Tr. J. of Mathematics
21 (1997) , 303 – 309.

PAINLEVÉ ANALYSIS AND INFINITE LIE SYMMETRIES OF THE COMPLEX MODIFIED KORTEWEG-DE VRIES-II EQUATION

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Abstract

The Painlevé analysis developed by Weiss et al. [9] for nonlinear partial differential equations is applied to the CMKdV-II equation. It has been shown that this equation passes the Painlevé test. By specializing the arbitrary functions that appear in the *phi* series expansions found in the test, one obtains a system of partial differential equations for the formally arbitrary data. For specific systems, and conjectured in general, these equations are integrable. The form of the resulting reduction enables the identification of integrable reductions of the original systems. Assuming $u_i = v_i = 0$, $i \geq 2$ we obtain conditions to have truncated series solutions. Then the data obtained by this truncation technique are used to develop some analytical solutions and infinite dimensional Lie symmetries of the equation.

1. Introduction

Quasilinear parabolic equations, or nonlinear reaction-diffusion systems arise in the modeling of phenomena in physics, chemistry, biology and other applied sciences. The complex Korteweg-de Vries-I equation (CMKdV-I)

$$w_t + \alpha(|w|^2 w)_x + \beta w_{xxx} = 0 \tag{1}$$

and the complex modified Korteweg-de Vries-II equation (CMKdV-II)

$$w_t - 6|w|^2 w_x + w_{xxx} = 0 \tag{2}$$

are among such equations. The CMKdV-I equation (1) arises in the asymptotic interpretation of one dimensional plane wave propagation in quadratic micropolar medium and was derived for the first time in [2]. Integrability and other properties of this equation has been studied elsewhere [5]. In this work, we investigate the CMKdV-II equation (2),

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which represents the interaction of two waves and, to our knowledge, appeared frist in [3].

Since the formulation of the Painlevé tests, there has been considerable interest in using the Painlevé property as a means of determining whether given differential equations, both partial and ordinary, are integrable. To apply the test to a partial differential equation we use the theory of complex functions with several complex variables.

The major difference between analytic functions of one complex variable and several complex variables is that, in general, the singularities of a function of several complex variables can not be isolated if $f = f(z_1, .., z_n)$ is a meromorphic function of n complex variables ($2n$ real variables), where the singularities of f occur along analytic manifolds of (real) dimension $2n - 2$. These manifolds are determined by conditions of the form

$$\phi(z_1, ..., z_n) = 0, \tag{3}$$

where ϕ is an analytic function of $(z_1, ..., z_n)$ in a neighborhood of the manifold.

With reference to the above, we say that *a partial differential equation has the Painlevé property when the solutions of the PDE are single valued about the movable, singularity manifolds.* For differential equations we require that the solution be a *single-valued functional* of the data, i.e. *arbitrary functions.* This is a formal property and not a restriction on the data itself.

To verify if a PDE has the Painlevé property we introduce a method for expanding a solution of a nonlinear PDE about a movable, singular manifold (3).

Let $u = u(z_1, ..., z_n)$ be a solution of the PDE and assume that

$$u = \phi^p \sum_{j=0}^{\infty} u_j \phi^j \tag{4}$$

where ϕ and $u_j = u_j(z_1, ..., z_n)$ are analytic functions of $(z_1, ..., z_n)$ in a neighborhood of manifold (3). Substitution of (4) into the PDE determines the possible values of p and defines the recursion relations for $u_j, j = 0, 1, 2, \ldots$. When p is a negative integer and (4) is a valid and general expansion about the manifold (3), then the solution has a single valued representation about (3). If this representation is valid for all allowed movable singularity manifolds, then the PDE has the Painlevé property. For a specific PDE it is necessary to identify all possible values for p and then find what form of the resulting *phi* series [4] becomes.

A point that will be emphasized is that the *phi* series for a nonlinear PDE contains a lot of information about the PDE. For the equations which have the Painlevé property, a method has been developed for finding Lie symmetries, Lax pairs and Bäcklund transformations [5-8, 10-12]. An outline and an application of the *singular manifold method* is presented in the next section. For equations that do not have the Painlevé property, it is still possible to obtain single valued expansions. This specialization leads to a system of partial differential equations for the formally arbitrary data. For specific systems, and conjectured in general, these equations are integrable. The form of the resulting reduction enables the identification of integrable reductions of the original systems.

Now we apply the Painlevé test to the complex modified Korteweg-de Varies equation-II (2).

Painlevé Analysis for CMKdV-II Equation

Since $|w|$ in (2) brings some difficulty in the calculations, we first let $w = u + iv$ and separate the real and imaginary parts in (2) and obtain the system

$$\begin{aligned} u_t - 6(u^2 + v^2)u_x + u_{xxx} &= 0, \\ v_t - 6(u^2 + v^2)v_x + v_{xxx} &= 0. \end{aligned} \tag{5}$$

Let $\phi(x,t) = 0$ be the solution singularity manifold of (3) and

$$\begin{aligned} u &= \phi^p \sum_{j=0}^{\infty} u_j \phi^j . \\ v &= \phi^p \sum_{j=0}^{\infty} v_j \phi^j . \end{aligned} \tag{6}$$

Substituting (6) into (5), we have $p = -1$,

$$u_0^2 + v_0^2 = \phi_x^2; \tag{7}$$

$$u_0 u_1 + v_0 v_1 = \frac{1}{2}\phi_{xx}. \tag{8}$$

The resonance points are $j = -1, 0, 1, 3, 4$ and $j = 5$. Clearly the resonance point $j = -1$ corresponds to the free singularity manifold function $\phi(x,t)$. At the resonance $j = 0$, we have the relation (7), hence u_0 or v_0 is arbitrary. For $j = 1$, we have the relation (8) and one of the u_1 or v_1 is arbitrary. We have the same for $j = 3, 4$ and $j = 5$. To write a general solution to this system, one needs six arbitrary functions for which we have enough arbitrary functions. Therefore the system of quasilinear partial differential equations (5) passes the Painlevé test for PDE's.

In [5] it has been shown that the CMKdV-I equation (1) has resonances at $j = -1, 0, 3$ and $j = 4$. At these resonances one has only the five arbitrary functions $\phi(t,x), v_0(t,x), u_3(t,x), v_3(t,x), v_4(t,x)$. To write down a general solution to (1) one needs six arbitrary functions. Therefore the CMKdV-I equation (1) does not pass the Painlevé test for PDE's.

Truncated Solution of the CMKdV-II Equation

Let us truncate the series in (6) at the second term and assume that $u_j = v_j = 0, j \neq 2$. Then

$$\begin{aligned} u &= \frac{u_0}{\phi} + u_1, \\ v &= \frac{v_0}{\phi} + v_1. \end{aligned} \tag{9}$$

If we let $u_2 = v_2 = 0$ we get

$$\begin{aligned} 6(u_1^2 + v_1^2)u_0\phi_x - 6\phi_x^2 u_{1x} + 3(\phi_{xx}u_{0x} - \phi_x u_{0xx}) - u_0(\phi_{xxx} + \phi_t) &= 0, \\ 6(u_1^2 + v_1^2)v_0\phi_x - 6\phi_x^2 v_{1x} + 3(\phi_{xx}v_{0x} - \phi_x v_{0xx}) - v_0(\phi_{xxx} + \phi_t) &= 0, \end{aligned} \tag{10}$$

while $u_3 = v_3 = 0$ implies

$$\begin{aligned} -6(u_1^2 + v_1^2)u_{0x} + 6\phi_{xx}u_{1x} + u_{0xxx} + u_{0t} &= 0, \\ -6(u_1^2 + v_1^2)v_{0x} + 6\phi_{xx}v_{1x} + v_{0xxx} + v_{0t} &= 0. \end{aligned} \tag{11}$$

On the other hand, $u_4 = v_4 = 0$ gives

$$\begin{aligned} u_{1t} - 6(u_1^2 + v_1^2)u_{1x} + u_{1xxx} &= 0, \\ v_{1t} - 6(u_1^2 + v_1^2)v_{1x} + v_{1xxx} &= 0, \end{aligned} \tag{12}$$

i.e. u_1, v_1 must be a solution of the original system of PDE's (5). Therefore relation (9) can be taken as an auto-Bäcklund transformation that relates two solutions u, v and u_1, v_1 of (5) if ϕ, u_0, v_0 and u_1, v_1 interrelated by (7) and (8) satisfy (10), (11) and (12).

Hence for a given solution u_1, v_1 solving (10) and (11) for $u_0(t,x), v_0(t,x)$ and $\phi(t,x)$, one obtains new particular solutions of the CMKdV-II equation through (9).

Although this is a nonlinear system of partial differential equations and it's solution is even more difficult than the original one, we are interested in only some particular solutions.

Infinite Symmetries by Truncated Expansions

It is shown in [6] that the CMKdV-II equation (5) admits Lie point symmetries with the infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t},$$

$$X_3 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v},$$

and

$$X_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

It is interesting to note that the CMKdV-I equation (1) has completely the same finite dimensional Lie point symmetry group above as the CMKdV-II equation (2) [5].

Now following [8], find an infinite dimensional symmetry of CMKdV-II equation (5). Let us rewrite the system (5) in the form

$$\begin{aligned} u_t &= F(u,v) = -u_{xxx} + 6(u^2+v^2)u_x \\ v_t &= G(u,v) = -v_{xxx} + 6(u^2+v^2)v_x. \end{aligned} \tag{13}$$

To find the symmetries, we linearize (13) about a solution u, v;

$$\begin{aligned} w_{1t} &= \frac{\partial}{\partial \varepsilon}F(u+\varepsilon w_1, v+\varepsilon w_2)|_{\varepsilon=0} \\ &= -w_{1xxx} + 12(uw_1 + vw_2)u_x + 6(u^2+v^2)w_{1x} \\ w_{2t} &= \frac{\partial}{\partial \varepsilon}G(u+\varepsilon w_1, v+\varepsilon w_2)|_{\varepsilon=0} \\ &= -w_{2xxx} + 12(uw_1 + vw_2)v_x + 6(u^2+v^2)w_{2x} \end{aligned} \tag{14}$$

Any solution w_1, w_2 of (14) yields a symmetry or infinitesimal transformation about u, v. That is the transformation

$$\begin{aligned} u^2 &= u + \varepsilon w_1 \\ v^2 &= v + \varepsilon w_2 \end{aligned} \tag{15}$$

leaves system (5) form-invariant [8].

Considering equations in (9) and (11), we observe that u_0, v_0 is the solution of the linearized system (14) about the solution u_1, v_1 of the original system (5). Since for each of the infinitely many solutions ϕ, one has a transformation

$$\begin{aligned} u^* &= u_1 + \varepsilon w_0 \\ v^* &= v_1 + \varepsilon w_0, \end{aligned} \tag{16}$$

which is admitted by system (5), we obtain further evidence for the integrability of the CMKdV-II equation (2).

Discussion

Painlevé analysis provides a new and powerful tool for constructing explicit solutions for nonintegrable as well as integrable dynamical systems. But it only gives possible solutions so one must check the results if they are actual solutions of the given nonlinear partial differential equation. On the other hand, in general, the necessary calculations are too tedious to do by hand. In those cases one needs to use some computer algebra systems like REDUCE or MATHEMATICA.

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Kompleks Modifiye Korteweg-de Varies-II Denkleminin Painlevé Analizi ve Sonsuz Lie Simetrileri

Özet

Bu çalışmada Weiss, Tabor ve Carnevale'ın [9] lineer olmayan kısmi diferansiyel denklemler için geliştirdikleri Painlevé analizi KMKdV-II denklemine uygulanmış ve bu denklemin Painlevé testini geçtiği gösterilmiştir. Test sırasında elde edilen *phi* serisi açılımında ortaya çıkan keyfi fonksiyonlara özel değerler verilmekle daha önce keyfi olan fonksiyonlar cinsinden, genellikle integre edilebilir oldukları gözlenen kısmi diferansiyel denklem sistemleri elde edilmektedir. Burada $u_i = v_i = 0, i \geq 2$ varsayılarak kesik seri şeklinde çözüm elde etme şartları bulunmuş ve bunlar sayesinde KMKdV-II denkleminin bazı özel analitik çözümleri ile sonsuz Lie simetrilerinin nasıl elde edilebileceği gösterilmiştir.

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Received 6.12.1995