

## AN APPLICATION OF LINEAR TOPOLOGICAL INVARIANTS

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### Abstract

We consider a possible isomorphism of cartesian product of two Dragilev spaces of infinite type, and by making use of Zahariuta invariants and some structural properties, we show that if there is such an isomorphism, then any factor on the left is nearly isomorphic to the corresponding factor on the right.

**Key Words and Phrases:** Linear topological invariants, Dragilev space, Dragilev function, rapidly increasing function.

### 1. Introduction

Linear topologic invariants (such as approximative and diametral dimensions) as a tool of isomorphic classification of non-normed linear topological spaces appeared in investigations of Petczynski [19], Kolmogorov [11], Bessaga, Petczynski, Rolewicz [3], Mitiagin [13] and Zahariuta [24], [26], [27]. They were also initiated by Gelfand [7].

The counting functions  $M_a(t, \tau)$  and  $m_a(t)$  of Mitiagin first appeared in [16], [17], [18] for the power series spaces  $E_\alpha(a)$ .

In [24], [30], [31], [32], by developing Mitiagin's techniques Zahariuta introduced invariants with two or more inequalities, and in [33], [34] and [35] by making use of some geometrical considerations Zahariuta introduced invariants which we call Zahariuta invariants.

V.P. Zahariuta has proved some isomorphic classification theorems in [25] by using Riesz theory. In [23], M. Yurdakul and V.P. Zahariuta have considered the cartesian products of Montel power series spaces of finite and infinite type and proved that if  $E_0(a) \times E_\infty(b)$ ,  $E_0(\tilde{a}) \times E_\infty(\tilde{b})$  are isomorphic, then  $E_0(a)$  is near isomorphic to  $E_0(\tilde{a})$  and  $E_\infty(b)$  is near isomorphic to  $E_\infty(\tilde{b})$  by using linear topological invariants. This result is slightly weaker than the result obtained in [25]. But it shows the power of linear topological invariants. In this work we try to show that in case of the cartesian product of two Dragilev spaces of infinite type similar (and again weaker) result holds.

We made use of properties of Dragilev spaces introduced by Ahonen (for their different types) in [1]. Also we consider the relations  $f \prec g$  and  $f \approx g$ . For each possible isomorphism, there are several possibilities of relations of the defining functions of  $L_f(a, \alpha)$  spaces. But we eliminate many of them by using the compactness theorem of Zahariuta in [25](Theorem 4).

In order to deal with the remaining cases, we considered the synthetic neighbourhoods as defined in [10] and saw that the properties in [1] can be regarded as the property  $\mathcal{D}_\varphi$  (for some special functions  $\varphi$ ) which was considered by Vogt [22] and Tidten [21] as well as Zahariuta and Goncharov [8], [29]. This gave us the freedom of using some interpolation methods as well.

## 2. Preliminaries

**Definition 1** *Let  $X$  and  $Y$  be locally convex spaces (lcs's). A linear operator  $T : X \rightarrow Y$  is called a near-isomorphism (see [25]), if the following conditions are satisfied:*

- a)  $T(X)$  is closed in  $Y$  and  $T$  is an open map from  $X$  onto  $T(X)$ ,
- b)  $\alpha(T) = \dim \text{Ker}T < \infty$ ,
- c)  $\beta(T) = \text{codim } T(X) = \dim Y/T(X) < \infty$  (see [20]).

*The lcs's  $X$  and  $Y$  are said to be nearly isomorphic if there exists a near isomorphism  $T$  from  $X$  into  $Y$ .*

If the locally convex spaces  $X$  and  $Y$  are isomorphic, we write  $X \simeq Y$  and if they are nearly isomorphic we write  $X \approx Y$ .  $\mathbb{N}$  denotes the set of all positive integers.

Let  $A = (a_{i,p})_{i \in I, p \in \mathbb{N}}$  be a matrix of real numbers such that  $0 \leq a_{i,p} \leq a_{i,p+1}$  for each  $p$  and for each index  $i$  in the countable set  $I$ . The Köthe space  $K(A)$ , defined by the matrix  $A$  is the locally convex spaces of all families  $x = (x_i)$  of scalars such that  $\|x\|_p := \sum_{i \in I} |x_i| a_{i,p} < \infty$ ,  $p \in \mathbb{N}$ , with the topology generated by the system of seminorms  $\{\|\cdot\|_p : p \in \mathbb{N}\}$ . It is known that every Fréchet space with an absolute basis is isomorphic to some Köthe space. Let  $I = \mathbb{N}$ , if  $A$  satisfies the inequality  $\frac{a_{i+1,p}}{a_{i+1,p+1}} \leq \frac{a_{i,p}}{a_{i,p+1}}$  for each  $i \in \mathbb{N}$  and for each  $p \in \mathbb{N}$ , then  $A$  and  $K(A)$  are called regular [6]. The cartesian product  $K(A) \times K(B)$  of the Köthe spaces  $K(A)$  and  $K(B)$  where  $A = (a_{ip})$ ,  $B = (b_{ip})$ ,  $i, p \in \mathbb{N}$  is naturally isomorphic to the space  $K(C)$  where  $C = (c_{ip})$ ,

$$c_{ip} = \begin{cases} a_{kp} & \text{if } i = 2k - 1 \\ b_{kp} & \text{if } i = 2k \end{cases} . \tag{1}$$

Two sequences  $(a_i)$ ,  $(b_i)$  of positive numbers are said to be *weakly equivalent* if there is a constant  $\Delta > 0$  such that  $\frac{1}{\Delta} \leq \frac{a_i}{b_i} \leq \Delta$ ,  $i \in \mathbb{N}$ . In that case we write  $a_i \asymp b_i$ .

Let  $f$  be an increasing odd function on  $(-\infty, +\infty)$  which is logarithmically convex on  $[0, +\infty)$  ( i.e. )  $\phi(w) = \ln f \exp w$  is convex on  $(-\infty, +\infty)$  (such a function is called

a Dragilev function),  $r_k \leq r_{k+1}$ ,  $\lim r_k = r$ ,  $-\infty < r \leq +\infty$ ,  $a = (a_n)$ ,  $a_n \leq a_{n+1}$ ,  $\lim a_n = \infty$ .

Put  $A := \{(a_n^k) = (\exp f(r_k a_n)), k = 1, 2, 3, \dots\}$ . Then the corresponding Köthe space  $\lambda(A)$  is called an  $L_f(a, r)$  space or a Dragilev space (defined by Dragilev in [6]), i.e.

$$L_f(a, r) = \{(x_n) : \|x_n\|_k = \sum_n |x_n| e^{f(r_k a_n)} < \infty, k = 1, 2, 3, \dots\}.$$

**Definition 2** We say that an increasing function  $\phi$  defined on  $[0, +\infty)$  increases rapidly if, for every  $a > 1$ ,

$$\lim_{t \rightarrow \infty} \frac{\phi(at)}{\phi(t)} = \infty,$$

and it increases slowly if for every  $a > 1$ ,

$$\lim_{t \rightarrow \infty} \frac{\phi(at)}{\phi(t)} = \tau_\phi(a) < \infty.$$

It is well known that (see [6]), if  $f$  is an increasing logarithmically convex function, then  $f$  is either rapidly increasing or slowly increasing in which case  $\lim_{a \rightarrow \infty} \tau_f(a) = \infty$ . Moreover  $L_f(a, r)$  is isomorphic to a power series space if and only if  $f$  is slowly increasing. So we shall assume that  $f$  is rapidly increasing. In this case, there are four classes of Dragilev spaces corresponding to  $r = \infty, 1, 0, -1$ . If  $\phi$  is a rapidly increasing function, then for each  $A > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\phi^{-1}(At)}{\phi^{-1}(t)} = 1.$$

**Lemma 1 ( Dragilev )** An increasing, logarithmically convex function is either rapidly increasing or slowly increasing.

**Lemma 2** Let  $h$  be a slowly increasing function. Then

$$\lim_{a \rightarrow \infty} \tau_h(a) = \infty.$$

We shall use the relations " $\prec$ " and " $\approx$ " between  $f_1, f_2$ , where  $f_1$  and  $f_2$  are functions satisfying the conditions of Definition 2 and  $-\infty < r \leq \infty$ . We shall write  $f_1 \prec f_2$ , if  $f_1^{-1} f_2$  is rapidly increasing, and  $f_1 \approx f_2$ , if both  $f_1^{-1} f_2$  and  $f_2^{-1} f_1$  are slowly increasing. We shall say that  $f_1$  and  $f_2$  are comparable if  $f_1 \prec f_2$  or  $f_1 \approx f_2$  or  $f_2 \prec f_1$ .

**Theorem 1** *Let  $f, g$  be rapidly increasing, logarithmically convex Dragilev functions, and  $f^{-1}g$  be logarithmically convex and slowly increasing function. Let  $r = \infty$  or  $0$ . Then*

$$L_f(a, r) \simeq L_g(b, r) \text{ if and only if } f^{-1}g(b_i) \asymp a_i.$$

**Proof** It is given in [4] and [5] that if a nuclear Fréchet space  $X$  (with the system of seminorms  $(\|\cdot\|_p)$ ) has two regular bases  $(x_i), (y_i)$ , then there exists a sequence of scalars  $(\lambda_i)$  such that

$$\sum t_i x_i \text{ converges} \iff \sum t_i \lambda_i y_i \text{ converges},$$

or equivalently  $\forall p \exists q = q(p) \exists C > 0$  such that

$$\|x_i\|_p \leq C |\lambda_i| \|y_i\|_q \text{ and } |\lambda_i| \|y_i\|_p \leq C \|x_i\|_q.$$

Let  $r = \infty$ . When  $L_f(a, \infty) \simeq L_g(b, \infty)$ ,  $\forall p \exists q = q(p) \exists C > 0$  such that

$$\begin{aligned} (1) \quad & e^{f(pa_i)} \leq C |\lambda_i| e^{g(qb_i)} \\ (2) \quad & |\lambda_i| e^{g(pb_i)} \leq C e^{f(qa_i)} \end{aligned}$$

In (1), let  $p = 1$ . Then

$$e^{f(a_i)} \leq C |\lambda_i| e^{g(q(1)b_i)} \implies \frac{1}{C} e^{f(a_i) - g(q(1)b_i)} \leq |\lambda_i|$$

for  $p = q(1)$ , let  $q = q(p + 1)$ . Then by (2) we have

$$|\lambda_i| e^{g((p+1)b_i)} \leq C' e^{f(qa_i)}.$$

Hence we obtain

$$e^{f(a_i) - g(q(1)b_i) + g((p+1)b_i)} \leq CC' e^{f(qa_i)}.$$

Therefore we get, there exists  $i_1$  such that

$$i \geq i_1 \implies g(pb_i) \leq f(qa_i) \implies \frac{f^{-1}g(b_i)}{a_i} \leq \frac{f^{-1}g(b_i)}{\frac{1}{q}f^{-1}g(pb_i)} \leq q.$$

Similarly, starting from (2) we get for  $p = q(1)$  and  $q = q(p + 1)$ , there exists  $i_2$  such that

$$i \geq i_2 \implies f(pa_i) \leq g(qb_i) \implies \frac{f^{-1}g(b_i)}{a_i} \geq \frac{f^{-1}g(b_i)}{\frac{1}{p}f^{-1}g(qb_i)} = \frac{p}{\tau_{f^{-1}g}(q)}.$$

Conversely, assume

$$\forall i \quad \frac{1}{C} \leq \frac{f^{-1}g(b_i)}{a_i} \leq C.$$

We will show that for each  $p$  there is a  $q$  such that for all large  $i$ ,

$$(I) : f(pa_i) \leq g(qb_i) \text{ and } (II) : g(pb_i) \leq f(qa_i).$$

For (I), given  $p$ , by Lemma 2 we find  $q$ , such that for all large  $i$ ,

$$\frac{f^{-1}g(qb_i)}{f^{-1}g(b_i)} \geq pC.$$

Then

$$\frac{f^{-1}g(qb_i)}{a_i} = \frac{f^{-1}g(qb_i)}{f^{-1}g(b_i)} \frac{f^{-1}g(b_i)}{a_i} \geq pC \frac{1}{C} = p \implies f(pa_i) \leq g(qb_i).$$

For (II), given  $p$ , we find  $q \geq \tau_{f^{-1}g}(p) C$ . Then for all  $i$

$$\frac{f^{-1}g(pb_i)}{a_i} = \frac{f^{-1}g(pb_i)}{f^{-1}g(b_i)} \frac{f^{-1}g(b_i)}{a_i} \leq \tau_{f^{-1}g}(p) C \leq q$$

which implies the desired inequality.

Let  $a = (a_i)$  be a positive sequence and  $1 \leq \tau < t < \infty$ . Then the counting functions  $M_a(t, \tau)$  and  $m_a(t)$  are defined as follows:

$$M_a(t, \tau) = |\{i : \tau < a_i \leq t\}| \text{ and } m_a(t) = |\{i : a_i \leq t\}|$$

where  $|\cdot|$  denotes the cardinality of the set. If  $a_i \nearrow \infty$ , then  $M_a(t, \tau) = m_a(t) - m_a(\tau)$ . Let  $b = (b_i)$  be another positive sequence. If there is a constant  $\Delta > 0$  such that

$$M_a(t, \tau) \leq M_b(\Delta t, \frac{\tau}{\Delta}) \text{ and } M_b(t, \tau) \leq M_a(\Delta t, \frac{\tau}{\Delta}),$$

then the counting functions  $M_a, M_b$  are said to be *equivalent* and we write  $M_a \approx M_b$  (see e.g. [14], [15]).

The following technical lemma will be used throughout this work. It was proved in [23]. Since it plays a crucial role in our work, we present its proof here.

**Lemma 3** *Let  $a = (a_i)$  and  $\tilde{a} = (\tilde{a}_i)$  be increasing sequences. Assume  $\exists C \geq 1 \exists \tau_0 > 0 : \tau_0 \leq \tau \leq t \implies M_a(t, \tau) \leq M_{\tilde{a}}(Ct, \frac{\tau}{C})$  and  $M_{\tilde{a}}(t, \tau) \leq M_a(Ct, \frac{\tau}{C})$ . Then there is  $k \in \mathbb{Z}$  such that  $a_{i+k} \asymp \tilde{a}_i$ . More precisely, there is an  $i_0$  such that*

$$i \geq i_0 \implies \frac{1}{C} \leq \frac{a_{i+k}}{\tilde{a}_i} \leq C.$$

**Proof** Since  $(a_i)$  and  $(\tilde{a}_i)$  are increasing, the inequalities are equivalent to:

$$\begin{aligned} \exists C \geq 1 \exists \tau_0 : \tau_0 \leq \tau \leq t \implies m_a(t) - m_a(\tau) &\leq m_{\tilde{a}}(Ct) - m_{\tilde{a}}(\frac{\tau}{C}) & (i) \\ m_{\tilde{a}}(t) - m_{\tilde{a}}(\tau) &\leq m_a(Ct) - m_a(\frac{\tau}{C}) & (ii) \end{aligned}$$

Putting  $\tau = \tau_0$  in (i), we get

$$\forall t \geq \tau_0 : m_a(t) - m_{\tilde{a}}(Ct) \leq m_a(\tau_0) - m_{\tilde{a}}\left(\frac{\tau_0}{C}\right).$$

So, the integer valued function  $m_a(\cdot) - m_{\tilde{a}}(C\cdot)$  is bounded above. So it attains its maximum  $k$  at some  $t = t_1 \geq \tau_0$ . Thus

$$m_a(t_1) - m_{\tilde{a}}(Ct_1) = k$$

and

$$\forall t \geq \tau_0 \quad m_a(t) \leq m_{\tilde{a}}(Ct) + k \quad \dots(I).$$

Putting  $\tau = Ct_1$  in (ii), we get

$$\forall t \geq Ct_1 \quad m_{\tilde{a}}(t) + k \leq m_a(Ct) \quad \dots(II)$$

Since the inequalities (I) and (II) are symmetrical in  $k$  and  $-k$ , we may assume  $k \geq 0$ . (I) can be written as

$$|\{i : a_{i+k} \leq t\}| = m_a(t) - k \leq m_{\tilde{a}}(Ct) = |\{j : \tilde{a}_j \leq Ct\}|.$$

Now let  $n$  be large enough so that  $a_{n+k} \geq \tau_0$  and let  $t = a_{n+k}$ . Then the set on the left hand side has  $n$  elements. Thus the set on the right hand side has at least  $n$  elements. Since  $(\tilde{a}_j)$  is increasing, we have  $\tilde{a}_n \leq Ct$ , i.e.  $\frac{1}{C} \leq \frac{a_{n+k}}{\tilde{a}_n}$ .

(II) means  $|\{i : \tilde{a}_i \leq t\}| + k \leq |\{j : a_j \leq Ct\}|$ . Let  $n$  be large enough so that  $\tilde{a}_n \geq Ct_1$  and let  $t = \tilde{a}_n$ . Then left hand side of (II) is  $n + k$ . So the set on the right hand side has at least  $n + k$  elements. Since  $(a_j)$  is increasing, we get  $a_{n+k} \geq Ct$ , i.e.  $\frac{a_{n+k}}{\tilde{a}_n} \leq C$ .

Let  $X$  be a lcs with an absolute basis  $e = (e_i)$  and  $a = (a_i)$  be a positive sequence of reals. For  $1 \leq p \leq \infty$ , we denote by  $B_p^e(a)$  the weighted  $l^p$ -ball in it with respect to the basis  $e$  and by  $B^e(a)$  the ball  $B_1^e(a)$ . If  $(e_i)$  is the sequence which is zero at each coordinate except the  $i$ th where it is 1, we omit  $e$  and simply write  $B_p(a)$  or  $B(a)$ . So

$$B_p^e(a) = B_p^e(a_i) = \{x = (\xi_i) \in X : \sum |\xi_i|^p a_i^p \leq 1\}.$$

### 3. Linear Topological Invariants

In this and the next section, we use Zahariuta invariant  $\beta$  together with some geometrical considerations initiated by Zahariuta in [33], [34], [35].

We will use the following characteristic function for a pair of absolutely convex subsets in  $X$  :

$$\beta(V, U) = \sup\{\dim L : L \cap U \subset V\}$$

where sup is taken over all finite dimensional subspaces  $L$  of  $X$ . Trivially

$$V_1 \subset V_2 \text{ and } U_2 \subset U_1 \implies \beta(V_1, U_1) \leq \beta(V_2, U_2)$$

and if  $T : X \rightarrow Y$  is an isomorphism, then  $\beta(V, U) = \beta(T(V), T(U))$ .

**Lemma 4** *Let the positive sequences  $a = (a_i), b = (b_i)$  be such that  $\frac{b_i}{a_i} \rightarrow \infty$  and  $V = B_p(b), U = B_p(a)$ . Then  $\beta(V, U) = |\{i : \frac{b_i}{a_i} \leq 1\}|$ .*

**Lemma 5** *Let  $U = B(a), V = B(b)$  and  $c = (c_i), d = (d_i)$  where*

$$c_i = \max\{a_i, b_i\}, d_i = \min\{a_i, b_i\}.$$

*Then  $B(c) \subset U \cap V \subset B(c/2)$  and  $B(d) = \bar{\Gamma}(U \cup V)$*

*where  $\bar{\Gamma}$  denotes the closed absolutely convex hull.*

**Lemma 6** *Consider the balls  $B(a^p), a^p = (a_{ip}), p = 1, 2, 3, 4$  in a lcs  $X$ . Then*

$$(i) \quad |\{i : \frac{\max\{a_{i3}, a_{i2}\}}{\min\{a_{i2}, a_{i1}\}} \leq 1\}| = |\{i : \frac{a_{i3}}{a_{i2}} \leq 1, \frac{a_{i2}}{a_{i1}} \leq 1\}| \\ \leq \beta(B^e(a^3) \cap B^e(a^2), \bar{\Gamma}(B^e(a^2) \cup B^e(a^1))).$$

$$(ii) \quad \beta(B^e(a^3) \cap B^e(a^2), \bar{\Gamma}(B^e(a^4) \cup B^e(a^1))) \\ \leq |\{i : \frac{\frac{1}{2} \max\{a_{i3}, a_{i2}\}}{\min\{a_{i4}, a_{i1}\}} \leq 1\}| \leq |\{i : \frac{a_{i3}}{a_{i4}} \leq 2, \frac{a_{i2}}{a_{i1}} \leq 2\}|.$$

The proofs of the above lemmas can be found in [34], [35].

**Remark** If the balls  $U_p = B^e(a^p), p = 1, 2, 3, V_q = B^f(b^q), q = 1, 2, 3, 4$  in a lcs  $X$  are such that  $V_1 \subset U_1, V_4 \subset U_2 \subset V_2, U_3 \subset V_3$  then by lemma 1.4 we have

$$|\{i : \frac{a_{i3}}{a_{i2}} \leq 1, \frac{a_{i2}}{a_{i1}} \leq 1\}| \leq |\{j : \frac{b_{j3}}{b_{j4}} \leq 2, \frac{b_{j2}}{b_{j1}} \leq 2\}|.$$

**4. An Application**

We shall use the following properties defined by Ahonen [1]

**Definition** A Köthe space  $K(A)$   $A = (a_{ip})$  is said to have the property

$$S_1^+(f), \text{ if } \exists p \forall M > 1 \forall q \exists r : \forall i \in \mathbb{N} M f^{-1} \log\left(\frac{a_{iq}}{a_{ip}}\right) \leq f^{-1} \log\left(\frac{a_{ir}}{a_{iq}}\right),$$

$$Q_1^+(f), \text{ if } \forall p \exists q \forall r \exists M > 1 : \forall i \in \mathbb{N} f^{-1} \log\left(\frac{a_{ir}}{a_{iq}}\right) \leq M f^{-1} \log\left(\frac{a_{iq}}{a_{ip}}\right).$$

The proof of the following lemma is immediate.

**Lemma 7** Let  $f_1 \approx f_2$ . Then

- (i)  $L_{f_1}(a, \infty)$  does not have  $S_1^+(f_2)$ ,
- (ii)  $L_{f_2}(\tilde{a}, \infty)$  does not have  $Q_1^+(f_1)$ .

One of the linear topological invariants that we will consider in this chapter is the property  $\mathcal{D}_\varphi$  (or  $\mathcal{DN}_\varphi$ ). This property was considered by Vogt [22] and Tidten [21] and also by Goncharov and Zahariuta [8], [29]. Let  $\varphi$  be continuous, increasing function. A Fréchet space  $(X, \|\cdot\|)$  is said to have property  $\mathcal{D}_\varphi$  if

$$\exists p \forall q \exists r \exists C > 0 : \|x\|_q \leq \varphi(s) \|x\|_p + \frac{C}{s} \|x\|_r \quad \forall s > 0, \forall x \in X.$$

**Proposition 1** Let  $X = K(a_{i,p})$  be a Schwarz Köthe space. Then the following are equivalent.

- (i)  $X$  has property  $\mathcal{D}_\varphi$
- (ii)  $\exists p \forall q \exists r \exists C > 0 : \frac{a_{i,q}}{a_{i,p}} \leq \varphi\left(\frac{a_{i,r}}{a_{i,q}}\right)$ .

Given a function  $\varphi$  as in the definition of property  $\mathcal{D}_\varphi$  and  $u > 0$ , we define

$$\Phi(u) = \inf_{s>0} \left( \varphi(s) + \frac{u}{s} \right)$$

**Proposition 2** Let  $p < q < r, C > 0, K > 0$  and  $2a_{iq} \leq a_{i,q+1}$  for some  $i$ . Then

- (i)  $K \frac{a_{i,q}}{a_{i,p}} \leq \varphi\left(C \frac{a_{i,r}}{a_{i,q}}\right) \Rightarrow K \frac{a_{i,q}}{a_{i,p}} \leq \Phi\left(C K \frac{a_{i,r}}{a_{i,p}}\right)$
- (ii)  $\frac{a_{i,q+1}}{a_{i,p}} \leq \Phi\left(C \frac{a_{i,r}}{a_{i,p}}\right) \Rightarrow \frac{a_{i,q}}{a_{i,p}} \leq \varphi\left(2C \frac{a_{i,r}}{a_{i,q}}\right)$ .

**Proposition 3** Let  $p < r$  and

$$V = \bar{\Gamma}\left(\bigcup_{k>0} \left(\frac{1}{\varphi(k)} U_p \cap kU_r\right)\right) \quad , \quad V' = B^e\left(a_{i,p} \Phi\left(\frac{a_{i,r}}{a_{i,p}}\right)\right)$$

and  $\Phi(u) = \inf_{s>0} \left( \varphi(s) + \left(\frac{u}{s}\right) \right)$  as defined before. Then  $V' \subset V \subset 3V'$ .



The proofs of above propositions can be found in [10].

**Note** For the space  $L_f(\alpha, \infty)$  if we take

$$\varphi_f = \exp f \frac{1}{M} f^{-1} \log,$$

we see that with the same quantifiers as in the definitions of  $S_1^+(f)$  (resp.  $Q_1^+(f)$ ), the inequality  $\frac{a_{i,q}}{a_{i,p}} \leq \varphi_f(\frac{a_{i,r}}{a_{i,q}})$  (resp.  $\varphi_f(\frac{a_{i,r}}{a_{i,q}}) \leq \frac{a_{i,q}}{a_{i,p}}$ ) holds.

**Theorem 2** *Let  $f_1, f_2, f_3, f_4$  be comparable Dragilev functions and assume that  $L_{f_1}(a, \infty) \times L_{f_2}(b, \infty)$  is isomorphic to  $L_{f_3}(\tilde{a}, \infty) \times L_{f_4}(\tilde{b}, \infty)$ . Then either  $L_{f_1}(a, \infty) \approx L_{f_3}(\tilde{a}, \infty)$  and  $L_{f_2}(b, \infty) \approx L_{f_4}(\tilde{b}, \infty)$  or  $L_{f_1}(a, \infty) \approx L_{f_4}(\tilde{b}, \infty)$  and  $L_{f_2}(b, \infty) \approx L_{f_3}(\tilde{a}, \infty)$ .*

**Proof** Let  $K(C) = L_{f_1}(a, \infty) \times L_{f_2}(b, \infty)$ ,  $K(D) = L_{f_3}(\tilde{a}, \infty) \times L_{f_4}(\tilde{b}, \infty)$  and  $T : K(D) \rightarrow K(C)$  be an isomorphism. Here  $C = (c_{ip})$ ,  $D = (d_{ip})$  where

$$c_{ip} = \begin{cases} e^{f_1(pa_k)} & \text{if } i = 2k - 1, \\ e^{f_2(pb_k)} & \text{if } i = 2k \end{cases}, \quad d_{ip} = \begin{cases} e^{f_3(p\tilde{a}_k)} & \text{if } i = 2k - 1, \\ e^{f_4(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}.$$

Here, the only possible relations among the functions are the followings ;

$$\begin{aligned} f_4 &\approx f_1 \prec f_2 \approx f_3, \\ f_3 &\approx f_1 \prec f_2 \approx f_4, \\ f_3 &\approx f_2 \prec f_1 \approx f_4, \\ f_4 &\approx f_2 \prec f_1 \approx f_3, \\ f_1 &\approx f_2 \approx f_3 \approx f_4 \end{aligned} \tag{2}$$

The other cases can be eliminated by using Theorem 4 in [25], since for example if  $f_1 \approx f_2 \prec f_3 \approx f_4$ , any isomorphism  $T : L_{f_1}(a, \infty) \times L_{f_2}(b, \infty) \rightarrow L_{f_3}(\tilde{a}, \infty) \times L_{f_4}(\tilde{b}, \infty)$  will be compact. We consider the first case only.

Denoting the neighborhoods of  $K(C)$  and  $K(D)$  by  $U_p$  and  $V_p$  respectively, given  $p_1$ , we find  $p, p_2, q_1, q, q_2, r_1, r, r_2, s_1, s, s_2$  such that

$$\begin{aligned} T(V_{p_1}) &\succ U_p \succ T(V_{p_2}) \\ T(V_{q_1}) &\succ U_q \succ T(V_{q_2}) \\ T(V_{r_1}) &\succ U_r \succ T(V_{r_2}) \\ T(V_{s_1}) &\succ U_s \succ T(V_{s_2}) \end{aligned}$$

Then for all  $t, \tau > 0$  we have

$$\begin{aligned} \beta_1 &:= \beta \left( \left( \bar{\Gamma} \left( \bigcup_{k>0} \left( \frac{1}{\varphi_{f_1}(k)} U_p \cap kU_r \right) \right) \cap tU_r \right) \cap U_q, \bar{\Gamma}(U_q \cup \tau U_s) \right) \leq \\ \beta_2 &:= \beta \left( C \left( \left( \bar{\Gamma} \left( \bigcup_{k>0} \left( \frac{1}{\varphi_{f_1}(k)} V_{p_1} \cap kV_{r_1} \right) \right) \cap tV_{r_1} \right) \cap V_{q_1} \right), \bar{\Gamma}(V_{q_2} \cup \tau V_{s_2}) \right) \end{aligned}$$

where  $\varphi_{f_1} = \exp f_1 \frac{1}{M} f_1^{-1} \log$  with  $M_1 > 1$  suitably chosen. Then

$$\begin{aligned} \beta_1 &\geq \beta \left( \left( B^e \left( c_{i,p} \Phi_{f_1} \left( \frac{c_{i,r}}{c_{i,p}} \right) \right) \right) \cap tB^e(c_{i,r}) \cap B^e(c_{i,q}), \bar{\Gamma} (B^e(c_{i,q}) \cap \tau B^e(c_{i,s})) \right) \\ &\geq \left| \left\{ i : \frac{\max \left\{ c_{i,p} \Phi_{f_1} \left( \frac{c_{i,r}}{c_{i,p}} \right), \frac{1}{t} c_{i,r} \right\}}{c_{i,q}} \leq 1, \frac{c_{i,q}}{\frac{1}{\tau} c_{i,s}} \leq 1 \right\} \right| \\ &= \left| \left\{ i : \Phi_{f_1} \left( \frac{c_{i,r}}{c_{i,p}} \right) \leq \frac{c_{i,q}}{c_{i,p}}, \frac{c_{i,r}}{c_{i,p}} \leq t, \frac{c_{i,s}}{c_{i,q}} \geq \tau \right\} \right| \\ &\geq \left| \left\{ i : \underbrace{\varphi_{f_1} \left( \frac{c_{i,r+1}}{c_{i,q-1}} \right)}_{Q_1^+(f_1)} \leq \frac{c_{i,q-1}}{c_{i,p}}, \frac{c_{i,r}}{c_{i,q}} \leq t, \frac{c_{i,s}}{c_{i,q}} \geq \tau \right\} \right| := |I_1| \end{aligned}$$

If  $i = 2k$  in  $I_1$ , then  $c_{i,p} = e^{f_2(pb_k)}$ , but  $L_{f_2}(p, \infty)$  does not have property  $Q_1^+(f_1)$ . So we have  $i = 2k - 1$  and  $c_{i,p} = e^{f_1(pa_k)}$ . Then the first inequality holds for all large  $i$  and the last two inequalities give

$$\begin{aligned} f_1(ra_k) - f_1(qa_k) &\leq \ln t, \quad f_1(sa_k) - f_1(qa_k) \geq \ln \tau \\ \iff f_1(ra_k) &\leq \ln t, \quad f_1((s-1)a_k) \geq \ln \tau \\ \iff \frac{f_1^{-1}(\ln \tau)}{s-1} &\leq a_k \leq \frac{f_1^{-1}(\ln t)}{r} \end{aligned}$$

and hence  $M_a \left( \frac{f_1^{-1}(\ln t)}{r}, \frac{f_1^{-1}(\ln \tau)}{s-1} \right) \leq |I_1| \leq \beta_1$ .

On the otherhand,

$$\begin{aligned} \beta_2 &\leq \beta \left( C \left( B^e \left( 3d_{i,p_1} \Phi_{f_1} \left( \frac{d_{i,r_1}}{d_{i,p_1}} \right) \right) \cap tB^e(d_{i,r_1}) \cap B^e(d_{i,q_1}) \right), \bar{\Gamma} (B^e(d_{i,q_2}) \cup \tau B^e(d_{i,s_2})) \right) \\ &\leq \left| \left\{ i : \frac{\frac{1}{2C} \max \left\{ \frac{1}{3} d_{i,p_1} \Phi_{f_1} \left( \frac{d_{i,r_1}}{d_{i,p_1}} \right), \frac{1}{t} d_{i,r_1} \right\}}{d_{i,q_2}} \leq 1, \frac{\frac{1}{2C} d_{i,q_1}}{\frac{1}{\tau} d_{i,s_2}} \leq 1 \right\} \right| \\ &= \left| \left\{ \varphi_{f_1} \left( \frac{d_{i,r_1}}{d_{i,q_2}} \right) \leq 6C \left( \frac{d_{i,q_2}}{d_{i,p_1}} \right), \frac{d_{i,r_1}}{d_{i,q_2}} \leq 2Ct, \frac{d_{i,s_2}}{d_{i,q_1}} \geq \frac{\tau}{2C} \right\} \right| \\ &\leq \left| \left\{ i : \underbrace{\varphi_{f_1} \left( \frac{d_{i,r_1}}{d_{i,q_2+1}} \right)}_{Q_1^+(f_1)} \leq \frac{d_{i,q_2+1}}{d_{i,p_1}}, \frac{d_{i,r_1}}{d_{i,q_2}} \leq 2Ct, \frac{d_{i,s_2}}{d_{i,q_1}} \geq \frac{\tau}{2C} \right\} \right| := |I_2| \end{aligned}$$

As before only  $i = 2k$  is possible in  $I_2$  in which case the first inequality holds automatically and from the last two inequalities we get

$$\beta_2 \leq |I_2| \leq M_{\tilde{b}} \left( \frac{f_4^{-1}(\ln 2Ct)}{r_1 - 1}, \frac{f_4^{-1}(\ln \frac{\tau}{2C})}{s_2} \right).$$

Since  $\beta_1 \leq \beta_2$ , we get

$$M_a \left( \frac{f_1^{-1}(\ln t)}{r}, \frac{f_1^{-1}(\ln \tau)}{s - 1} \right) \leq M_{\tilde{b}} \left( \frac{f_4^{-1}(\ln 2Ct)}{r_1 - 1}, \frac{f_4^{-1}(\ln \frac{\tau}{2C})}{s_2} \right).$$

Letting  $T = \frac{f_1^{-1}(\ln t)}{r}$ ,  $S = \frac{f_1^{-1}(\ln \tau)}{s - 1}$ , we obtain

$$\begin{aligned} M_a(T, S) &\leq M_{\tilde{b}} \left( \frac{f_4^{-1}(\ln 2C + f_1(rT))}{r_1 - 1}, \frac{f_4^{-1}(f_1((s - 1)S) - \ln 2C)}{s_2} \right) \\ &\leq M_{\tilde{b}} \left( \frac{f_4^{-1}(f_1((r + 1)T))}{r_1 - 1}, \frac{f_4^{-1}(f_1((s - 2)S))}{s_2} \right) \\ &\leq M_{\tilde{b}} \left( K f_4^{-1} f_1(T), \frac{f_4^{-1} f_1(S)}{K} \right) \end{aligned}$$

if  $K$  is chosen such that

$$K > \max \left\{ \lim_{x \rightarrow \infty} \frac{f_4^{-1} f_1((r + 1)x)}{(r_1 - 1) f_4^{-1} f_1(x)}, \lim_{x \rightarrow \infty} \frac{s_2 f_4^{-1} f_1(x)}{f_4^{-1} f_1((s - 2)x)} \right\}.$$

Next choosing  $L > \lim_{x \rightarrow \infty} \frac{f_1^{-1} f_4(Kx)}{f_1^{-1} f_4(x)}$  we get

$$M_a(T, S) \leq M_{f_1^{-1} f_4(\tilde{b})} \left( LT, \frac{S}{L} \right) \text{ for } T > S \geq S_0.$$

We also have the symmetrical inequality

$$M_{f_1^{-1} f_4(\tilde{b})}(T, S) \leq M_a \left( LT, \frac{S}{L} \right).$$

From these two inequalities, we obtain

$$a_i \asymp f_1^{-1} f_4(\tilde{b}_{j+k_1})$$

for some fixed integer  $k_1$ .

Next we proceed similarly for

$$\begin{aligned} & \beta \left( tU_s \cap U_q, \bar{\Gamma} \left( U_q \bar{\Gamma} \left( \bigcup_{k>0} \frac{1}{\varphi_{f_2}(k)} U_p \cap kU_r \right) \cup \tau U_r \right) \right) \\ & \leq \beta \left( tV_{s_1} \cap V_{q_1}, \bar{\Gamma} \left( V_{q_2} \bar{\Gamma} \left( \bigcup_{k>0} \frac{1}{\varphi_{f_2}(k)} V_{p_2} \cap kV_{r_2} \right) \cup \tau V_{r_2} \right) \right) \end{aligned}$$

where  $\varphi_{f_2} = \exp f_2 \frac{1}{M} f_2^{-1}$ . This time we use the property  $S_1^+$  to eliminate some indices and obtain

$$M_b \left( \frac{f_2^{-1}(\ln t)}{s}, \frac{f_2^{-1}(\ln \tau)}{r-1} \right) \leq M_{\bar{a}} \left( \frac{f_3^{-1}(\ln 2Ct)}{s_1-1}, \frac{f_3^{-1}(\ln \frac{\tau}{2C})}{r_2} \right)$$

which gives a constant  $M > 1$  such that

$$M_b(T, S) \leq M_{f_2^{-1}f_3(\bar{a})} \left( MT, \frac{S}{M} \right) \text{ for } T > S \geq S_0.$$

This inequality together with its symmetrical inequality implies  $b_i \asymp f_2^{-1}f_3(\bar{a}_{i+k_2})$  for some fixed integer  $k_2$ .

## References

- [1] H. Ahonen, *On nuclear Köthe spaces defined by Dragilev functions*, Ann. Acad. Sci. Fenn. ser. A, 1. Math. Diss. 38 (1981).
- [2] J. Bergh - J. Löfström, *Interpolation spaces : An introduction*. Springer-Verlag Berlin Heidelberg New York (1976).
- [3] C. Bessega, A. Petczynski, R. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Pol. Sci. **9** (1961), 677-683.
- [4] L. Crone, W.B. Robinson, *Every nuclear Fréchet space with a regular basis has the quasi-equivalence property*, Studia Math. T.**LII**. (1975), 203-207.
- [5] P.B. Djakov, *A short proof of the theorem of Crone and Robinson on quasi-equivalence of regular bases*, Studia Math., 53 (1975), 269-271.
- [6] M.M. Dragilev, *On regular bases in nuclear spaces*, Amer. Math. Soc. Transl. 93, 1970, pp. 61-82.
- [7] I.M. Gelfand, *On some problems of functional analysis* (Russian), Usp. Mat. Nauk **6** (1956), 3-12.
- [8] A.P. Goncharov, V.P. Zahariuta, *Linear topological invariants and spaces of infinitely differentiable functions* (Russian), Math. Analiz i ego priloz, Rostov State Univ., 18-27 (1985).

- [9] A. Goncharov, T. Terzioğlu, V. Zahariuta, *On isomorphic classification of tensor products  $E_\infty(a) \widehat{\otimes} E'_\infty(b)$* , to appear in *Dissertationes Math.*
- [10] M. Kocatepe, V.P. Zahariuta, *Köthe spaces modeled on spaces of  $C^\infty$  functions*. *Studia Math.* **121**, 1-14 (1996).
- [11] A.N. Kolmogorov, *On the linear dimension of topological vector spaces* (Russian), *DAN USSR* **120** (1958), 239-341.
- [12] V.P. Kondakov, V.P. Zahariuta, *On bases in spaces of infinitely differentiable functions on special domains with Cup*. *Note di Matematica* **12**, 99-106 (1992).
- [13] B.S. Mitiagin, *Approximate dimension and bases in nuclear spaces*, *Russian Math. Surveys* **16** no.4 (1961), 59-127.
- [14] B.S. Mitiagin, *The equivalence of bases in Hilbert scales*, (in Russian), *Studia Math.* **37**, 111-137 (1970).
- [15] B.S. Mitiagin, *Non-Schwartzian power series spaces*, *Math. Z.* **182**, 303-310 (1983).
- [16] B.S. Mitiagin, *The equivalence of bases in Hilbert scales*, (In Russian), *Studia Math.*, **37**, 111-137 (1970).
- [17] B.S. Mitiagin, *Geometry of linear spaces and linear operators*, (In Russian), *School on theory of operators in function spaces*, preprint-Novosibirsk (1975).
- [18] B.S. Mitiagin, *Non-Schwartzian power series spaces*, *Math. Z.*, **182**, 303-310 (1983).
- [19] A. Petczynski, *On the approximation of  $S$ -spaces by finite dimensional spaces*, *Bull. Acad. Pol. Sci.* **5** (1957), 879-881.
- [20] H.H. Schaefer, *Über singulare Integralgleichungen und eine Klasse von Homomorphismen in lokalkonvaxen Räumen*, *Mat. Zeit.*, **66**, p2 (1971), pp. 147-163.
- [21] M. Tidten, *An example of a continuum of pairwise non-isomorphic spaces of  $C^\infty$  functions*. *Studia Math.* **78** 3, 267-274 (1984).
- [22] D. Vogt, *Some results on continuous linear maps between Fréchet spaces*, *Functional Analysis: Survey and Recent Results 3* (Ed: Beirstedt and B. Fuchssteiner), *North Holland Math. Studies* **90**, 349-381 (1984).
- [23] M. Yurdakul, V.P. Zahariuta, *Linear topological invariants and isomorphic classification of cartesian products of locally convex spaces*. *Turkish J. Math.* **19**, 1, 37-47 (1995).
- [24] V.P. Zahariuta, *Linear topological invariants and isomorphisms of spaces of analytic functions* (Russian), *Matem. analiz i ego pril.*, Rostov-on-Don, Rostov Univ. **2** 3-13 (1970), **3** 176-180 (1971)
- [25] V.P. Zahariuta, *On the isomorphism of cartesian products of locally convex spaces*, *Studia Math. T.* **XLVI**. 201-221 (1973)
- [26] V.P. Zahariuta, *Generalized Mitiagin invariants and continuum of pairwise nonisomorphic spaces of analytic functions* (Russian), *Funk. analiz i ego pril.* **11** 24-30 (1977)
- [27] V.P. Zahariuta, *Synthetic diameters and linear topological invariants* (Russian), "School on theory of operators in functional spaces (abstracts of reports)", Minsk 1978, 51-52.
- [28] V.P. Zahariuta, *Linear topological invariants and their application to generalized power spaces*. To appear in *Turkish J. Math.*.

- [29] V.P. Zahariuta, *On isomorphic classification of  $F$ -spaces*, Linear and Complex Analysis Problem Book. 199 Research Problems. Lecture Notes in Mathematics 1043, 34-37 (1984).
- [30] V.P. Zahariuta, *On isomorphisms and quasiequivalence of bases of power Köthe spaces*, (in Russian), DAN USSR **221**, 772-774 (1975).
- [31] V.P. Zahariuta, *On isomorphisms and quasiequivalence of bases of power Köthe spaces*, Proceedings of 7-th winter school in Drogobych, Moskow, 101-126 (1977).
- [32] V.P. Zahariuta, *Compact operators and isomorphisms of Köthe spaces*, (in Russian), Aktualniye voprosy matem. analiza, Rostov-on-Don, 62-71 (1978).
- [33] V.P. Zahariuta, *Synthetic diameters and linear topological invariants*, (in Russian), School on theory of operators in function spaces (abstract of reports), 51-52 Minsk (1978).
- [34] V.P. Zahariuta, *Linear topological invariants and their applications to power Köthe spaces*, (in Russian), Manuscript, Rostov-on-Don Universty, (1979).
- [35] V.P. Zahariuta, *Linear topologic invariants and their applications to isomorphic classification of generalized power spaces*, to appeared in Turkish Journal of Math.

### Doğrusal Topolojik Değişmezlerin Bir Uygulaması

#### Özet

Sonsuz türde iki Dragilev uzayının kartezyen çarpımlarının mümkün olabilen bir izomorfizmini gözönüne aldık. Zahariuta invaryantlarını ve bazı yapısal özellikleri kullanarak eğer bu şekilde bir izomorfizma varsa sol taraftaki çarpanlardan herbirinin sağ tarafta karşılık gelen bir çarpana hemen hemen izomorfik olduğunu gördük.

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