroup in $\mathcal{M}^1(G)$. Then the generating functional A of $(\mu_t)_{t\in \mathbb{R}_+^*}$ on $C_{(2)}(G)$ exists, i.e.*

$$C_{(2)}(G) \subset D(A).$$

Proof. Let $f \in C_{(2),n}(G)$ $(n \in \mathbf{N})$ and set

$$g(a) := f(a) - f(e) - \sum_{i=1}^{n} z_i(a) \cdot X_i f(e) \text{ for all } a \in G,$$

where the functions $z_i, i = 1, 2, \ldots, n$ are as above. Then $g \in C_{(2),n}(G)$ with $g(e) = 0, X_j g(e) = X_j f(e) - \sum_{i=1}^{n} X_j z_i(e) \cdot X_i f(e) = X_j f(e) - \sum_{i=1}^{n} \delta_{ij} \cdot X_i f(e) = 0$. The Taylor expansion of g in a neighborhood $W \subset U_e$ gives

$$g(a) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_i(a) d_j(a) X_i X_j g(\bar{a}), \quad a \in W.$$

Thus there is a constant $k_1 \in \mathbf{R}_+^*$ such that

$$|g(a)| \leq k_1 \cdot \|g\|_2 \cdot \Phi_n(a) \text{ for all } a \in W.$$

It follows from Proposition 2.8 that

$$\sup_{t\in \mathbf{R}_+^*} |\frac{1}{t} \int_W g d\mu_t| \leq k_1 \cdot \|g\|_2 \cdot \sup_{t\in \mathbf{R}_+^*} \int \Phi_n d\mu_t < \infty. \tag{1}$$

Clearly, $|\frac{1}{t} \int_{W^c} g d\mu_t| \leq \|g\|_2 \cdot \frac{1}{t}\mu_t(W^c)$, and $\sup_{t\in \mathbf{R}_+^*} |\frac{1}{t} \int_{W^c} g d\mu_t| < \infty$. Hence, there exists a constant $k_2 \in \mathbf{R}_+^*$ independent of t such that

$$|\frac{1}{t} \int_{W^c} g d\mu_t| \leq k_2 \cdot \|g\|_2 \text{ for all } t \in \mathbf{R}_+^*. \tag{2}$$

Adding the inequalities (1) and (2) we get

$$|\frac{1}{t}[T_{\mu_t} f(e) - f(e)] - \frac{1}{t} \sum_{i=1}^{n} X_i f(e) \cdot T_{\mu_t} z_i(e)| \leq k_3 \cdot \|f\|_2, \text{ for all } t \in \mathbf{R}_+^*.$$

where k_3 is a constant (independent of t). Since $z_i \in D(N)$ and $z_i(e) = 0$, we have $\sup_{t\in \mathbf{R}_+^*} |\frac{1}{t} T_{\mu_t} z_i(e)| < \infty$ for all $i = 1, 2 \ldots, n$.

Hence we obtain a constant $k(n) \in \mathbf{R}_+^*$ depending only on n such that

$$|\frac{1}{t}(T_{\mu_t}f(e) - f(e))| \leq k(n) \cdot \|f\|_2$$

for all $t \in \mathbf{R}_+^*$ and $f \in C_{(2),n}(G)$. By the Banach-Steinhaus Theorem the limit

$$\lim_{t\downarrow 0} \frac{1}{t}[T_{\mu_t}f(e) - f(e)]$$

exists for every $f \in C_{(2)}(G)$. □

Remark 2.10 Let G be *commutative* Hilbert-Lie group and $(\mu_t)_{t\in\mathbf{R}_+^*}$ a convolution semigroup in $\mathcal{M}^1(G)$. As in the proof of Theorem 2.9, we can find a constant $k(n) \in \mathbf{R}_+^*$ (independent of $a \in G$ and $t \in \mathbf{R}_+^*$) such that

$$\begin{aligned} |\frac{1}{t}[T_{\mu_t}f(a) - f(a)]| &= |\frac{1}{t}[T_{\mu_t}(L_a f)(e) - (L_a f)(e)]| \\ &\leq k(n) \cdot \|L_a f\|_2 = k(n) \cdot \|f\|_2 \end{aligned}$$

for all $f \in C_{(2),n}(G)$ and $a \in G$. The Banach-Steinhaus Theorem now yields the existence of the limit

$$Nf(a) = \lim_{t\downarrow 0} \frac{1}{t}[T_{\mu_t}f(a) - f(a)]$$

uniformly in $a \in G$. This implies existence of the infinitesimal generator N on $C_{(2)}(G)$.

Remark 2.11 Let $G = H$ be a separable Hilbert space and $C_u^{(2)}(H)$ the space of all twice Fréchet differentiable functions $f \in C_u(H)$ such that $\|f'\| := \sup_{x\in H} \|f'(x)\| < \infty$, $\|f''\| := \sup_{x\in H} \|f''(x)\| < \infty$ and f'' is uniformly continuous in x. Then we have $C_u^{(2)}(H) \subset D(N)$ (cf. [6]) and $C_2(H) = C_u^{(2)}(H)$.

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Hilbert-Lie Grubu Üzerinde Diferensiyellenebilir Fonksiyonlar ve Generatörler

Özet

Lie gruplarında olasılık ölçümü teorisinde, konvolüsyon yarıgrupları önemli rol oynamaktadır. Temel problem, yarıgrubu Lévy-Khinchine formülü olarak ifade etmektir. Hilbert-Lie grubu G üzerinde olasılık ölçümlerinin sürekli bir yarıgrubu $(\mu_t)_{t\in\mathbf{R}_+^*}$ ise,

$$T_{\mu_t} f := \int f_a \mu_t(da)(f \in C_u(G), t > 0).$$

ile $C_u(G)$ uzayı üzerinde N infinitezimal generatörüne sahip sürekli operatör yarıgrubu $(T_{\mu_t})_{t\in\mathbf{R}_+^*}$ tanımlanır. Bu yarıgrup için doğurucu fonksiyonel A, $Af := \lim_{t\downarrow 0} \frac{1}{t}(T_{\mu_t} f(e) - f(e))$ biçiminde tanımlanır. Buna göre problem, A doğurucu fonksiyonelinin tanımlı olacağı $C_u(G)$ nin bir $C_{(2)}(G)$ alt uzayını oluşturmaktadır. Bu sonuç, daha sonra Hilbert-Lie gruplarında Lévy-Khinchine formülünün elde edilmesinde kullanılacaktır.

Erdal COŞKUN
Hacettepe Üniversitesi,
Eğitim Fakültesi, Fen Bilimleri Bölümü,
06532 Beytepe, Ankara-TURKEY
E-mail: coskun@eti.cc.hun.edu.tr

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