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ON THE STABILITY RESULTS FOR THIRD ORDER DIFFERENTIAL-OPERATOR EQUATIONS

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Abstract

Sufficient conditions for the stability and the global asymptotic stability of the zero solution of third order linear differential- operator equations are established.

Key words: Differential-Operator Equations, Stability, Global asymptotic stabilty

1 Introduction

Several problems of mathematical physics are leading to the initial- boundary value problems for evolutionary partial differential equations of third order which can be realized as third order differential-operator equations in some Hilbert space (see [3], [4], [7], [8], and the references therein).

There are many results on solvability of the Cauchy problem for the higher order differential-operator equations (see [1], [2], [2], [6], [9]). In the literature, there are many articles devoted to the stability and instabilty of solutions of the first and second order differential-operator equations. But little is known about the third order equations.

Our aim is to study the problem of stability and global asymptotic stability of the zero solution of third order linear differential- operator equations.

2 Main Results

Let H be a real Hilbert space with the inner product $(.,.)$ and norm $\| \, . \, \|$. We will consider in H the following third order equation:

$$u''' + Au'' + Bu' + Cu = 0, \tag{1}$$

where A, B and C are linear (not necessarely bounded), positive- definite and self adjoint aperators. The domains of definition of these operators $D(A), D(B)$ and $D(C)$ are dense linear subspaces of H. The symbol "$'$" stands for differentitation with respect to t.

Our first result is the following,

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Theorem 2.1 *Let the operatos A, B and C be as above. Suppose that $D(B) \subseteq D(C)$ and there exist positive numbers σ and γ_1 such that*

$$\alpha\gamma_1 > 1 \tag{2}$$

and

$$\alpha \parallel u \parallel^2 \leq (Au, u), \forall u \in D(A) \tag{3}$$

$$(Cu, u) \leq \gamma_1^{-1}(Bu, u), \forall u \in D(B) \tag{4}$$

Then the zero solution of the equation (1) is stable in the sense of the norm

$$\parallel u'' \parallel^2 + \parallel A^{1/2}u' \parallel^2 + \parallel B^{1/2}u' \parallel^2 + \parallel C^{1/2}u \parallel^2 . \tag{5}$$

Proof. Assume that $u = u(t)$ is an arbitrary solution of the equation (1). Taking the inner product in H of (1) with $u'' + \varepsilon u'$, we obtain

$$\begin{aligned} 0 &= (u''' + Au'' + Bu' + Cu, u'' + \varepsilon u') \\ &= \frac{1}{2}\frac{d}{dt}\|u''\|^2 + \|A^{1/2}u''\|^2 + \frac{1}{2}\frac{d}{dt}\|B^{1/2}u'\|^2 + (Cu, u'') + \\ &+ \varepsilon(u''', u') + \frac{\varepsilon}{2}\frac{d}{dt}\|A^{1/2}u'\|^2 + \varepsilon\|B^{1/2}u'\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\|C^{1/2}u\|^2 \end{aligned}$$

where ε is a positive number which will be specified below. It follows that

$$\begin{aligned} \frac{d}{dt}[\frac{1}{2}\|u''\|^2 &+ \frac{\varepsilon}{2}\|A^{1/2}u'\|^2 + \frac{1}{2}\|B^{1/2}u'\|^2 + \frac{\varepsilon}{2}\|C^{1/2}u\|^2 + \\ &+ \varepsilon(u'', u') + (C^{1/2}u, C^{1/2}u')] + \|A^{1/2}u''\|^2 + \\ &+ \varepsilon\|B^{1/2}u'\|^2 - \varepsilon\| u''\|^2 - \|C^{1/2}u'\|^2 = 0. \end{aligned} \tag{6}$$

Let us denote by $\Phi(u(t))$ the following expression

$$\begin{aligned} \Phi(u(t)) &= \frac{1}{2}\|u''\|^2 + \frac{\varepsilon}{2}\|A^{1/2}u'\|^2 + \frac{1}{2}\|B^{1/2}u'\|^2 + \\ &+ \frac{\varepsilon}{2}\|C^{1/2}u\|^2 + \varepsilon(u'', u') + (C^{1/2}u, C^{1/2}u'). \end{aligned} \tag{7}$$

Using the standart inequality: $ab \leq \frac{\mu}{2}a^2 + \frac{1}{2\mu}b^2$, the Schwarz's inequality and conditions (3), (4) we can get:

$$\varepsilon|(u'',u'| \leq \frac{1}{2(1+\varepsilon_0)}\| u''\|^2 + \frac{\varepsilon^2(1+\varepsilon_0)}{2\alpha}\|A^{1/2}u'\|^2 \tag{8}$$

$$\begin{aligned} |(C^{1/2}u, C^{1/2}u')| &\leq \|C^{1/2}u\| \, \|B^{1/2}u'\|\gamma_1^{-1/2} \\ &\leq \frac{1-\varepsilon_1}{2}\|B^{1/2}u'\|^2 + \frac{1}{2\gamma_1(1-\varepsilon_1)}\|C^{1/2}u\|^2, \end{aligned} \tag{9}$$

where ε_0 and ε_1 are positive constants to be choosen below.

Hence due to (8) and (9) we find the following estimation of $\Phi(u(t))$ from below:

$$\begin{aligned} \Phi(u(t)) &\geq \frac{1}{2}\frac{\varepsilon_0}{1+\varepsilon_0}\| u''\|^2 + \frac{1}{2}(\varepsilon - \frac{\varepsilon^2(1+\varepsilon_0)}{\alpha})\,\|A^{1/2}u'\,\|^2 + \\ &+ \frac{\varepsilon_1}{2}\|B^{1/2}u'\|^2 + \frac{1}{2}(\varepsilon - \frac{1}{\gamma_1(1-\varepsilon_1)})\,\|C^{1/2}u\,\|^2. \end{aligned} \tag{10}$$

From our main condition (2) it is clear that there is a positive number $\alpha' < \alpha$ such that

$$\alpha'\gamma_1 > 1. \tag{11}$$

We can also choose ε_0 and ε_1 such that

$$d_0 \equiv \frac{1}{2}[\alpha' - \frac{(\alpha')^2(1+\varepsilon_0)}{\alpha}], d_1 \equiv \frac{1}{2\gamma_1}[\alpha'\gamma_1 - \frac{1}{1-\varepsilon_1}]$$

are positive. So taking $\varepsilon = \alpha'$ in (10) we obtain:

$$\begin{aligned} \Phi(u(t)) \geq \frac{1}{2}\frac{\varepsilon_0}{1+\varepsilon_0}\,\|u''\,\|^2 + \frac{\varepsilon_1}{2}\,\|\,B^{1/2}u'\,\|2\, + \,d_0\,\|\,A^{1/2}u'\,\|^2 + \\ + \,d_1\,\|C^{1/2}u\,\|^2. \end{aligned} \tag{12}$$

Therefore Φ is a Lyapunov functional for (1) and the zero solution of (1) is stable. This completes the proof of Theorem 1. □

Theorem 2.2 *Suppose that all contitions of the Theorem 2.1 are satisfied. Assume also that $D(B) = D(C) \subseteq D(A)$ and there exist positive numbers β and γ_2 such that*

$$\beta(Au,u) \leq (Bu,u), \quad \forall u \in D(B) \tag{13}$$

$$\gamma_2(Bu,u) \leq (Cu,u), \quad \forall u \in D(B) \tag{14}$$

Then the zero solution of the equation (1) *is globally asymptotically stable in the sense of the norm*

$$\| u'' \|^2 + \| B^{1/2}u' \|^2 + \| C^{1/2}u \|^2. \tag{15}$$

Moreover every solution of the Cauchy problem for the equation (1) *tends to zero with an exponential rate.*

Proof. Let us take the scalar product in H of (1) with u:

$$\frac{d}{dt}[(u'',u) - \frac{1}{2}\|u'\|^2 + (A^{1/2}u', A^{1/2}u) + \frac{1}{2}\|B^{1/2}u\|^2] + \\ + \| C^{1/2}u \|^2 - \|A^{1/2}u'\|^2 = 0. \tag{16}$$

Assume that η is a pasitive parameter. Multiply (16) by η and add to (6):

$$\frac{d}{dt}[\Phi(u) + \eta(u'',u) - \frac{\eta}{2}\|u'\|^2 + \eta(A^{1/2}u', A^{1/2}u) + \frac{\eta}{2}\|B^{1/2}u\|^2] + \\ + \| A^{1/2}u'' \|^2 + \alpha' \| B^{1/2}u' \|^2 - \alpha' \|u''\|^2 - \|C^{1/2}u'\|^2 + \\ + \eta \|C^{1/2}u\|^2 - \eta\|A^{1/2}u'\|^2 = 0. \tag{17}$$

Detonte by $\Psi(u(t))$ the following expression

$$\Psi(u(t)) \equiv \Phi(u) + \eta(u'',u) - \frac{\eta}{2}\|u'\|^2 + \eta(A^{1/2}u', A^{1/2}u) + \\ + \frac{\eta}{2}\| B^{1/2}u \|^2. \tag{18}$$

Using the Schwarz's inequality and the conditions (3), (13), (14) for sufficiently small η we have

$$\Psi(u(t)) \geq \nu_1(\| u'' \|^2 + \| B^{1/2}u' \|^2 + \|C^{1/2}u \|^2) \tag{19}$$

where ν_1 is a positive parameter depending on α, β, γ_1 and γ_2. It is also not difficult to see that there exists a positive paramater $\nu_2 = \nu_2(\alpha, \beta, \gamma_1, \gamma_2)$ such that

$$\Psi(u(t)) \leq \nu_2(\|u''\|^2 + \| B^{1/2}u' \|^2 + \| C^{1/2}u\|^2) \tag{20}$$

for each solution $u(t)$ of the equation (1).

By using (3), (13) and (14) we can get from (17), (18) and (20), the following inequality

$$\begin{aligned}\frac{d}{dt}\Psi(u(t))+\delta\Psi(u(t)) &= \delta\Psi(u(t))-\parallel A^{1/2}u''\parallel^2-\alpha'\parallel B^{1/2}u'\parallel^2+\alpha'\parallel u'\parallel^2+\\ &+\parallel C^{1/2}u'\parallel^2\eta\parallel C^{1/2}u\parallel^2+\eta\parallel A^{1/2}u'\parallel^2\\ &\leq -(\alpha-\alpha'-\delta\nu_2)\parallel u''\parallel^2-(\eta-\delta\nu_2)\parallel C^{1/2}u\parallel^2-\\ &-(\alpha'-\eta\beta^{-1}-\gamma_1^{-1}-\delta\nu_2\parallel B^{1/2}u'\parallel^2\end{aligned}$$

where δ is some positive parameter.

Since the parameters α,α' satisfy (2) and (11) respectively we can choose η and δ so small that:

$$\frac{d}{dt}\Psi(u(t))+\delta\Psi(u(t))\leq 0. \tag{21}$$

From the enequalities (19) and (21) it follows that Ψ is a Lyapunov functional for (1). Moreover this inequalities imply:

$$\begin{aligned}\parallel u''(t)\parallel^2+\parallel B^{1/2}u'(t)\parallel^2+\parallel C^{1/2}u(t)\parallel^2 &\leq \frac{\nu_2}{\nu_1}e^{-\delta t}(\parallel u''(o)\parallel^2+\\ &+\parallel B^{1/2}u'(0)\parallel^2+\parallel C^{1/2}u(0)\parallel^2).\end{aligned}$$

Thus the zero solution of (1) is globally asymtoticaly stable and every solution of the Cauchy problem for the equation (1) is tending to zero with an exponential rate. Hence the proof of Theorem 2 is completed. □

3 Applications

The above results give us possibilty to investigate various partial differential equations and systems of partial differential equations.

Example 3.1 Let $\Omega\subset R^n$ be bounded domain with sufficiemtly smooth boundary $\partial\Omega$. Consider in $\Omega\times(0,\infty)$ the equation

$$u_{ttt}+a_1u_{tt}-a_2\Delta u_t-a_3\Delta u=0. \tag{22}$$

where a_1,a_2 and a_3 are some positive constants.

This equation is one of the mathematical models describing small movements of compressible relaxing medium ([8]).

Theorem 2.2 gives us possibility to prove that under some condition on a_1,a_2,a_3 every solution of (22) satisfying the boundary condition

$$u\,|_{\partial\Omega} = 0 \tag{23}$$

tends to zero with an exponential rate.

In fact (22), (23) can be written in the from (1), where $A = a_1 I, B = -a_2\Delta, C = -a_3\Delta, H = L_2(\Omega), D(A) = L_2(\Omega)$ and $D(B) = D(C) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. Since

$$(A(u(.,t), u(.,t)) = a_1 \int_\Omega u^2(x,t)dx = a_1 \parallel u(.,t)) \parallel^2,$$

it is clear that the inequality (3) holds with $\alpha = a_1$. Due to the Poincare Friedrichs inequality

$$\int_\Omega |\nabla u(x,t)|^2 dx \geq \lambda_1 \int_\Omega u^2(x,t)dx,$$

we obtain

$$\begin{aligned}(Bu(.,t), u(.,t)) &= -a_2 \int_\Omega (\Delta u(x,t), u(x,t))dx = a_2 \int_\Omega |\nabla u(x,t)|^2 dx \\ &\geq a_2\lambda_1 \int_\Omega u^2(x,t)dx = a_2\lambda_1 \parallel u(.,t) \parallel^2 \\ &= \frac{a^2}{a_1}\lambda_1(Au(.,t), u(.,t))\end{aligned}$$

where $\lambda_1 > 0$ is the first eigenvalue of the problem

$$-\Delta\psi = \lambda\psi, \quad \psi|\partial\Omega = 0. \tag{24}$$

Therefore the inequality (13) holds with $\beta = \frac{a_2}{a_1}\lambda_1$.

Finally, since $B = \frac{a_2}{a_3}C$ the conditions (4) and (14) are satisfied with $\gamma_1^{-1} = \gamma_2 = \frac{a_2}{a_3}$. Thus we reach at the statement of the following:

Corollary 3.1 *If the positive constants a_1, a_2 and a_3 satisfy the condition $a_1a_2 - a_3 > 0$, then the zero solution of the equation (22) is globally asymptoticaly stable in the sense of the norm*

$$\parallel u_{tt}(.,t) \parallel^2 + \parallel \nabla u_t(.,t) \parallel^2 + \parallel \nabla u(.,t) \parallel^2 . \tag{25}$$

Remark Let $\psi_k(x)$ be the $k-th$ eigenfunction corresponding to the eigenvalue λ_k of the problem (24) and suppose that $v_k(t)$ is the solution of the equation

$$v_k^{'''}(t) + a_1 v_k^{''}(t) + \lambda_k a_2 v_k^{'}(t) + a_3 \lambda_k v_k(t) = 0.$$

Then it is clear that the function $u_k(x,t) = v_k(t)\psi_k(x)$ is the solution of $(22)-(23)$. The standart analysis of the equation (26) allows us to get the following assertion:

If a_1, a_2 and a_3 satisfy the condition $a_1 a_2 - a_3 < 0$, then the zero solution of the problem $(22)-(23)$ is unstable.

Example 3.2 Consider now in $\Omega \times (0,\infty)$ the equation

$$u_{ttt} - a_1 \Delta u_{tt} - a_2 \Delta u_t - a_3 \Delta u = 0. \tag{26}$$

Proceeding as in Example 3.1 we obtain the following:

Corollary 3.2 *If the coefficients a_1, a_2, a_3 are positive constants and satisfy the contition $\lambda_1 a_1 a_2 > a_3$, then the zero solution of the equation* (27) *is ghobally asymptoticaly stable in the sense of the norm* (25).

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Üçüncü Basamaktan Diferensiyel-Operatör Denklemlerin Kararlılık Sonuçları Üzerine

Özet

Bu çalışmada üçüncü basamaktan lineer diferensiyel-operatör denklemlerin sıfır çözümünün kararlılık ve global asimtotik kararlılığına ilişkin yeter koşullar verildi.

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