Tr. J. of Mathematics
21 (1997) , 277 – 284.

ON THE CURVES OF CONSTANT BREADTH IN E^4 SPACE

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Abstract

In this paper, the concepts concerning the space curves of constant breadth were extended to E^4-space. The integral of third curvature of the curve was obtained as $\int_0^{2\pi} \sigma ds = 2k\pi (k \in Z)$. In addition, the relation $\int_0^{2\pi} g(\kappa, \tau, \sigma) ds = 0$ was obtained between the curvatures of curves of constant breadth in E^4.

Key words and phrases: Curvature, Constant Breadth, Integral Characterization of Curve, Spherical Curves.

1. Introduction

Curves of constant breadth were introduced by L. Euler [4]. F. Reuleaux gave a method obtaining some curves of constant breadth and has found use in the kinematics of machinary [11]. In mathematics, many geometers have obtained the geometric properties of plane curves of constant breadth [3], [8], [10], [14].

Furthermore, W. Blascke defined the curve of constant breadth on the sphere [1] and M. Fujivara had obtained a problem to determine whether there exist "space curve of constant breadth" or not, an he defined "breadth" for space curces and obtained these curves on a surface of constant breadth [5]. Ö. Köse presented some concepts for space curves of constant breadth [9]. M. Sezer investigated differential equations characterizing space curves of constant breadth and gave a criterion for these curves [12]. But, the work of these papers are in E^2 or E^3. A. R. Forsyt had given the theory of curves in E^4[6].

In this paper, this kind of curves were extented to the E^4-space and some characterizations were obtained.

*AMS Subject classification number: Primary 53A04.

2. The Curves of Constant Breadth

Let $\vec{X} = \vec{X}(s)$ be a simple closed curve in E^4-space. These curves will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other then P. We call the point Q the oppsite point of P. We consider a curve in the class Γ as in [5] having parallel tangets $\vec{T}$ and $\vec{T}^*$ in opposite directions at the opposite points X and X^* of the curve. A simple closed curve of cnostant breadth having parallel tangents in opposite directions at opposite points can be represented by the quation

$$\vec{X}^*(s) = \vec{X}(s) + m_1(s)\vec{T} + m_2(s)\vec{N} + m_3(s)\vec{B} + m_4(s)\vec{E} \;\; (0 \leq s \leq L), \tag{1}$$

where $\vec{X}$ and $\vec{X}^*$ are opposite points and $\vec{T}, \vec{N}, \vec{B}, \vec{E}$ denote the Frenet - Serret frame in E^4-space [7]. We have from equation (1)

$$\begin{aligned} \frac{dX^*}{ds} &= \frac{dX^*}{ds^*}\frac{ds^*}{ds} = \vec{T}^*\frac{ds^*}{ds} = (1 + \frac{dm_1}{ds} - m_2\kappa)\vec{T} \\ &+(m_1\kappa + \frac{dm_2}{ds} - m_3\tau)\vec{N} + (\frac{dm_3}{ds} + m_2\tau - m_4\sigma)\vec{B} + (\frac{dm_4}{ds} + m_3\sigma)\vec{E}, \end{aligned} \tag{2}$$

where κ, τ and σ are that first, the second and the third curvatures of the curve, respectively [7]. Since $\vec{T}^* = -\vec{T}$, we obtain

$$\begin{aligned} 1 + \frac{dm_1}{ds} - m_2\kappa &= -\frac{ds^*}{ds} \\ m_1\kappa + \frac{dm_2}{ds} - m_3\tau &= 0 \\ \frac{dm_3}{ds} + m_2\tau - m_4\sigma &= 0 \\ \frac{dm_4}{ds} + m_3\sigma &= 0. \end{aligned} \tag{3}$$

If we call ϕ as the angle between the tangent of the curve (C) at point $X(s)$ with a given fixed direction and consider $\frac{d\phi}{ds} = \kappa$, we can rewrite equations (3) as

$$\begin{aligned} \frac{dm_1}{d\phi} &= m_2 - f(\phi) \\ \frac{dm_2}{d\phi} &= -m_1 + \rho\tau m_3 \\ \frac{dm_3}{d\phi} &= -\rho\tau m_3 + \rho\tau m_4 \\ \frac{dm_4}{d\phi} &= -\rho\sigma m_3, \end{aligned} \tag{4}$$

where $f(\phi) = \rho + \rho^* \cdot \rho = \frac{1}{\kappa}$ and $\rho^* = \frac{1}{\kappa^*}$ denote the radii of curvatures at X and X^*, respectively. If m_2, m_3, m_4 and their derivatives are eliminated in equations (4), we obtain the following equation with respect to m_1:

$$\frac{d}{d\phi}\left\{\frac{d}{d\phi}\left[\frac{1}{\rho\tau}\left(\frac{d^2m_1}{d\phi^2}+m_1\right)\right]\frac{\tau}{\sigma}\frac{dm_1}{d\phi}\right\} + \frac{\sigma}{\tau}\left(\frac{d^2m_1}{d\phi^2}+m_1\right) \tag{5}$$
$$+\frac{d}{d\phi}\left[\frac{1}{\rho\sigma}\frac{d}{d\phi}\left(\frac{1}{\rho\tau}\frac{df}{d\phi}\right)+\frac{\tau}{\sigma}f\right] + \frac{\sigma}{\tau}\frac{df}{d\phi} = 0.$$

This equation is a characterization for X^*. If the distance between the opposite points of (C) and (C^*) is constant, then

$$||X^* - X||^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 = k^2, \ k \in \Re.$$

Hence, we write

$$m_1\frac{dm_1}{d\phi} + m_2\frac{dm_2}{d\phi} + m_3\frac{dm_3}{d\phi} + m_4\frac{dm_4}{d\phi} = 0. \tag{6}$$

By considering system (4),

$$m_1\left(\frac{dm_1}{d\phi} - m_2\right) = 0. \tag{7}$$

Thus, we write $m_1 = 0$ or $\frac{dm_1}{d\phi} = m_2$. If $\frac{dm_1}{d\phi} = m_2$ then $f(\phi) = 0$. In this case, (C^*) is translated by the constant vector

$$\vec{l} = m_1\vec{T} + m_2\vec{N} + m_3\vec{B} + m_4\vec{E} \tag{8}$$

of (C). We write from system (4)

$$f(\phi) = 0, \ m_3 = \frac{m_1}{\rho\tau}, \ \frac{dm_3}{d\phi} = \rho\sigma m_4, \ \frac{dm_4}{d\phi} = -\rho\sigma m_3 \tag{9}$$

by letting $m_1 =$ *constant* $= c$ and $m_2 = 0$. The change of variable $t(\phi) = \int_0^\phi \rho\sigma d\phi$ gives

$$\frac{d^2m_3}{dt^2} + m_3 = 0$$

General solution for $m_3(t)$ is

$$m_3 = A\cos\int_0^\phi \rho\sigma dt + B\sin\int_0^\phi \rho\sigma dt$$

[15]. If we consider m_3, then

$$m_4 = -A\sin\int_0^{\phi}\rho\sigma dt + B\cos\int_0^{\phi}\rho\sigma dt,$$

where A and B are arbitrary constants. Therefore, the general solution set of the system (9) is

$$\begin{aligned}\{m_1 = c, m_2 = 0, m_3 &= A\cos\int_0^{\phi}\rho\sigma dt + B\sin\int_0^{\phi}\rho\sigma dt,\\ m_4 &= -A\sin\int_0^{\phi}\rho\sigma dt + B\cos\int_0^{\phi}\rho\sigma dt\}.\end{aligned} \tag{10}$$

Thus, equation (1) becomes

$$\begin{aligned}\vec{X}^* = \vec{X} + c\vec{T} &+ (A\cos\int_0^{\phi}\rho\sigma dt + B\sin\int_0^{\phi} dt)\vec{B}\\ &+ (-A\sin\int_0^{\phi}\rho\sigma dt + B\cos\int_0^{\phi}\rho\sigma dt)\vec{E}.\end{aligned} \tag{11}$$

The distance between the opposite point of these curves is $\sqrt{c^2 + A^2 + B^2}$. Since $m_3 = \frac{m_1}{\rho\tau} = \frac{c}{\rho\tau} = c\frac{\kappa}{\tau}$, we obtain

$$\frac{\kappa}{\tau} = A_1\cos\int_0^{\phi}\rho\sigma dt + B_1\sin\int_0^{\phi}\rho\sigma dt, \tag{12}$$

where $A_1 = A/C$ and $B_1 = B/C$.

In the case $m_1 = 0$, we write from equations (4)

$$\begin{aligned} m_1 = f(\phi), &\quad f(\phi) \neq 0\\ \frac{dm_2}{d\phi} &= \rho\tau m_3\\ \frac{dm_3}{d\phi} &= -\rho\tau m_2 + \rho\sigma m_4\\ \frac{dm_4}{d\phi} &= -\rho\sigma m_3.\end{aligned} \tag{13}$$

Let us consider system (13) with $\lambda = \rho\tau$, $\mu = \rho\sigma$ and $u = \int_0^{\phi}\mu(t)dt$. Hence, we obtain the differential equation

$$\frac{d^2 m_3}{du^2} + m_3 = -\frac{d}{du}\left(\frac{\lambda}{\mu} m_2\right). \tag{14}$$

General solution of (14) is

$$m_3 = F_1(\phi) - \int_0^{\phi} \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt, \tag{15}$$

where

$$F_1(\phi) = A_2 \cos \int_0^{\phi} \rho\sigma dt + B_2 \sin \int_0^{\phi} \rho\sigma dt \tag{16}$$

[2]. From equations (13) and (15), we write

$$m_4 = F_2(\phi) + \int_0^{\phi} \sin[u(t) - u(\phi)]\rho(t)\tau(t)f(t)dt, \tag{17}$$

where

$$F_2(\phi) = -A_2 \sin \int_0^{\phi} \rho\sigma dt + B_2 \cos \int_0^{\phi} \rho\sigma dt.$$

Therefore, the general solution set of system (9) is

$$\begin{aligned} \{m_1 = 0, m_2 = f(\phi), m_3 &= F_1(\phi) - \int_0^{\phi} \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt, \\ m_4 &= F_2(\phi) + \int_0^{\phi} \sin[u(t) - u(\phi)]\rho(t)\tau(t)f(t)dt\}. \end{aligned} \tag{18}$$

Consequently, we obtain by using (18) in (1) the curve (C^*) as

$$\begin{aligned} \vec{X}^* &= \vec{X} + \{\int_0^{\phi} \rho\tau[F_1(\phi) - \int_0^{\phi} \cos[u(\phi) - u(t)]\rho(t)\tau(t)f(t)dt]d\phi\}\vec{N} \\ &+\{F_1(\phi) - \int_0^{\phi} \cos[u(\phi) - u(t)\rho(t)\tau(t)f(dt)dt\}\vec{B} \\ &+\{F_2(\phi) + \int_0^{\phi} \sin[u(t) - u(\phi)\rho(t)\tau(t)f(t)dt\}\vec{E}. \end{aligned} \tag{19}$$

If we write $m_1 = 0$ in (5),

$$\frac{d}{d\phi}\left[\frac{1}{\rho\sigma}\frac{d}{d\phi}\left(\frac{1}{\rho\tau}\frac{df}{d\phi}\right)+\frac{\tau}{\sigma}f\right]+\frac{\sigma}{\tau}\frac{df}{d\phi}=0. \tag{20}$$

By means of transformation of the independent variable ϕ of the form $w=\int_0^\phi \rho\tau d\phi$, we rewrite (20) $(\tau\neq 0,\rho\neq 0)$

$$\frac{d}{dw}\left\{\frac{\tau}{\sigma}\left[\frac{d^2f}{dw^2}+f\right]\right\}+\frac{\sigma}{\tau}\frac{df}{dw}=0. \tag{21}$$

The function f of (19) satisfies (21). Since the curves of constant breadth is simple closed we have the fact that $X^*(0)=X^*(2\pi)$, $0\leq\phi\leq 2\pi$. Therefore, we give the following results.

Corollary 1. *Let (C^*) be a curve in E^4-space, such that $\kappa>0,\tau$ and σ are continuous periodic functions. If (C^*) is a curve of constant breadth, then*

$$\int_0^{2\pi}\sigma ds=2k\pi,\quad k\in Z \tag{22}$$

Corollary 2. *For the curves of constant breadth in E^4-space*

$$\begin{aligned}\int_0^{2\pi}\cos[u(2\pi)-u(t)]\rho(t)\tau(t)f(t)dt &= 0,\\ \int_0^{2\pi}\sin[u(t)-u(2\pi)]\rho(t)\tau(t)f(t)dt &= 0.\end{aligned}$$

Corollary shows that, the curvatures of the curves of constant breadth in E^4-space satisfy

$$\int_0^{2\pi}g(\kappa,\tau,\sigma)ds=0. \tag{23}$$

With respect to this result, if $\int_0^{2\pi}\cos[u(2\pi)-u(t)]\rho(t)\tau(t)f(t)dt=0$ or $\int_0^{2\pi}g(\kappa,\tau,\sigma)ds=0$ is used in (19), we can write

$$f^2+\{A_2\cos\int_0^\phi\rho\sigma dt+B_2\sin\int_0^\phi\rho\sigma dt\}^2+\{-A_2\sin\int_0^\phi\rho\sigma dt+B_2\cos\int_0^\phi\rho\sigma dt\}^2=C^2.$$

A curve satisfying this condition lies on an E^4 sphere of radius C [13]. Thus $\int_0^{2\pi}g(\kappa,\tau,\sigma)ds=0$ characterizes that the curve of constant breadth lies on E^4 sphere.

Corollary 3. *If $\frac{\tau}{\sigma}$ is a constant in equation (21), we write*

$$a\frac{d}{dw}\left[\frac{d^2f}{dw^2}+f\right]+\frac{1}{a}\frac{df}{dw}=0$$

or

$$\frac{d^3f}{dw^3}+K^2\frac{df}{dw}=0,\ K^2=\frac{1+a^2}{a^2},\ K\neq\mp1. \tag{24}$$

From the equation (24), we obtain

$$f=A_3\sin\int_0^{\phi}K\rho\tau d\phi+B_3\cos\int_0^{\phi}K\rho\tau d\phi+D, \tag{25}$$

where A_3, B_3 and D are constants [14]. In this case, the curvature of the (C^*) is

$$\rho^*=-\rho+A_3\sin\int_0^{\phi}K\rho\tau d\phi+B_3\cos\int_0^{\phi}K\rho\tau d\phi+D.$$

We find from (25) for the curves of constant breadth that

$$\int_0^{2\pi}\tau ds=2\frac{k}{K}\pi,\ \ k\in Z.$$

The final equality shows that total torsion of (C^*) is a constant. From corollary 3, if $\frac{\tau}{\sigma}=a$ (constant) then we write $\int_0^{2\pi}\tau ds=a\int_0^{2\pi}\sigma ds=2k\pi,\ k\in Z$.

$\int_0^{2\pi}\tau ds=2k\pi,\ k\in Z$ was obtained for the curve of constant breadth on E^3 [9]. That is, corallary 1 generalize this, on the other hand this result shows that the integral of third curvature of such a curve depends only on an integer k.

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E^4 Uzayında Sabit Genişlikli Eğriler Üzerine

Özet

Bu çalışmada, sabit genişlikli eğriler E^4 uzayına genişletilerek sabit genişlikli bir eğrinin toplam üçüncü eğriliği $\int_0^{2\pi} \sigma ds = 2k\pi$ olarak elde edildi. Ayrıca sabit genişlikli eğrilerin eğriliklerinin $\int_0^{2\pi} g(\kappa, \tau, \sigma) ds = 0$ eşitliğini sağladıkları gösterildi.

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Received 25.9.1995
Revised 1.7.1997