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THE HESSIAN TENSOR ON A HYPERSURFACE IN EUCLIDEAN SPACE AND OTSUKI'S LEMMA

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Abstract

The purpose of this paper is to obtain a condition for a hypersurface in Euclidean space with belongs to Hessian Tensor and is to give an alternative proof of Otsuki's lemma by applying this condition.

1. Introduction

Let M^n be n-dimensional manifold and let $h : M^n \to R$ be a differentiable function. The linear operator *Hess* $h : T_pM \to T_pM$ given by

$$(Hess\ h)Y = \nabla_Y grad\ h, \ \ Y \in T_pM,$$

is called hessian of h at $p \in M$, where ∇ is the Riammanian connection of M. If $X, Y \in T_pM$, then the Hessian Tensor of h is defined by

$$(HessH)(X,Y) = < (Hess\ h)X, Y > .$$

We observe that a Hessian Tensor has the property

$$HessH \leq 0$$

at a point of maximum of h.

2. A Condition of Hessian Tensor on a Manifold

In this section, we shall prove a lemma which give a relation between Hessian Tensor and the second fundamental form of an immersion on a hypersurface in R^{n+1}. Here we shall denote the connection on M^n by ∇ and the connection on R^{n+1} by ∇.

Lemma 2.1. *Let M^n be a n-dimensional hypersurface in R^{n+1}. Let x denote the position vector in R^{n+1} and consider the distance function $f(x) =< x, x >$ on M^n. Then at any $x \in M^n$ and for any unit vector $V \in T_xM$, the Hessian of f at x in the direction V is*

$$Hess f(V,V) = 2 < B(V,V), x > +2$$

where B denotes the second fundamental form of M^n in R^{n+1}.

Proof. Let $X \in T_xM$. For the distance function $g : R^{n+1} \to R$ which is $f = g|_M$, we have

$$< grad\ f, X > (x) = df(x)X =< grad\ g, X > (x). \tag{1}$$

Since, at x,

$$grad\ g = (grad\ g)^T + (grad\ g)^\perp,$$

by (1) we get

$$(grad\ g)^T = grad\ f$$

where $(grad\ g)^T \subset T_xM$ and $(grad\ g)^\perp \perp T_xM$. Using this fact, for $V, W \in T_xM$, at x,

$$\begin{aligned}
Hess\ f(V,W) &= < \nabla_V grad\ f, W > \\
&= < \overline{\nabla}_V grad\ f - B(grad\ f, V), W > \\
&= < \overline{\nabla}_V grad\ f, W > - < B(grad\ f, V), W > \\
&= < \overline{\nabla}_V grad\ f, W > \\
&= V < grad\ f, W > - < grad\ f, \overline{\nabla}_V W > \\
&= V < grad\ f, +grad\ g^\perp, W > - < grad\ f, \overline{\nabla}_V W > \\
&= V < grad\ g, W > - < grad\ f, \overline{\nabla}_V W > \\
&\quad + < grad\ g^\perp, \overline{\nabla}_V W > + < grad\ g^\perp, \overline{\nabla}_V W > \\
&= < \overline{\nabla}_V grad\ g, W > + < grad\ g, \overline{\nabla}_V W > \\
&\quad - < grad\ g, \overline{\nabla}_V W > + < grad\ g^\perp, \overline{\nabla}_V W > \\
&= < \overline{\nabla} grad\ g, W > + < grad\ g^\perp, \overline{\nabla}_V W > \\
&= \overline{Hess}\, g(V,W) + < grad\ g^\perp, \overline{\nabla}_V W > \\
&= \overline{Hess}\, g(V,W) + < grad\ g^\perp, B(V,W) > \\
&= \overline{Hess}\, g(V,W) + < grad\ g, B(V,W) >
\end{aligned}$$

Not, let $\alpha(t) = x + tV$ be a curve in R^{n+1}. Clearly, $\alpha(0) = x$ and $\alpha'(0) = V$. We can restrict g to the curve α and the directional derivative with respect to the vector V as

$$dg(x)V = \frac{d(go\alpha)}{dt}|_{t=0}.$$

But, since

$$\begin{aligned} go\alpha(t) &= <\alpha(t), \alpha(t)>=<x+tV, x+tV> \\ &= <x,x> +2t<x,V> +t^2<V,V> \end{aligned}$$

we have

$$\begin{aligned} <grad\ g, V>(x) &= dg(x)V\frac{d(go\alpha(t))}{dt}|_{t=0} \\ &= 2<x,V>=<2x,V> \end{aligned}$$

and we obtain

$$grad\ g(x) = 2x.$$

Hence, at x

$$Hess\ \ f(V,W) = \overline{Hess}\ \ g(V,W) + 2 < x, B(V,W) > \tag{2}$$

Now let us compute the Hessian of g at $x \in R^{n+1}$ in direction unit vector $V \in T_x R^{n+1}$.

$$\begin{aligned} \overline{Hess}\, g(V,V) &= <\overline{\nabla}_V grad\ g, V> \\ &= V<2x,V> - <grad\ g, \overline{\nabla}_V V> \end{aligned}$$

Let $\beta(x) = x + tV$ in R^{n+1}. Then

$$\begin{aligned} V<2x,V> &= V<grad\ g,V>=\frac{d}{dt}(<grad\ g,V>(\beta(t)))|_{t=0} \\ &= \frac{d}{dt}<2\beta(t),V>|_{t=0} = 2<\beta'(t),V>|_{t=0} \\ &= 2 \end{aligned}$$

Moreover,

$$<grad\ g, \overline{\nabla}_V V>= x$$

because $\beta(t)$ is a geodesic, $V(\beta)) = \beta'$. So, if we put in equation (2) these results, then lemma follows. □

3. An Alternative Proof of Otsuki's Lemma

Let M^n be a compact hypersurface in R^{n+1} and X_0 be a unit vector in T_pM such that $||B(V,V)||$ attains its minimum value for all unit vectors $V \in T_pM$. Since the normal space N_pM has dimension 1 has $B(X_0, X_0) \neq 0$ by lemma for compact manifold M^n, therefore the kernel of $B(X_0, .) : T_pM \to N_pM$ has dimension $n-1$. Hence we can write tangent space T_pM, at p of M as

$$T_pM = KerB(X_0, 0) \oplus < X_0 > .$$

Let $\eta \in N_pM, ||\eta|| = 1$. The linear operator $S_\eta : T_pM \to M_pM$ given by

$$< S_\eta(x), Y >= H_\eta(X,Y) =< (B(X,Y), \eta > \text{ for } X, Y \in T_pM$$

is symmetric. Moreover,

$$S_\eta(KerB(K_0, .)) \subset KerB(X_0, .)$$

because, if $Y \in KerB(X_0, 0)$ then $< S_\eta(Y), X_0 >= 0$, i.e. $S_\eta(Y) \perp < X_0 >$. This implies $S_\eta(Y) \in KerB(X_0, 0)$.

Let $Y_1, ..., Y_{n-1}$ be an orthonormal basis in $Ker\, B(X_0, .)$ diagonalizing $S_\eta|_{KerB(X_0),.)}$. Since $< S_\eta(X_0), X_0 > \neq 0$ we have $S_\eta(X_0) = \lambda_0 X_0$. Therefore $X_0, Y_1, ..., Y_{n-1}$ is an orthonormal basis composed of principle directions of B in T_pM.

Let η be unit normal vector in T_pM. For the orthonormal basis $X_0, Y_1, ..., Y_{n-1}$ of principal directions of B in T_pM with $||X_0|| = ||Y_i|| = 1, i = 1, 2, ..., n-1$ and the principal curvature $\lambda_0, ..., \lambda_{n-1}$ at p, we have $B(X_0, Y_i) = 0, i = 1, 2, ..., n-1$ and

$$\begin{aligned} < B(X_0, X_0), \eta > &= < S_\eta(X_0), X_0 > \\ &= < \lambda_0 X_0, X_0 > \\ &= \lambda_0. \end{aligned}$$

Hence

$$B(X_0, X_0) = \lambda_0 \eta. \tag{3}$$

Similarly, for $i = 1, 2, ..., n-1$ we find

$$B(Y_i, Y_i) = \lambda_i \eta. \tag{4}$$

If we suppose $||B(X_0, X_0)|| \leq ||B(X, X)||$ for all $X \in T_pM$ then we observe that it is possible to give an alternative proof to Otsuki's lemma as a theorem.

Theorem 3.1. *If* $B(X_0, X_0) \neq 0$ *then*

(i) $X_0 \perp KerB(X_0, .)$

(ii) *for any* $Y_i \in KerB(X_0, .)$ *we have*

$$\sum_{i=1}^{n-1} < B(X_0, X_0), B(Y_i, Y_i) > \geq \sum_{i=1}^{n-1} ||B(X_0, X_0)||^2.$$

Proof. **(i)** We saw this in the above.

(ii) Since M^n is compact, there is a maximum point p of f. Hence for any unit vector $V \in T_pM$ we have $Hess\ f(p)(V, V) \leq 0$. First we show that $\lambda_0 \lambda_i > 0, i = 1, ..., n-1$. By Lemma 2.1,

$$Hess\ f(p)(X_0, X_0) = 2 < B(X_0, X_0), p > +2 \leq 0$$

$$\Rightarrow \lambda_0 < \eta, p > = < B(X_0, X_0), p > \leq -1 \qquad \text{by (3)}$$
$$\Rightarrow \lambda_0 < \eta, p > \leq -1.$$

Similarly, $Hess\ f(p)(Y_i, Y_i) \leq 0$ with (4) implies that

$$\lambda_i < \eta, p > \leq -1 \quad 1 \leq i \leq n-1.$$

Then we have

$$\lambda_0 \lambda_i < \eta, p >^2 > 0 \quad 1 \leq i \leq n-1.$$

Hence we obtain for $i = 1, ..., n-1$

$$\lambda_0 \lambda_i > 0. \tag{5}$$

Using our assumption and (5), for some i, we have

$$\begin{aligned} < B(X_0, X_0), B(Y_i, Y_i) > &= \lambda_0 \lambda_i = |\lambda_0 \lambda_i| = |\lambda_0|\,|\lambda_i| \\ &\geq |\lambda_0|\,|\lambda_0 = \lambda_0^2 = ||B(X_0, X_0)||^2. \end{aligned}$$

The proof is verfied. □

This theorem with maximum principle leads to an important result due to Leung [4]. Let M^n be a n-dimensional compact connected hypersurface in R^{n+1} such that $M^n \subset B(r)$, where $B(r)$ denotes a closed ball centered at the origin with radius r in R^{n+1}. If for any $p \in M^n$ and for any unit vector $V \in T_pM$ we have $Ric(V, V) \leq$

$(n-1)/r^2$, then M^n must be boundary of $B(r)$. The reason of this claim is as follows. We take a point p in M^n realizing the maximum of the distance function f to the origin. The upper bound on the Ricci curvature implies that $f(p) = r$ (=radius of the ball $B(r)$) and p is umbilic. Then we show that f is subharmonic in a small neighbourhood of p. By the maximum principle, f is constant in this neighbourhood. By connectivennes of M, f is constant in M and the proof of claim is done.

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Öklid Uzayı'nda Bir Hiperyüzey Üzerinde Hessian Tensörü ve Otsuki Yardımcı Teoremi

Özet

Bu makalenin amacı, Öklid uzayında bir hiperyüzey için Hessian Tensörü'ne ait bir koşul elde etmek ve bu koşulu kullanarak Otsuki yardımcı teoreminin değişik bir ispatını vermektir.

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