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CERTAIN MEROMORPHICALLY STARLIKE FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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Abstract

In this paper we consider the class $\Sigma S^*_{o,c}(\alpha)$ consisting of meromorphically starlike univalent functions with positive coefficients and fixed second coefficients. The object of the present paper is to show coefficient estimates and closure theorems for this class. Also, we obtain the radius of convexity for functions belonging to the class $\Sigma S^*_{o,c}(\alpha)$.

Key words: Meromorphic, univalent, starlike.

1. Introduction

Let $\Sigma_{p,o}$ denote the class of functions of the form

$$f(z) = \frac{a_o}{z} + \sum_{n=1}^{\infty} a_n z^n (a_o > 0; a_n \geq 0) \tag{1.1}$$

which are regular and univalent in the punctured disc $U^* = \{z : 0 < |z| < 1\}$. Denote by $\Sigma S_0^*(\alpha)$ the class of functions $f(z) \in \Sigma_{p,o}$ which satisfy the conditions

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for $z \in U^*$ and $0 \leq \alpha < 1$. The functions in $\Sigma S_o^*(\alpha)$ is clearly meromorphic starlike univalent of order α in U^*. The class $\Sigma S_o^*(\alpha)$ was introduced by Mogra [2]. We note that for $a_o = 1$, the class $\Sigma S_o^*(\alpha) = \Sigma_M^*(\alpha)$, was studied by Juneja and Reddy [1].

We begin by recalling the following lemma due to Mogra [2].

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Lemma 1. *Let the function $f(z)$ be defined by (1.1). Then f(z) is in the class $\Sigma S_o^*(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}[(n+\alpha)a_n] \leq (1-\alpha)a_o. \tag{1.3}$$

In view of (1.3), we can see that the functions $f(z)$ defined by (1.1) in the class $\Sigma S_o^*(\alpha)$ satisfy the coefficient inequality

$$a_1 \leq \frac{(1-\alpha)}{(1+\alpha)}a_o. \tag{1.4}$$

Hence we may take

$$a_1 = \frac{(1-\alpha)a_o c}{(1+\alpha)}, \ \ 0 \leq c \leq 1. \tag{1.5}$$

Making use of (1.5), we now introduce the following class of functions:

Let $\Sigma S_{o,c}^*(\alpha)$ denote the subclass of $\Sigma S_o^*(\alpha)$ consisting of functions of the form

$$f(z) = \frac{a_o}{z} + \frac{(1-\alpha)a_o c}{(1+\alpha)}z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.6}$$

where

$$a_o > 0, a_n \geq 0 \text{ and } 0 \leq c \leq 1.$$

In this paper we obtain coefficient inequalities for the class $\Sigma S_{o,c}^*(\alpha)$ and closure theorems. Further, the radius of convexity is obtained for the class $\Sigma S_{o,c}^*(\alpha)$. Techniques used are similar to those of Silverman and Silvia [3] and Uralegaddi [4].

2. Coefficient Inequalities

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z)$ is in the class $\Sigma S_{o,c}^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}[(n+\alpha)a_n] \leq (1-\alpha)a_o(1-c). \tag{2.1}$$

The result is sharp.

Proof. Putting

$$a_1 = \frac{(1-\alpha)a_o c}{(1+\alpha)}, \ \ 0 \leq c \leq 1, \tag{2.2}$$

in (1.3) and simplifying we get the result. The result is sharp for the function

$$f(z) = \frac{a_o}{z} + \frac{(1-\alpha)a_o c}{(1+\alpha)} z + \frac{(1-\alpha)a_o(1-c)}{(n+\alpha)} z^n, \ (n \geq 2). \tag{2.3}$$

□

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $\Sigma S^*_{o,c}(\alpha)$. Then*

$$a_n \leq \frac{(1-\alpha)a_o(1-c)}{(n+\alpha)} \ (n \geq 2). \tag{2.4}$$

The result is sharp for the function $f(z)$ given by (2.3).

Corollary 2. *If $0 \leq c_1 \leq c_2 \leq 1$, then*

$$\Sigma S^*_{o,c_2}(\alpha) \subseteq \Sigma S^*_{o,c_1}(\alpha).$$

3. Closure Theorems

Using Theorem 1, we can prove the following Theorems:

Theorem 2. *Let the functions*

$$f_j(z) = \frac{a_{o,j}}{z} + \frac{(1-\alpha)a_{a,j}c}{(1+\alpha)} z + \sum_{n=2}^{\infty} a_{n,j} z^n (a_{o,j} > 0, a_{n,j} \geq 0) \tag{3.1}$$

be in the class $\Sigma S^*_{o,c}(\alpha)$ for every $j = 1, 2, ..., m$. Then the function

$$g(z) = \frac{b_o}{z} + \frac{(1-\alpha)b_o c}{(1+\alpha)} z + \sum_{n=2}^{\infty} b_n z^n (b_o > 0; b_n \geq 0) \tag{3.2}$$

is also in the same class $\Sigma S^*_{o,c}(\alpha)$, where

$$b_o = \frac{1}{m} \sum_{j=1}^{m} a_{o,j} \text{ and } b_n = \frac{1}{m} \sum_{j=1}^{m} a_{n,j} (n = 1, 2, ...). \tag{3.3}$$

Theorem 3. *The class $\Sigma S^*_{o,c}(\alpha)$ is closed under convex linear combination.*

4. Radius of Convexity

Theorem 4. *Let the function $f(z)$ defined by (1.6) be in the class $\Sigma S^*_{o,c}(\alpha)$. Then $f(z)$ is meromorphically convex of order $\rho(0 \leq \rho < 1)$ in $0 < |z| < r_1 = r_1(\alpha, c, \rho)$, where $r_1(\alpha, c, \rho)$ is the largest value for which*

$$\frac{(3-\rho)(1-\alpha)c}{(1+\alpha)} r^2 + \frac{n(n+2-\rho)(1-\alpha)(1-c)}{(n+\alpha)} r^{n+1} \leq 1-\rho \tag{4.1}$$

for $n \geq 2$. The result is sharp for the function

$$f_n(z) = \frac{a_o}{z} + \frac{(1-\alpha)a_o c}{(1+\alpha)} z + \frac{(1-\alpha)a_o(1-c)}{(n+\alpha)} z^n \quad \textit{for some } n. \tag{4.2}$$

Proof. It is sufficient to show that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1-\rho (0 \leq \rho < 1) \text{ for } 0 < |z| < r_1(\alpha, c, \rho).$$

Note that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq \frac{\frac{2(1-\alpha)a_o c}{(1+\alpha)} r^2 + \sum_{n=2}^{\infty} n(n+1)a_n r^{n+1}}{a_o - \frac{(1-\alpha)a_o c}{(1+\alpha)} r^2 - \sum_{n=2}^{\infty} n\, a_n r^{n+1}} \leq 1-\rho \tag{4.3}$$

for $0 < |z| < r$ if and only if

$$\frac{(3-\rho)(1-\alpha)a_o c}{(1+\alpha)} r^2 + \sum_{n=2}^{\infty} n(n+2-\rho)a_n r^{n+1} \leq (1-\rho)a_o. \tag{4.4}$$

Since $f(z)$ is in the class $\Sigma S^*_{o,c}(\alpha)$, from (2.4) we may take

$$a_n = \frac{(1-\alpha)a_o(1-c)\lambda_n}{(n+\alpha)} \quad (n \geq 2), \tag{4.5}$$

where $\lambda_n \geq 0$ $(n \geq 2)$ and

$$\sum_{n=2}^{\infty} \lambda_n \leq 1. \tag{4.6}$$

For each fixed r, we choose the positive integer $n_o = n_o(r)$ for which $\frac{n(n+2-\rho)}{(n+\alpha)} r^{n+1}$ is maximal. Then it follows that

$$\sum_{n=2}^{\infty} n(n+2-\rho)a_n r^{n+1} \leq \frac{n_o(n_o+2-\rho)(1-\alpha)a_o(1-c)}{(n_o+\alpha)} r^{n_o+1}. \tag{4.7}$$

Then $f(z)$ is convex of order ρ in $0 < |z| < r_1(\alpha, c, \rho)$ provided that

$$\frac{(3-\rho)(1-\alpha)a_o c}{(1+\alpha)} r^2 + \frac{n_o(n_o+2-\rho)(1-\alpha)a_o(1-c)}{(n_o+\alpha)} r^{n_o+1} \leq (1-\rho)a_o. \tag{4.8}$$

We find the value $r_o = r_o(\alpha, c, \rho)$ and the corresponding integer $n_o(r_o)$ so that

$$\frac{(3-\rho)(1-\alpha)c}{(1+\alpha)} r_o^2 + \frac{n_o(n_o+2-\rho)(1-\alpha)(1-c)}{(n_o+\alpha)} r_o^{n_o+1} = 1-\rho. \tag{4.9}$$

Then this value r_o is the radius of meromorphically convex of order ρ for functions belonging to the class $\Sigma S^*_{o,c}(\alpha)$. □

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Özet

Başlıkta adı geçen sınıf için katsayı kestirimleri, kapanış teoremleri elde edilmiştir.

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