Tr. J. of Mathematics
21 (1997) , 285 - 294.


# ON A GENERALISATION OF LIE IDEALS IN PRIME RINGS*

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**Abstract**

Let $R$ be a prime ring of characteristic 3, $\sigma$ and $\tau$ automorphisms of $R$, $U$ a nonzero $(\sigma,\tau)$-Lie ideal of $R, d$ a nonzero derivation of $R$ such that $\sigma d = d\sigma, \tau d = d\tau, d(U) \subseteq U$, and $d^2(U) \subseteq Z$, the center of $R$. Then we prove that $U \subseteq Z$. This provides a proof of the Theorem in [4], when char $R = 3$.

## 1. Introduction

Let $R$ be a ring, $U$ an additive subgroup of $R$, and $\sigma$ and $\tau$ two automorphisms of $R$. For each pair $x, y$ in $R$, we set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$ and $Z_{\sigma,\tau} = \{r \in R | r\sigma(x) = \tau(x)r$, for all $x \in R\}$. $U$ is called a $(\sigma,\tau)$-Lie ideal of $R$ if both $[U,R]_{\sigma,\tau}$ and $[R,U]_{\sigma,\tau}$ are in $U$.

In this paper, we will prove the following theorem, which will provide a proof of [4, Theorem] when char $R = 3$.

**Theorem.** *Let $R$ be a prime ring of characteristic 3, $\sigma$ and $\tau$ two automorphisms of $R, U$ a nonzero $(\sigma,\tau)$-Lie ideal of R. If $Z$ is the center of $R$ and $d$ is a nonzero derivation of $R$ such that $\sigma d = d\sigma, \tau d = d\tau, d(U) \subseteq U$, and $d^2(U) \subseteq Z$, then $U \subseteq Z$.*

In what follows, $R, Z, U, \sigma, \tau$ and $d : x \longmapsto x'$ have the same meaning as in the above theorem.

We need the following lemmas.

**Lemma 1.** *Let $S$ be a prime ring of characteristic not 2, $D : x \longmapsto x'$ a nonzero derivation of $S$ with $D^3 = 0$, $C$ the center of $S$, and $A$ a nonzero subset of $S$ such that $A'' \subseteq C$.*

*(a) If $A$ is a right ideal of $S$, then $S'' \subseteq C$ or $A'A = 0$.*

*AMS subject classification (1991 revision): Primary: 16 W 25, Secondary: 16 U 80. This work has been supported by TÜBİTAK, Ankara (TBAG-Ç2).

*(b) If $A$ is an ideal of $S$, then $S'' \subseteq C$.*

**Proof.** (a) Assume that $S'' \not\subseteq C$. For any $a \in A$ and $s \in S, a''$ and $a''s'' = (as'')''$ are central, so $A'' = 0$ and hence

$$0 = (as')'' = a's''. \tag{1}$$

Replacing $s$ by $s'r$ in (1) we get $a's'r'' = 0$ and substituting $ar'$ for $a$ in (1) and using the last equation we obtain $ar''s'' = 0$ for all $a \in A,\ r,\ s \in S$. Since $A$ is a right ideal, follows that $r''s'' = 0$. Taking $ra$ in place of $r$ here and noting that $a's'' = 0$, one has $r''As'' = 0$ for all $r, s \in S$. Thus, $r''A = 0$, so $0 = (ra)''b = r'a'b$ and hence

$$0 = (rs)'a'b = r'sa'b \text{ for all } r, s, \in S, a, b, \in A.$$

Therefore $A'A = 0$.

(b) Suppose that $S'' \not\subseteq C$. We have seen in part (a) that $r''A = 0$ for all $r \in S$, which implies that $S'' = 0$, a contradiction. □

**Lemma 2.** *Let $S$ be a prime ring with center $C$ and $D : x \longmapsto x'$ a nonzero derivation of $S$.*

*(a) If $r''S's'' = 0$ for all $r, s \in S$, then there is no $u \in S$ such that $0 \neq u'' \in C$.*

*(b) Let $A$ be a nonzero ideal of $S$, char $S \neq 2$, and $D^3 = 0$. If $A''b = 0$ for some $b \in S$, then either there is no $u \in S$ such that $0 \neq u'' \in C$ or $b = 0$.*

**Proof.** (a) If there were $u \in S$ such that $0 \neq u'' \in C$, then by hypothesis we would get

$$0 = u''S'u'' = u''S'Su'' = u''S' = u''SS' = S',$$

contradicting $D \neq 0$.

(b) For any $r, s, t \in S, a \in A, 0 = (r'a)''b = r''a'b$ and hence $0 = r''(ts''a)'b = r''t's''ab$. Thus, either $r''S's'' = 0$ for all $r, s \in S$ or $Ab = 0$. The former case implies, by part (a), that there is no $u \in S$ such that $0 \neq u'' \in C$. The latter case forces $b$ to be zero. □

**Lemma 3.** *Let $S$ be a ring with center $C$ and $D : x \longmapsto x'$ a nonzero derivation of $S$ with $D^3 = 0$. Let $b \in S$ and $c \in C$. Suppose that*

$$[[s', b'], b'] + c[s', b'] = 0 \text{ for all } s \in S, \tag{2}$$

*then*

*(a) $2[s'', b']^2 = 0$ for all $s \in S$.*

*(b) If $S''$ is commutative and*

$$[r'', [s'', t']] = 0 \text{ for all } r, s, t \in S, \tag{3}$$

*then*

$$[r'', [r'', b]]b'' = 0 \text{ for all } r \in S.$$

*If $b''$ is not a zero-divisor and $S$ is semiprime, then $2S'' \subseteq C$.*

**Proof.** (a) By (2), we write

$$[[s'', b], b'] + c[s'', b'] = 0 \text{ for all } s \in S. \tag{4}$$

For this and (2) we get

$$0 = [[(r's'')', b'], b'] + c[(r's'')', b']$$

$$= r''([[s'', b'], b'] + c[s'', b']) + ([[r'', b'], b'] + c[r'', b'])s''$$

$$+2[r'', b'][s'', b'],$$

so

$$2[r'', b'][s'', b'] = 0 \text{ for all } r, s \in S.$$

In particular, $2[s'', b']^2 = 0$ for all $s \in S$.

(b) Now by (3) and by the commutativity of $S''$ we have

$$0 = [r'', [r'', (st')']] = 2[r'', s'][r'', t'] + [r'', [r'', s]]t'' \tag{5}$$

for all $r, s, t \in S$. We substitute $b$ for $s$ and $t$ in the last equation and employ part (a) to obtain

$$[r'', [r'', b]]b'' = 0 \text{ for all } r \in S.$$

If $b''$ is not a zero-divisor, then $[r'', [r'', b]] = 0$ for all $r \in S$. Next, by (3) and by the commutativity of $S''$ we can write

$$0 = [r'', [r'', (s'b)']] = 2[r'', s'][r'', b'] \text{ for all } r, s \in S.$$

Hence (5) implies that $[r'', [r'', s]] = 0$. Thus

$$0 = [r'', [r'', st]] = 2[r'', s]][r'', t],$$

so,

$$0 = 2[r'', st][r'', s] = 2[r'', s]t[r'', s] \text{ for all } r, s, t \in S.$$

Since $S$ is semiprime we obtain $2[r'', s] = 0$ for all $r, s \in S$, that is, $2S'' \subseteq C$. □

After these general lemmas, we consider our ring $R$, and collect some facts in a lemma for convenience.

**Lemma 4.** *If char $R \neq 2$ and $U \not\subseteq Z$, then*

*(a) $U''$ and $Z$ are not zero.*

*(b) $[R, U]_{\sigma,\tau}, U'$, and $R''$ are not contained in $Z$.*

*(c) There exists a nonzero ideal $M$ of $R$ such that*

$$[R, M]_{\sigma,\tau} \subseteq U \quad \textit{but} \quad [R, M]_{\sigma,\tau} \not\subseteq Z_{\sigma,\tau}.$$

*(d) $[U, a] = 0$ implies that $a \in Z$ for each $a \in R$.*

**Proof.** (a) follows from [3, Theorem 3.5.7] and the fact that $U'' \subseteq Z$, (b) follows from Lemmas 1 and 2 in [4] and from Theorem 3 in [2], and (c) and (d) follow from Lemma 3.5.2, Theorems 2.6.7 and 2.6.8 in [3]. □

In the sequel, $M$ will stand for the ideal guaranteed by Lemma 4(c).

We note the important identities in $R$.

$$[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z),$$

$$[xy, z]_{\sigma,\tau} = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y.$$

**Lemma 5.** *Let char $R = 3$ and $U \not\subseteq Z$.*

*(a) If $U'' = 0$, then $d^3 = 0$.*

*(b) If $d^3 = 0$, then $Z' = 0$.*

**Proof.** (a) It is well-known ([1]) that $d^3$ is also a derivation of $R$, so $[r, u]_{\sigma,\tau} \in U$ implies that $[r''', u)_{\sigma,\tau} = [r, u]'''_{\sigma,\tau} = 0$, that is $[r''', u]_{\sigma,\tau} = 0$ for all $r \in R, u \in U$. Replacing $r$ by $rv, v \in U$, here we obtain $r'''[v, \sigma(u)] = 0$ and replacing $r$ by $rs, s \in R$, we get

$$r'''s[v, \sigma(u)] = 0 \text{ for all } r, s \in R,\ u, v \in U.$$

Since $R$ is prime, either $d^3 = 0$ or $[U, \sigma(U)] = 0$. The latter implies, by Lemma 4($d$), that $U \subseteq Z$, a contradiction.

(b) We first show that $Z'' = 0$. Suppose not. For any $r \in R, m \in M$, we have

$$Z \ni [r, m]''_{\sigma,\tau} = [r'', m]_{\sigma,\tau} + 2[r', m']_{\sigma,\tau} + [r, m'']_{\sigma,\tau}. \tag{6}$$

Replacing $r$ by $rc', c \in Z$, and noting that $d^3 = 0$ we obtain $[r,m]'_{\sigma,\tau}c'' \in Z$ and since $Z'' \neq 0$, it follows that $[r,m]'_{\sigma,\tau} \in Z$. Again by replacing $r$ by $rc', c \in Z$, here we have $[r,m]_{\sigma,\tau} \in Z$ for all $r \in R, m \in M$. We now substitute $r\sigma(m)$ for $r$ in the last relation and obtain $[r,m]_{\sigma,\tau}\sigma(m) \in Z$ which implies that either $[r,m]_{\sigma,\tau} = 0$ for all $r \in R$ or $\sigma(m) \in Z$. We set

$K = \{m \in M | [r,m]_{\sigma,\tau} = 0 \text{ for all } r \in R\}$ and $L = \{m \in M | \sigma(m) \in Z\}$. Then $M = K \cup L$, so by Brauer's Trick, either $M = K$ or $M = L$. In the letter case, $M \subseteq Z$ which forces $R$ to be commutative, since a prime ring having a commutative one-sided ideal must be commutative. This is impossible because $U \not\subseteq Z$. In the former case, $[R,M]_{\sigma,\tau} = 0 \subseteq Z_{\sigma,\tau}$, a contradiction to the choice of $M$. Thus $Z'' = 0$.

Next, we show that $Z' = 0$. Suppose not. For all $r \in R, u \in U$,

$$Z \ni [r,u]''_{\sigma,\tau} = [r'',u]_{\sigma,\tau} + 2[r',u']_{\sigma,\tau} + [r,u'']_{\sigma,\tau} \tag{7}$$

and for any $c \in Z$,

$$Z \ni [rc,u]''_{\sigma,\tau} = ([r'',u]_{\sigma,\tau} + 2[r',u']_{\sigma,\tau} + [r,u'']_{\sigma,\tau})c + 2[r,u]'_{\sigma,\tau}c'$$

imply that

$$[r,u]'_{\sigma,\tau} \in Z, \text{ and so } Z \ni [rc,u]'_{\sigma,\tau} = [r,u]'_{\sigma,\tau}c + [r,u]_{\sigma,\tau}c'$$

gives us that $[R,U]_{\sigma,\tau} \subseteq Z$, which contradicts Lemma 4(b). Therefore $Z' = 0$. □

**Lemma 6.** *Let char $R = 3, d^3 = 0$, and $U \not\subseteq Z$. Then*

*(a) $\sigma(c) = \tau(c)$ for all $c \in Z$, hence $[R,Z]_{\sigma,\tau} = 0$.*

*(b) $[Z,M'']_{\sigma,\tau} = 0$.*

**Proof.** By Lemma 5(b), $Z' = 0$.

(a) Replacing $r$ by $c \in Z$ in (6) we get

$$[c,m'']_{\sigma,\tau} \in Z \quad \text{for all} \quad c \in Z, \quad m \in M. \tag{8}$$

Since $Z \neq 0$ by Lemma 4(a), it follows that $\sigma(m'') - \tau(m'') \in Z$ for all $m \in M$. Hence

$$\sigma(m''c) - \tau(m''c) = \sigma((mc)'') - \tau((mc)'') \in Z$$

implies that $\sigma(m'')(\sigma(c) - \tau(c)) \in Z$. Thus either $M'' \subseteq Z$ or $\sigma(c) - \tau(c) = 0$ for all $c \in Z$. The former case, together with Lemma 1(b), yields a contradiction to $R'' \not\subseteq Z$ (see Lemma 4(b)). Thus $\sigma(c) = \tau(c)$ for all $c \in Z$, from which we deduce immediately that $[R,Z]_{\sigma,\tau} = 0$.

(b) By (8)

$$[c, m'']_{\sigma,\tau}(\tau(m'') + \sigma(m'')) = [c, (mm'')'']_{\sigma,\tau} \in Z,$$

so either $[c, m'']_{\sigma,\tau} = 0$ for all $c \in Z$ or $\sigma(m'') + \tau(m'') + \tau(m'') \in Z$. Suppose that $[c_0, m_0'']_{\sigma,\tau} \neq 0$ for some $c_0 \in Z$ and $m_0 \in M$. Then

$$\sigma(m_0'') + \tau(m_0'') \in Z. \tag{9}$$

Also by (8), $[c_0, m_0'']_{\sigma,\tau} \in Z$, which implies that $c_0(\sigma(m_0'') - \tau(m_0'')) \in Z$. Since $c_0 \neq 0$, it follows that $\sigma(m_0'') - \tau(m_0'') \in Z$. This, together with (9), gives us that $m_0'' \in Z$ and hence by part (a), $[c_0, m_0'']_{\sigma,\tau} = 0$, a contradiction. Therefore, $[Z, M'']_{\sigma,\tau} = 0$. □

**Lemma 7.** *If char* $R = 3, d^3 = 0$, *and* $U \not\subseteq Z$, *then*

*(a)* $[r, s'']_{\sigma,\tau} = [r, \sigma(s'')]$ *for all* $r, s \in R$.

*(b)* $\sigma(r'') = \tau(r'')$ *for all* $r \in R$.

**Proof.** (a) By Lemma 6(b),

$$0 = [c, m'']_{\sigma,\tau} = c(\sigma(m'') - \tau(m'')) = \sigma(m'') - \tau(m'')$$

for all $c \in Z$ and $m \in M$. Hence $[r, m'']_{\sigma,\tau} = [r, \sigma(m'')]$. Thus,

$$[r, (ms'')'']_{\sigma,\tau} = [r, \sigma((ms'')'')]$$

which implies that

$$\sigma(m'')([r, s'']_{\sigma,\tau} - [r, \sigma('')]) = 0 \text{ for all } r, s \in R,\ \ m \in M.$$

We fix $r$ and $s$ and set $b = \sigma^{-1}([r, s'']_{\sigma,\tau} - [r, \sigma(s'')])$. Then $M''b = 0$. Since $0 \neq U'' \subseteq Z$, it follows from Lemma 2(b) that $b = 0$.

(b) Substitution $0 \neq c \in Z$ for $r$ in part (a) yields immediately the required result. □

**Lemma 8.** *Let char* $R = 3, d^3 = 0$, *and* $U \not\subseteq Z$. *Then*

*(a)* $R''$ *is commutative.*

*(b)* $[r_0'', u_0']_{\sigma,\tau} \neq 0$ *for some* $r_0 \in R$, $u_0 \in U$.

**Proof.** (a) By (6) and by Lemma 7(a), $[r'', \sigma(m'')] = [r'', m'']_{\sigma,\tau} \in Z$, so $[r'', m''] \in Z$ for all $r \in R$, $m \in M$. Hence,

$$[r', (mr'')''] = [r'', m'']r'' \in Z,$$

which implies that either $[r'', m''] = 0$ for all $m \in M$ or $r'' \in Z$. In either case, $[r'', m''] = 0$ for all $r \in R$ and $m \in M$. Thus,

$$0 = [r'', (ms'')''] = [r'', m''s''] = m''[r'', s''], r, s \in R, \; m \in M.$$

Since $0 \neq U'' \subseteq Z$, it follows from Lemma 2 (b) that $[r'', s''] = 0$.

(b) Suppose, to the contrary, that $[r'', u']_{\sigma,\tau} = 0$ for all $r \in R$, $u \in U$. Since $U'' \subseteq Z$, $[R, U'']_{\sigma,\tau} = 0$ by Lemma 6(a). So,

$$Z \ni [r''\sigma(u), u]''_{\sigma,\tau} = [r'', u]_{\sigma,\tau}\sigma(u'') \text{ for all } r \in R, \; u \in U.$$

Hence either $u'' = 0$ or $[r'', u]_{\sigma,\tau} \in Z$ for all $r \in R$. Let $K = \{u \in U | u'' = 0\}$ and $L = \{u \in U | [r'', u]_{\sigma,\tau} \in Z \text{ for all } r \in R\}$ then $U = K \cup L$ and Brauer's Trick shows that either $U = K$ or $U = L$. The former case is impossible since $U'' \neq 0$ by Lemma 4(a). Thus, $U = L$ and hence $[r'', u]_{\sigma,\tau} \in Z$ for all $r \in R$, $u \in U$. Now we deduce from (7) and $[R, U'']_{\sigma,\tau} = 0$ that $[r', u]_{\sigma,\tau} \in Z$. Hence,

$$Z \ni [(r''s)', u']_{\sigma,\tau} = r''[s', \sigma(u')], \text{ that is, } r''[s', u'] \in Z \tag{10}$$

for all $r, s \in R, u \in U$. Since $U'' \neq 0$, letting $r$ be in $U$ we get $[s', u'] \in Z$, so that (10) implies that either $R'' \subseteq Z$, which contradics Lemma 4(b), or $[s', u'] = 0$ for all $s \in R$ and $u \in U$, which yields that $U' \subseteq Z$, contradicting Lemma 4(b) again. Therefore, $[r''_0, u'_0]_{\sigma,\tau} \neq 0$ for some $r_0 \in R$ and $u_0 \in U$. □

**Lemma 9.** *Let char* $R = 3$ *and* $d^3 = 0$. *Then* $U \subseteq Z$.

**Proof.** Suppose, to the contrary, that $U \not\subseteq Z$. We first show that $\sigma(u') = \tau(u')$ for all $u \in U$. Recalling here the fact that $[R, Z]_{\sigma,\tau} = 0$ and that $\sigma(r'') = \tau(r'')$ for all $r \in R$ (see Lemmas 6 and 7) we get

$$Z \ni [u, v']''_{\sigma,\tau} = u''(\sigma(v') - \tau(v')) \text{ for all } u, v \in U.$$

Hence

$$\sigma(v') - \tau(v') \in Z \text{ for all } v \in Z.$$

On the other hand, since $U' \subseteq U$ we can write

$$Z \ni [r, u']''_{\sigma,\tau} = [r'', u']_{\sigma,\tau}. \tag{11}$$

We fix $u$ and set $c = \sigma(u') - \tau(u')$. Then

$$Z \ni [(vs'')'', u']_{\sigma,\tau} = v''[s'', \sigma(u')] + cv''s'' \text{ for all } s \in R, \; v \in U.$$

Since $R''$ is commutative by Lemma 8, it follows that

$$0 = [r'', v''[s'', \sigma(u')] + cv''s''] = v''[r'', [s'', \sigma(u')]],$$

that is,

$$[r'', [s'', \sigma(u'')]] = 0 \text{ for all } r, s \in R, \; u \in U. \tag{12}$$

We fix $s$ and $u$ and set $a = [s'', \sigma(u')]$ in (12), then $[r'', a] = 0$ for all $r \in R$ and

$$0 = [(y\sigma(u')'', a] = y''[\sigma(u'), a] + 2[y', a]\sigma(u''),$$

that is, $y''[\sigma(u'), a] = [y', a]\sigma(u'')$. Substituting $\sigma(u')y$ for $y$ here we obtain

$$\sigma(u'')([y, a]\sigma(u'') + [\sigma(u'), a]y' + y'[\sigma(u'), a]) = 0 \text{ for all } y \in R.$$

Let $\sigma(u'') \neq 0$. Then

$$[y, a]\sigma(u'') + [\sigma(u'), a]y' + y'[\sigma(u'), a] = 0 \text{ for all } y \in R. \tag{13}$$

Since, for any $r \in R$,

$$[r, u']_{\sigma,\tau} = [r, \sigma(u')] + cr, \tag{14}$$

it is easy to see, by (11), that $[\sigma(u'), a] = ca$. Hence, by (13)

$$\sigma(u'')[y, a] + cay' + cy'a = 0.$$

Taking $y'$ in place of $y$ and recalling that $ay'' = y''a$, we have

$$2cay'' + \sigma(u'')[y', a] = 0 \quad \text{or} \quad cay'' = \sigma(u'')[y', a].$$

Now, substituting $[s'', \sigma(u')]$ for $a$ and employing (14) and (11) one obtains

$$\begin{aligned} cy''[s'', \sigma(u')] &= \sigma(u'')[s'', \sigma(u')]] = \sigma(u'')[y', (s'', u']_{\sigma,\tau} - cs''] \\ &= -c\sigma(u'')[y', s''], \end{aligned}$$

hence

$$c(y''[s'', \sigma(u')] + \sigma(u'')[y', s'']) = 0. \tag{15}$$

Let $c \neq 0$, then

$$y''[s'', \sigma(u')] + \sigma(u'')[y', s''] = 0 \text{ for all } y, s \in R.$$

Commuting this with $r''$ and using the fact that $R''$ is commutative we obtain, by (12), that $\sigma(u'')[r'',[s'',y']]=0$, and since $\sigma(u'')\neq 0$, it follows that

$$[r'',[s'',y']]=0 \text{ for all } r,s,y\in R. \tag{16}$$

Furthermore, $[R,Z]_{\sigma,\tau}=0$ by Lemma 6(a), so

$$Z\ni [r''\sigma(u),u]''_{\sigma,\tau}=[r'',u]_{\sigma,\tau}\sigma(u'')-[r'',u']_{\sigma,\tau}\sigma(u').$$

We commute this element with $\sigma(u')$ and use (11) to obtain $[[r'',u]_{\sigma,\tau},\sigma(u')]\sigma(u'')=0$ for all $r\in R$, and since $u''\neq 0$ we get $[[r'',u]_{\sigma,\tau},\sigma(u')]=0$. Set $b=\sigma(u)$. Since $[R,U'']_{\sigma,\tau}=0$, it follows from $[r,u]''_{\sigma,\tau}\in Z$ that $[r'',u]_{\sigma,\tau}-[r',u']_{\sigma,\tau}\in Z$. Hence, by (14),

$$0=[[r'',u]_{\sigma,\tau}-[r',u']_{\sigma,\tau},b'][[r',u']_{\sigma,\tau},b']=[[r',b']+cr',b'],$$

which implies that $[[r',b'],b']+c[r',b']=0$ for all $r\in R$. This, together with (16), and Lemma 3(b) give us that $R''\subseteq Z$, which contradicts Lemma 4(b). Consequently, when $\sigma(u'')\neq 0$ in (13), then $c$ in (15) must be zero, that is, for each $u\in U$, either $u''=0$ or $\sigma(u')=\tau(u')$. Let $K=\{u\in U|u''=0\}$ and $L=\{u\in U|\sigma(u')=\tau(u')\}$. Then $U=K\cup L$ and by Brauer's Trick, either $U=K$, which gives us a contradiction: $U''=0$, or $U=L$, which means that $\sigma(u')=\tau(u')$ for all $u\in U$, as claimed.

We can now complete the proof of this lemma. By (11),

$$Z\ni [r'',u']_{\sigma,\tau}=[r'',\sigma(u')],\ \text{that is, } [r'',u']\in Z$$

for all $r\in R, u\in U$, so

$$Z\ni [(rr'')'',u']=2r''[r'',u'],\ \text{hence } r''[r'',u']\in Z.$$

This means that either $r''\in Z$ or $[r'',u']=0$. In either case, $[r'',u']=0$ for all $r\in R$, $u\in U$. Hence $[r'',u']_{\sigma,\tau}=[r'',\sigma(u')]=0$ for all $r\in R$, $u\in U$. This contradicts Lemma 8(b). Therefore $U\subseteq Z$. □

We are now ready to prove our theorem.

**Proof of the Theorem.** Suppose, to the contrary, that $U\not\subseteq Z$. By Lemmas 5 and 9, it is sufficient to show that $U'''=0$. But this follows from the proof of [4, Theorem]. We give that proof here for convenience. For any $r\in R$, $u$, $v\in U$, $[r,v]''_{\sigma,\tau}\in Z$ implies that

$$Z\ni [ru'',v]''_{\sigma,\tau}=[r,v]_{\sigma,\tau}u^{(iv)}-([r',v]_{\sigma,\tau}+[r,v']_{\sigma,\tau})u''.$$

Hence, from

$$Z \ni [rw'', v]_{\sigma,\tau} u^{(iv)} - ([(rw'')', v]_{\sigma,\tau} + [rw'', v']_{\sigma,\tau}) u'',$$

it follows that $u'''w'''[r, v]_{\sigma,\tau} \in Z$ for all $u, v, w \in U, r \in R$. So, if $U''' \neq 0$, then $[R, U]_{\sigma,\tau} \in Z$, contradicting Lemma 4 (b). $U''' = 0$.

**Acknowledgement.**

The author thanks to the referee for his remarks.

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## Asal Halkalarda Lie İdeallerin Bir Genelleşmesi Üzerine

**Özet**

Bu çalışmada, $R$ karakteristiği 3 ve merkezi $Z$ olan asal bir halka, $\sigma$ ve $\tau$ $R$ nin iki otomorfizması, $U$ $R$ nin sıfırdan farklı bir $(\sigma, \tau)$-Lie ideali, $d$ $R$ nin $d\sigma = \sigma d$, $d\tau = \tau d, d(U) \subseteq U, d^2(U) \subseteq Z$ olarak şekilde bir türemesi olmak üzere $U \subseteq Z$ olduğu gösterilmiştir.

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Received 6.7.1995
Revised 8.11.1996