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THE HARDY-LITTLEWOOD-SOBOLEV INEQUALITY FOR NON-ISOTROPIC RIESZ POTENTIALS

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Abstract

In this study the inequality of Hardy-Littlewood-Sobolev type are established for non-isotropic generalized Riesz potential depending on λ -distance.

In this paper we establish analogues well known Hardy-Littlewood-Sobolev inequality (see[3]) for Riesz potentials with non-isotropic kernel depended on λ-distance. Note that different problems for convolution type integrals with kernels, depending on λ-distance were considered in [1] and [2].

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$ and $||x||_\lambda = \left(|x_1|^{\frac{1}{\lambda_1}} + \cdots |x_n|^{\frac{1}{\lambda_n}} \right)^{\frac{|\lambda|}{n}}, x \in R^n$. The expression $||x-y||_\lambda, x, y \in R^n$ is called the λ-distance between the points x and y . It can be seen that for $\lambda_j = \frac{1}{2}$ j=1,2,...,n the λ-distance become ordinary Euclidean distance $|x-y|$.

Now we define the generalized non-isotropic Riesz potentials. We set

$$\Lambda_\alpha f(x) = \int_{R^n} ||x-y||_\lambda^{\alpha-n} f(y)dy \quad 0 < \alpha < n \tag{1}$$

This integral is called non-isotropic or λ -Riesz potential. We show that this potential has a weak (p,q)- type for some p and q in the following sense [3]: There exsist a positive constant $C_{p,q}$ independent on function f such that for any $\beta > 0$ the inequality

$$m\{x : |\Lambda_\alpha f(x)| > \beta\} \le \left(C_{p,q} \frac{||f||_p}{\beta} \right)^q \tag{2}$$

is hold. The following theorem is our main result.

Theorem (Hardy-Littlewood-Sobolev type). Let

$$1 \le p < q < \infty \text{ and } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \tag{3}$$

a) If $f \in L_p(R^n)$, then the integral $\Lambda_\alpha f$ is absolutely convergent almost everywhere.

b) If $p>1$, then

$$||\Lambda_\alpha f||_{L_q} \le A_{p,q}||f||_{L_p} \tag{4}$$

c) If $f\in L_1(R^n)$ then potential $\Lambda_\alpha f$ has weak $(1,q)$ type, where $q=1-\frac{\alpha}{n}$.

Proof. a) it is obvious that

$$\begin{aligned}\Lambda_\alpha f(x) &= \int_{R^n} f(x-y)R_1(y)dy+\int_{R^n} f(x-y)R_\infty(y)dy\\ &= I_1(x)+I_2(x)\end{aligned} \tag{5}$$

where for some positive μ

$$R_1(y)=\begin{cases}||y||_\lambda^{\alpha-n} & \text{if} \quad ||y||_\lambda \le\mu\\ 0 & \text{if} \quad ||y||_\lambda >\mu\end{cases}$$

$$R_\infty(y)=\begin{cases}0 & \text{if} \quad ||y||_\lambda \le\mu\\ ||y||_\lambda^{\alpha-n} & \text{if} \quad ||y||_\lambda >\mu\end{cases}$$

Applying the generalized Minkowsky inequality we can see that

$$||I_1||_{L_p} \le ||R_1||_{L_1}||f||_{L_p}$$

Note that the norm $||R_1||_{L_1}$ can be easy calculated. Namely passing to spherical coordinates by transformation $x_j=(\gamma\theta_j)^{\lambda_j}, j=1,2,\cdots,n$, where θ_j are the coordinates of the point θ on unit sphere, we can see that the jacobian of this transformation is $\gamma^{2|\lambda|-1}\Omega(\theta)$ and $\Omega(\theta)$ depend only on angles. Consequently, denoting $C_1=\int\limits_{S^{n-1}}\Omega(\theta)d\theta$, where S^{n-1} is the unite sphere in R^n, we obtain

$$||R_1||_{L_1}=\int\limits_{||y||_\lambda\le\mu} ||y||_\lambda^{\alpha-n}dy \le C_1\mu^{\frac{2|\lambda|}{n}\alpha}. \tag{6}$$

Therefore this it follows that

$$||I_1||_{L_p}\le C_1\mu^{\frac{2|\lambda|}{n}\alpha}||f||_{L_p}<\infty \tag{7}$$

By Hödler inequality, we have

$$|I_2(x)| \le ||R_\infty||_{L_{p'}}||f||_{L_p} \tag{8}$$

where $p' = \frac{p}{p-1}$

Now we will show that $||R_\infty||_{L_{p'}}$ is finite. We have

$$||R_\infty||_{L_{p'}} = \left(\int\limits_{||y||_\lambda > \mu} ||y||_\lambda^{(\alpha-n)p'} dy \right)^{\frac{1}{p'}} = C_2 \left(\rho^{\frac{2|\lambda|}{n}(\alpha-n)p'+2|\lambda|} \right)^{\frac{1}{p'}} \Bigg|_\mu^\infty .$$

By (3) it follows that

$$\frac{2|\lambda|}{n}(\alpha - n) + \frac{2|\lambda|}{p'} = 2|\lambda|(\frac{\alpha}{n} - \frac{1}{p}) < 0$$

and that is, $||R_\infty||_{L_{p'}} < C_3$ and $I_2(x)$ is also finite. Due to (5) and (7), $\lambda-$ Riesz potentials is finite almost everywhere.

We now prove the part c)- Obviously, it is sufficient to prove this fact in case $||f||_{L_p} = 1$ and with 2β replaced β in (2).

Since $\Lambda_\alpha(x) = I_1(x) + I_2(x)$ in view of (5) we have the inequality

$$m\{x : |\Lambda_\alpha f(x)| > 2\beta\} \leq m\{x : |I_1(x)| > \beta\} + m\{x : |I_2(x)| > \beta\}. \tag{9}$$

Consider the right side of this inequality. We have

$$\beta^p m\{x : |I_1(x)| > \beta\} \leq \int\limits_{\{x:|I_1(x)|>\beta\}} I_1(x)|^p dx \leq \int\limits_{R^n} |I_1(x)|^p dx = ||I_1||_{L_p}^p$$

and therefore

$$m\{x : |I_1(x)| > \beta\} \leq \frac{||I_1||_{L_p}^p}{\beta^p}.$$

Since

$$||I_1||_{L_p} \leq ||R_1||_{L_1}||f||_{L_p} = ||R_1||_{L_1}$$

and by (6) the inequality

$$m\{x : |I_1(x)| > \beta\} \leq \frac{||R_1||_{L_1}}{\beta^p} \leq C_1 \frac{\mu^{\frac{2|\lambda|\alpha}{n}}}{\beta^p} \tag{10}$$

holds.

Now by (8) we have the inequality

$$|I_2(x)| \leq ||R_\infty||_{L_{p'}} ||f||_{L_p} = C_3 \mu^{\frac{2|\lambda|}{n}(\alpha - n) + \frac{2|\lambda|}{p'}} = C_3 \mu^{-\frac{2|\lambda|}{q}}$$

so it follows that if we choose

$$\mu = (C_3^{-1}\beta)^{-\frac{q}{2|\lambda|}}$$

then for all x $|I_2(x)| \leq \beta$ and $m\{x : |I_2(x)| > \beta\} = 0$.
By (10), we have

$$m\{x : |\Lambda_\alpha f(x)| > 2\beta\} < C_1 \left(\frac{\mu^{\frac{2|\lambda|}{n}\alpha}}{\beta}\right)^p = C_1 C_4 \left(\mu^{\frac{2|\lambda|}{n}\alpha + \frac{2|\lambda|}{q}}\right)^p$$

$$= C_5 \left(\mu^{2|\lambda|\left(\frac{\alpha}{n} - \frac{1}{q}\right)}\right)^p = C_5 \left(\mu^{2|\lambda|\frac{1}{p}}\right)^p = C_5 \mu^{2|\lambda|} = C_5 C_6 \beta^{-q} = C_7 \left(\frac{||f||_{L_P}}{\beta}\right)^q.$$

Consequently under condition $(3)\Lambda_\alpha f(x)$ has a weak (p,q) type.

b)- To prove this part we use the Marcinkiewicz Interpolation theorem [3]. By part (c) the operator $\Lambda_\alpha f(x)$ is the weak type (p,q) where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. In special case $p = 1$ this operator is the weak type $(1,q)$ where $\frac{1}{q} = 1 - \frac{\alpha}{n}$. Using the Marcinkiewicz Interpolation theorem between (p_0, q_0) and (p_1, q_1) where $p_0 = 1, q_0 = \left(1 - \frac{\alpha}{n}\right)^{-1}$ $p_1 = p_1, q_1 = \left(\frac{1}{p_1} - \frac{\alpha}{n}\right)^{-1}$. We have that for potential $\Lambda_\alpha f(x)$ hods (4) and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The proof is completed. □

Remark. The condition (3) is also the necessary for (4). To prove this we assume that (4) holds for every function $f \in L_p(R^n)$ and consider the function $g(x) = f(t^\lambda x)$, where $t^\lambda x = (t^{\lambda_1}x_1, \cdots, t^{\lambda_n}x_n)$. Then the simple calculations show that

$$||g||_{L_p} = \left(\int_{R^n} \left|f\left(t^\lambda x\right)\right|^p dx\right)^{1/p} = t^{-\frac{|\lambda|}{p}} ||f||_{L_p}$$

and

$$||\Lambda_\alpha g||_{L_q} = t^{-\left(\frac{|\lambda|}{q} + \alpha\frac{|\alpha|}{n}\right)} ||\Lambda_\alpha f||_{L_q}.$$

Hence, by (4) we have

$$||\Lambda_\alpha f||_{L_q} \leq A_{p,q} t^{|\lambda|[\frac{1}{q}-(\frac{1}{p}-\frac{\alpha}{n})]} ||f||_{L_p} \tag{11}$$

The contradiction, which can be obtained from this enequality when $t \to 0$(if $\frac{1}{q} > \frac{1}{p}-\frac{\alpha}{n}$) and when $t \to \infty$(if $\frac{1}{q} < \frac{1}{p} - \frac{\alpha}{n}$) show that (4) holds only for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.
Note that (4) does not hold for $p = q$. Really from the inequality (11) it may be seen that in the case $p = q$

$$||\Lambda_\alpha f||_{L_p} \leq A_{p,q} t^{\frac{|\lambda|}{n}\alpha} ||f||_{L_p}.$$

But this is possible only when $\alpha = 0$. That is the potential $\Lambda_\alpha f$ can not acting from $L_p(R^n)$ to $L_q(R^n)$.
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İsotropic Olmayan Riesz Potansiyelleri İçin Hardy-Litleewood Sobolev Tipi Eşitsizlik

Özet

Bu çalışmada λ- uzaklığa bağlı Riesz potansiyeli tanımlanarak bu potansiyeller için Hardy-Litleewood-Sobolev tipinde bir teorem ispatlanmaktadir.

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