Tr. J. of Mathematics
21 (1997) , 381 – 386.

INTEGRAL CLOSURE OF AN IDEAL RELATIVE TO A MODULE AND Δ-CLOSURE

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Abstract

The aim in this paper is to give the relation between the Δ-closure of an ideal I in a commutative Noetherian ring R, (see [3]), and the integral closure of the ideal I relative to a Noetherian R-module (see (1.1). Definition) and to give the closure cancellation law.

1. Introduction

The important ideas of reduction and integral closure of an ideal in a commutative Noetherian ring R (with identity) were introduced by Northcott and Rees [2]; a brief and direct approach to their theory is given in [4, (1.1)] and it is appropriate for me to begin by briefly summarizing some of the main aspects.

Let a be an ideal of R. We say that a is a reduction of the ideal b of R if $a \subseteq b$ and there exists $s \in N$ such that $ab^s = b^{s+1}$ (We use N to denote the set of positive integers.). An element x of R is said to be integrally dependent on a if there exists $n \in N$ and elements $c_1, ..., c_n \in R$ with $c_i \in a^i$ for $i = 1, ..., n$ such that

$$x^n + c_1x^{n-1} + \cdots c_{n-1}x + c_n = 0.$$

In fact, this is the case if and only if a is a reduction of $a + Rx$; moreover,

$$\overline{a} = \{y \in R : y \text{ is integrally dependent on } a\}$$

is an ideal of R, called the integral closure of a, and is the largest ideal of R which has a as a reduction in the sense that a is a reduction of $\overline{a}$ and any ideal of R which has a as a reduction must be contained in $\overline{a}$.

In [6], Sharp, Tıraş and Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R (with identity) relative to a Noetherian R-module

1991 Mathematics Subject Classification: 13A99-13B22 13B21-13E05

M, and they showed that these concepts have properties which reflect those of the classical concepts outlined in the last paragraph. Again, it is appropriate for me to provide a brief review.

Definition 1.1. *We say that I is a reduction of the ideal J of R relative to M if $I \subseteq J$ and there exists $s \in N$ such that $I \cdot J^S \cdot M = J^{S+1}M$. An element x of R is said to be integrally dependent on I relative to M if there exists $n \in N$ such that*

$$x^n \cdot M \subseteq \left(\sum_{i=1}^{n} x^{n-i} I^i \right) \cdot M.$$

In fact, this is the case if and only if I is a reduction of $I + Rx$ relative to M [6, (1.5) (iv)]; moreover, $I^- = \{y \in R : y$ is integrally dependent on I relative to $M\}$ is an ideal of R, called the integral closure of I relative to M, and is the largest ideal of R which has I as a reduction relative to M. In this paper, I shall indicate the dependence of I^- on the Noetherian R-module M by means of the extended notation $I^{-(M)}$.

The current paper is concerned with the integral closure of an ideal I of a commutative Noetherian ring R relative to M and the Δ-closure of the ideal I. Specifially, for a multiplicatively closed set Δ of non-zero ideals of a commutative Noetherian ring R, I define the Δ-closure I_Δ of an ideal I of R and prove that, if Δ is the multiplicatively closed set defined in theorem (2.4) below, then show $I_\Delta = I^{-(M)}$ and also the closure cancellation law:

$$\text{If } (IK)^{-(M)} = (JK)^{-(M)} \text{ and } K \in \Delta \text{ then } I^{-(M)} = J^{-(M)}$$

2. The Closure-Cancellation Law

Throughout R will be a Noetherian ring and M will be an non-zero finitely generated R-module. I begin with a definition which will be very useful for my aims.

Definition 2.1. *Let I be an ideal in R and Δ a multiplicatively closed set of non-zero ideals of R. The ascending chain condition guarantees that the set $\{(IKM : KM) : K \in \Delta\}$ has maximal elements, and since for K and J in Δ $(IJKM : JKM)$ contains both $(IJM : JM)$ and $(IKM : KM)$, we see that the set under consideration in fact contains a unique maximal element. Let I_Δ, Δ-closure of I, denote that unique maximal element.*

The following theorem gives some useful properties of the Δ-closure of any ideal of R.

Theorem 2.2. *Let I and J be ideals of R. Then*

a) $I \subseteq I_\Delta$

b) *If* $I \subseteq J$ *then* $I_\Delta \subseteq J_\Delta$

c) $I_\Delta J_\Delta \subseteq (IJ)_\Delta$

Proof. (a) and (b) are very clear so I omit their proof. For (c), let $x \cdot y \in I_\Delta I_\Delta$ with $x \in I_\Delta$ and $y \in I_\Delta$. Then there exist ideals K_1 and K_2 in Δ such that $x \in IK_1M : K_1M$ and $y \in JK_2M : K_2M$. Therefore $xyK_1K_2M \subseteq IJK_1K_2M$, so $xy \in (IJK_1K_2M : K_1K_2M) \subseteq (IJ)_\Delta$, so it follows that (c) holds.

Netx I give the first result, which I promised in the introductory section, in two steps. □

Theorem 2.3. *Let Δ be a multiplicatively closed set of ideals of R such that each ideal in Δ contains an element of R which is a non-zerodivisor on M. Let I_Δ be as in (2.1). Then $I_\Delta \subseteq I^{-(M)}$.*

Proof. Let $I_\Delta = (IKM : KM)$ for a suitable $K \in \Delta$ and let $x \in I_\Delta$. Suppose that KM is generated by $a_1, ..., a_n$. Then for $x \in I_\Delta$ and $1 \leq i \leq n$, we have

$$x \cdot a_i = \sum_{j=i}^{n} b_{ij} a_j \text{ with } b_{ij} \in I.$$

Now by [5, (13.15)] and since $K \in \Delta$, a standard determinant argument shows that

$$x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \in (O :_R M),$$

where $c_i \in I^i$. This means $\overline{x}$ is integrally dependent on $\overline{I}$ where "$-$" refers to the natural ring homomorphism $R \longmapsto R/O :_M$. Thus $\overline{x} \in (\overline{I})^-$, the classical integral closure of $\overline{I} \left(= \frac{I+O:_R M}{O:_R M} \right)$ in $\overline{R}$. Now the result follows from [6, (1.6)]. □

Theorem 2.4. *Let $\Delta = \{J : J$ is an ideal of R which contains a non-zerodivisor on $M\}$. Assume that $I \in \Delta$. Let I_Δ be as in (2.3).*
Then

$$I_\Delta = I^{-(M)}.$$

Proof. Let $x \in I^{-(M)}$. Then by [6, (1.5) (iv)], I is a reduction of $I + Rx$ relative to M. Then there exists $n \in N$ such that $I(I+Rx)^n = (I+Rx)^{n+1}M$.
Suppose $I_\Delta = (IKM : KM)$ for a suitable $K \in \Delta$. Then

$$x \cdot (I + Rx)^n \cdot M \subseteq I \cdot (I + Rx)^n \cdot M$$

Since $(I+Rx)^n \in \Delta$ and by the maximality of I_Δ, we get $x \in I_\Delta$. Now the result follows from (2.3). □

Theorem 2.5. *Let* Δ *and* I *be as in (2.4). Then*

$$I_\Delta = I_\Delta KM : KM \text{ for all } K \in \Delta.$$

Proof. By the definition of I_Δ and (2.4), it is readily seen that $I_\Delta KM : KM \subseteq (I_\Delta)_\Delta = (I^{-(M)})^{-(M)}$. Thus $I_\Delta KM : KM \subseteq I_\Delta$ by [6, (1.5) (ix)]. This completes the proof since the reverse is always true.

The following proposition gives another description of I_Δ and it will be used in the proof of the closure cancellation law (2.8). □

Proposition 2.6. *Let* Δ *and* I *be as in (2.4).*
Then

$$I_\Delta = I_\Delta KM : KM = (IK)_\Delta M : KM \text{ for all } K \in \Delta.$$

Proof. $I_\Delta = I_\Delta KM : KM \subseteq (IK)_\Delta M : KM$ by (2.5) and (2.2) (c). Let $x \in (IK)_\Delta M : KM$. Then $xKM \subseteq (IK)_\Delta M$. By the definition $(IK)_\Delta = IKJM : JM$ for a suitable $J \in \Delta$. Thus we get $x \in I_\Delta$. This completes the proof. □

Remark 2.7. Let Δ and I be as in (2.4). Also let "$-$" refer to the natural ring homomorphism $R \to R/O :_R M$.

$$Let \Delta' = \left\{ \overline{J} = \frac{J + O :_R M}{O :_R M} : J \in \Delta \right\}.$$

Then it is easy to see that $\overline{I}_\Delta = (\overline{I})_{\Delta'}$
From (2.6) we can easily get that

$$(\overline{I})_{\Delta'} = (\overline{I})_{\Delta'} \overline{K} M : (\overline{IK})_{\Delta'} M : \overline{K} M \text{ for all } \overline{K} \in \Delta'.$$

Now I am in the position to give the main theorem which I promised earlier:

Theorem 2.8. (Closure-cancellation law). *Let* Δ *and* I *be as in (2.4). Also let* $J \in \Delta$. *If* $(IK)^{-(M)} = (JK)^{-(M)}, K \in \Delta$, *then* $I^{-(M)} = J^{-(M)}$.

Proof. Let "$-$" and Δ' be as in (2.7)
Suppose that $(IK)^{-(M)} = (JK)^{-(M)}$.
Let $x \in I^{-(M)}$. Then by [6, (1.6)], $\overline{x} \in \overline{I}^{-(M)} = \left(\frac{I + O:_R M}{O:_R M}\right)^{-}$, the integral closure of the ideal $\overline{I}$ ind $\overline{R}$. Then, as is mentioned in the introductory section, $\overline{I}$ is a reduction of $(\overline{I} + \overline{R}\overline{x})$. Thus there exists $s \in N$ such that $\overline{I} \cdot (\overline{I} + \overline{R}\overline{x})^s = (\overline{I} + \overline{R}\overline{x})^{s+1}$.

Therefore we get

$$\overline{x}(\overline{I}+\overline{R}\overline{x})^s \subseteq \overline{I}(\overline{I}+\overline{R}\overline{x})^s.$$

Hence

$$\overline{x}\overline{K}(\overline{I}+\overline{R}\overline{x})^s M \subseteq \overline{IK}(\overline{I}+\overline{R}\overline{x})^s M \text{ for all } \overline{K} \in \Delta'.$$

Thus

$$\overline{x} \in (\overline{IK}(\overline{I}+\overline{R}\overline{x})^s M : \overline{K}(\overline{I}+\overline{R}\overline{x})^s M).$$

Since $(IK)^{-(M)} = (JK)^{-(M)}, (\overline{IK})_{\Delta'} = (\overline{JK})_{\Delta'}$ by (2.4) and (2.7). Then $\overline{x} \in \big((\overline{IK})_{\Delta'}(\overline{I}+\overline{R}\overline{x})^s M : \overline{K}(\overline{I}+\overline{R}\overline{x})^s M\big)$ by (2.2) (a). Thus $\overline{x} \in \big((\overline{JK})_{\Delta'}(\overline{I}+\overline{R}\overline{x})^s M : \overline{K}(\overline{I}+\overline{R}\overline{x})^s M\big)$. Now by (2.7) we get $x \in J_\Delta = J^{-(M)}$. Therefore it follows by symmetry that $I^{-(M)} = J^{-(M)}$ as desired.

As stronger converse is true as will be shown in the following theorem. □

Theorem 2.9. *Let* Δ, I *and* J *be as in (2.8). Then the following are equivalent:*

a) $ILM = JLM$ *for some* $L \in \Delta$

b) $(IK)^{-(M)} = (JK)^{(-(M)}$ *for all* $K \in \Delta$

c) $I^{-(M)} = J^{-(M)}$

Proof. **a)** $\rightarrow$ **b)** This is easy by (2.2) (b), (2.4) and [6, (1.5) (ix)].

b) $\rightarrow$ **c)** This is clear by (2.8).

c) $\rightarrow$ **a)** $I^{-(M)} = I_\Delta = IF_1M : F_1M = J_\Delta = J^{-(M)} = JF_2M : F_2M$ for suitable $F_1, F_2 \in \Delta$. Let $L = F_1F_2$. Then $F_1F_2 \in \Delta$ and $ILM = (ILM : LM)LM = (JLM : LM)LM = JLM$. This completes the proof. □

Acknowledgements

In am exteremely grateful to Prof. R. Y. Sharp, The University of Sheffield-England, for his valuable advice and suggestions on this work.

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Bir İdealin Bir Modüle Göre İntegral Kapanışı ve Δ-Kapanış

Özet

Bu makalede temel amaç Noetherian bir halka üzerindeki bir I idealinin [3]'de tanımlanan Δ-kapanışı ile I idealinin bir Noetherian M modülüne göre (1.1) Tanımda verilen integral kapanışı arasındaki ilişki ve ayrıca kapanış sadeleştirme kuralını vermektir.

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Received 14.10.1996