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# QUOTIENTS OF REAL ALGEBRAIC SETS VIA FINITE GROUPS

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**Abstract**

In this paper, we will study finite algebraic group actions on real algebraic sets and compare the topological quotient $X/G$ with the algebraic quotient $X//G$. We will give a different and shorter proof of a result of Procesi and Schwarz, stating that if the order of the group $G$, acting algebraically on a real algebraic set $X$, is odd then $X/G$ is equal to $X//G$. In the case of even order groups, we will a give sufficient condition (and a necessary condition in the case $G = \mathbb{Z}_2$) for the $X/G$ to be equal to $X//G$.

## 1. Introduction and Preliminaries

The problem of real algebraic realization of topological or smooth objects has been studied by many authors. In [11], Seifert showed that any closed smooth submanifold $M \subseteq \mathbb{R}^n$ with trivial normal bundle is isotopic to a nonsingular component of a real algebraic set $X \subseteq \mathbb{R}^n$. Nash showed that every closed smooth manifold is diffeomorphic to a component of a nonsingular real algebraic set in some $\mathbb{R}^N$ ([7]). Later, Tognoli proved that any closed smooth manifold is diffeomorphic to a nonsingular real algebraic set in some $\mathbb{R}^N$ ([13]). In [2, 3] Akbulut and King improved Tognoli's result by showing that any closed smooth submanifold of $\mathbb{R}^n$ can be isotoped to the nonsingular points of an algebraic subset of $\mathbb{R}^n$. Dovermann and Masuda showed that, in some cases, smooth manifolds with group actions can be realized as equivariant nonsingular algebraic sets ([6]).

In this paper, we will study finite algebraic group actions on real algebraic sets and compare the topological quotient $X/G$ with the algebraic quotient $X//G$. If the order of the group $G$, acting algebraically on a real algebraic set $X$ is odd, then $X/G$ is canonically equal to $X//G$ (Theorem 2.1). When the order $|G|$ is even, in general, this is not true (see the counterexample after Theorem 2.1). In the case of even order groups, we will a give sufficient condition (and a necessary condition in the case $G = \mathbb{Z}_2$) for the $X/G$ to be equal to $X//G$ (Theorem 2.2 and Theorem 2.3).

In [9] Procesi and Schwarz had proved Theorem 2.1.a in the case of linear $G$ actions. However, the proof we give is shorter and does not require linear $G$ actions.

**Definition 1.1.** *1) Let $X \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^m$ be semialgebraic sets. A map $F : X \longrightarrow Z$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_n]$, $i = 1, \dots, m$, such that each $g_i$ vanishes nowhere on $X$ and*

$$F = (f_1/g_1, \dots, f_m/g_m).$$

*We say $X$ and $Z$ are isomorphic if there are entire rational maps $F : X \longrightarrow Z$ and $G : Z \longrightarrow X$ such that $FoG = id_Z$ and $GoF = id_X$.*

*2) Let $X \subseteq \mathbb{R}^n$ be a semialgebraic set and $G$ be a finite group acting on $X$. Then $G$ is said to be acting algebraically on $X$, if for each $g \in G$ the map $g : X \longrightarrow X$, $x \longmapsto g \cdot x$ is the restriction of some polynomial map $P_g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.*

**Remark.** By the last theorem in Section 9 in [6], in the case that $X$ is an algebraic set, any algebraic $G$ action on $X$ is equivalent to a linear action if we are willing to replace $X$ with an isomorphic copy of it possibly in a larger Euclidean space. Nevertheless, our proofs work in the polynomial case and hence we will assume that the action is given by polynomial maps.

Let $X$ be an algebraic set in $\mathbb{R}^n$. Suppose that $G$ is a finite group acting algebraically on $X$. Let $S = \mathbb{R}[x_1, \dots, x_n]/J(X)$ be the ring of polynomial functions on $X$ where $J(X) \subseteq \mathbb{R}[x_1, \dots, x_n]$ is the ideal of vanishing polynomials on $X$. We define a $G$ action on $S$ as follows: for $g \in G$ and $f \in S$, let $g \cdot f = fog^{-1}$. Consider the subring $T$ of $S$ defined by

$$T = S^G = \{f \in S \mid g \cdot f = f, \ \forall g \in G\}.$$

Both $T$ and $S$ are $\mathbb{R}$ algebras. Moreover, it is well known that $S$ is an algebraic extension of $T$ and therefore $T$ is also a finitely generated $\mathbb{R}$ algebra (Exercise 5.12 and Proposition 7.8 in [4]). Say $T$ is generated by $y_1, \dots, y_m$ over the reals. Consider the complexification $T_\mathbb{C}$ and $S_\mathbb{C}$ of $T$ and $S$ defined by

$$T_\mathbb{C} = T \otimes_\mathbb{R} \mathbb{C} \quad \text{and} \quad S_\mathbb{C} = S \otimes_\mathbb{R} \mathbb{C}.$$

Clearly, these are finitely generated $\mathbb{C}$ algebras and $S_\mathbb{C}$ is the ring of polynomial functions on the complexification $X_\mathbb{C}$ of $X$. The above action of $G$ on $S$ extends to $S_\mathbb{C}$, linearly over $\mathbb{C}$. With this definition of $G$ action, immediately, we have that $S_\mathbb{C}^G = T_\mathbb{C}$.

Consider the maps $F_\mathbb{C} : X_\mathbb{C} \longrightarrow \mathbb{C}^m$ and $F = F_{\mathbb{C}|X} : X \longrightarrow \mathbb{R}^m$, where both are given by

$$x \longrightarrow (y_1(x), \dots, y_m(x)).$$

If $Z$ is the complex algebraic set corresponding to the $\mathbb{C}$ algebra $T_\mathbb{C}$ (i.e. the embedding of the maximal spectrum of $T_\mathbb{C}$ into $\mathbb{C}^m$ via $F_\mathbb{C}$), then $F_\mathbb{C} : X_\mathbb{C} \longrightarrow Z$ is the map corresponding to the inclusion of $\mathbb{C}$ algebras $i : T_\mathbb{C} \longrightarrow S_\mathbb{C}$ and $F_\mathbb{C}(X_\mathbb{C}) = Z$.

Let $Y_0 = F(X) \subseteq \mathbb{R}^n$ and $Y$ denote the Zariski closure of the semialgebraic set $Y_0$ in $\mathbb{R}^m$. We endow $Y_0$ with its subspace topology and $X/G$ with the quotient topology. Now we state a well known result which we will need later. We refer the reader to Section 1.3 in [5] or Chapter 1 in [8] or [10] for a proof this lemma.

**Lemma 1.2.** $F : X \longrightarrow Y_0$ induces a homeomorphism $\overline{F} : X/G \longrightarrow Y_0$. Moreover, if $X$ is nonsingular and the $G$ action on $X$ is free, then $Y_0$ is a subset of nonsingular points of its Zariski closure $Y$ and the induced map $\overline{F} : X/G \longrightarrow Y_0$ becomes a diffeomorphism.

**Lemma 1.3.** $Y = Z \cap \mathbb{R}^m$ or $Z = Y_{\mathbb{C}}$.

**Proof.** Since $Y_0 = F(X) \subseteq F_{\mathbb{C}}(X_{\mathbb{C}}) \cap \mathbb{R}^m = Z \cap \mathbb{R}^m$ we see that $Y \subseteq Z \cap \mathbb{R}^m \subseteq Z$. Hence, $Y_{\mathbb{C}} \subseteq Z$, where $Y_{\mathbb{C}}$ is the complexification of the real algebraic set $Y$. Also, since $X \subseteq {F_{\mathbb{C}}}^{-1}(F_{\mathbb{C}}(X)) \subseteq {F_{\mathbb{C}}}^{-1}(Y) \subseteq {F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})$, we have $X_{\mathbb{C}} \subseteq {F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})$. Therefore, $Z = F_{\mathbb{C}}(X_{\mathbb{C}}) \subseteq F_{\mathbb{C}}({F_{\mathbb{C}}}^{-1}(Y_{\mathbb{C}})) \subseteq Y_{\mathbb{C}} \subseteq Z$ and thus $Y_{\mathbb{C}} = Z$ and $Y = Z \cap \mathbb{R}^m$. □

This is nothing but the first paragraph of the proof of Proposition 2.10.3 in [1]. In the literature, $Y$ is usually denoted by $X//G$, the algebraic quotient of $X$ by $G$.

## 2. Results

**Theorem 2.1.** *Let $X$ be an irreducible real algebraic set and $G$ is a finite group acting algebraically on $X$. Let $Y$ be the Zariski closure of $Y_0$. Then:*

***a)*** *if $|G|$ is odd, then $Y_0 = Y$; i.e. $Y_0$ is algebraic;*

***b)*** *if $G$ acts freely on $X$ and $X$ is nonsingular, then $Y_0$ is a union of topological components of $Nonsing(Y)$ ($G$ might have even order);*

***c)*** *if $|G|$ is odd, $X$ is nonsingular and $G$ acts freely on $X$, then $Y_0$ is a nonsingular algebraic set.*

**Proof.** First let us prove (a): Assume that the conclusion of the theorem is not true. So there exists a point $p \in Y - Y_0$. Let $\{q_1, \dots, q_l\}$ be the preimage of $p$ under $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$. Since $G$ acts transitively on the fibers $l = |G|/Stab_G(q_1)$. But $G$ is of odd order and hence $l$ is an odd number. However, since $p \in Y - Y_0$ none of the $q_i$'s is contained in the real part of $X_{\mathbb{C}}$, because all the real points of $X_{\mathbb{C}}$ are sent to $Y_0$. Moreover, since $F_{\mathbb{C}}$ is a real polynomial map, and $X_{\mathbb{C}}$ is defined over the reals, the complex conjugation of $\mathbb{C}^n$ preserves the fibers of $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ over real points. In particular, it preserves the fiber over $p$. So $l$ should be an even number, which is a contradiction. Hence, $Y = Y_0$.

For part (b) consider the map $F : X \longrightarrow Y$. Since this map is a local diffeomorphism at each point of $X$ (Lemma 1.2) and $\dim(X)=\dim(Y)$ we have that $Y_0$ is an open subset of $Nonsing(Y) \subseteq Y$. The map $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ is a closed mapping ([8] or Corollary A on page 49 in [12]). Now since $X \subseteq X_{\mathbb{C}}$ is closed, $Y_0 = F_{\mathbb{C}}(X)$ is closed in $Y_{\mathbb{C}}$ and hence in $Y$. So $Y_0$ is a union of topological components of $Nonsing(Y)$. Finally, part (c) follows from parts (a) and (b). □

**Remark.** The following example shows that Theorem 2.1.a. does not hold for groups of even order. Let $X$ be the zero set of the irreducible polynomial $x^4 + y^4 - 1$ in $\mathbb{R}^2$. Then, $X$ is a nonsingular irreducible algebraic set diffeomorphic to the unit circle $S^1$. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the involution of $\mathbb{R}^2$ given by

$$f((x,y)) = (-x,-y).$$

Note that $G = \{id, f\}$ acts freely on $X$. Let $S$,$T$ be as in Section 1. Then,

$$S = \mathbb{R}[x,y]/(x^4 + y^4 - 1) \quad \text{and} \quad T = \mathbb{R}[x^2, y^2, xy]/(x^4 + y^4 - 1).$$

The map $F : X \longrightarrow \mathbb{R}^3$ is given by $(x,y) \longrightarrow (x^2, y^2, xy)$, and, as before, let us denote its image by $Y_0$, which is a smooth manifold diffeomorphic to $S^1$. The Zariski closure $Y$ of $Y_0$ in $\mathbb{R}^3$ is given by the polynomial equations $t_1^2 + t_2^2 - 1 = 0$ and $t_1 t_2 - t_3^2 = 0$. In $Y_0$ the first two coordinates are always non negative, whereas in $Y$ these two coordinates can be negative. Actually, $Y_0 \cup Y_0' = Y$ (Example 2 after Theorem 2.2), where $Y_0' = \{(t_1, t_2, t_3) \mid (-t_1, -t_2, -t_3) \in Y_0\}$ and $Y_0 \cap Y_0' = \emptyset$. Therefore, $Y_0$ can not be a Zariski open set and thus $Y_0 = F(X)$ is not algebraic.

**Even order group case:** To be able to get a result similar to Theorem 2.1 in the case of even order groups we will assume that $X$ is a nonsingular real algebraic set and the $G$ action on $X$ is free. First let us consider the case where $G = \mathbb{Z}_2 =< g >$. By Theorem 2.1.b $Y_0$ is a union of the components of $Nonsing(Y)$. Let us look at $Y$ carefully and see when $Y_0$ is an algebraic set or a Zariski open subset of $Y$.

The points in $Y - Y_0$ are coming from the non real points of $X_{\mathbb{C}}$. So, the preimage of any point in $Y - Y_0$ under $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ consists of two complex conjugate points in $X_{\mathbb{C}} - X$. Let $W$ be the set of the points in all such fibers. Note that $W$ is nothing but the subset of $X_{\mathbb{C}}$ on which complex conjugation and $g$ agree. So it is a real algebraic subset of $X_{\mathbb{C}}$.

$G$ acts freely on $X$ and complex conjugation acts trivially on $X$ and thus $X \cap W = \emptyset$. Since $F_{\mathbb{C}} : X_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ is a finite to one map, the semialgebraic set $F_{\mathbb{C}}(W) = Y - Y_0$ has the same dimension as $W$. So, $\dim(W) \leq \dim(Y) = k$. If $\dim(W) \leq k - 1$, then $Y - Y_0$ is contained in $Sing(Y)$ and therefore $Y_0$ is a Zariski open subset of $Y$. If $W = \emptyset$ then $Y_0 = Y$ and hence is algebraic in $\mathbb{R}^m$. So we have proved the following theorem.

**Theorem 2.2.** *Let $G = \mathbb{Z}_2 =< g >$ act freely on a $k$ dimensional nonsingular real algebraic set $X \subseteq \boldsymbol{R}^n$. Let $W$ be as above, then $Y = Y_0 \cup F^{\mathbb{C}}(W)$. In particular,*

***a)*** *if $W = \emptyset$, then $Y_0 = Y$ and hence $Y_0$ is algebraic in $\boldsymbol{R}^m$;*

***b)*** *if $\dim(W) \leq k - 1$, then $Y_0$ is a Zariski open subset of $Y$;*

***c)*** *if $\dim(W) = k$, then $Y_0$ is not a Zariski open subset of $Y$.*

**Remark.** Assume that the above $G$ action is linear. So, by a linear change of coordinates we have

$$g(x_1, \dots, x_n) = (x_1, \dots, x_{j-1}, -x_j, \dots, -x_n)$$

for some $1 \leq j \leq n$. In this case $W = X_{\mathbb{C}} \cap V_g$, where $V_g$ is the linear subspace of $\mathbb{R}^{2n} = \mathbb{C}^n$ given by

$$x_l = 0, \quad l = j, \dots, n \quad \text{and} \quad y_l = 0, \quad l = 1, \dots, j-1,$$

where $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$ ($\mathbb{R}^{2n} = \mathbb{C}^n$ by $z_l = x_l + iy_l$). By identifying

$$(x_1, 0, x_2, 0, \dots, x_{j-1}, 0, 0, y_j, 0, y_{j+1}, \dots, 0, y_n) \text{ with } (x_1, \dots, x_{j-1}, y_j, \dots, y_n),$$

we can identify $W$ with the algebraic subset $\tilde{W} \subseteq \mathbb{R}^n$, given by

$$\tilde{W} = \{(X, Y) = (x_1, \dots, x_{j-1}, y_j, \dots, y_n) \mid f(X, iY) = 0 \;, \forall f \in J(X)\}.$$

**Examples.** **1)** Let $X = S^k \subseteq \mathbb{R}^{k+1}$ be the standard $k$-sphere and $g$ be the antipodal map. Then,

$$\tilde{W} = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid (ix_1)^2 + \dots + (ix_{k+1})^2 = 1\} = \emptyset.$$

So $W = \emptyset$. The coordinate functions of $F$ are $x_i x_j$ for $i \leq j = 1, \dots, k+1$. So $\mathbb{R}P^k$ sits in $\mathbb{R}^{(k+1)(k+2)/2}$ as an algebraic set.

**2)** Let us look at the counterexample following Theorem 2.1 once more. $W \subseteq \mathbb{R}^4 = \mathbb{C}^2$ is equal to

$$W = \{(0, ix, 0, iy) \mid x, y \in \mathbb{R},\ x^4 + y^4 = 1\},$$

so that, $W$ has dimension one. Moreover, $W$ is sent onto $Y_0'$ by the map $F_{\mathbb{C}}(x, y) = (x^2, y^2, xy)$. Therefore $Y = Y_0 \cup Y_0'$.

**3)** Let $X \subseteq \mathbb{R}^2$ be given by $x^4 + y^2 = 1$. $X$ is diffeomorphic to $S^1$. Let $g$ be as above. Then, $Y_0$ is diffeomorphic to $S^1$ but, the Zariski closure $Y$ of $Y_0$ contains an unbounded curve. Hence, $Y$ may not be compact even if $X$ is compact.

A weaker version of Theorem 2.2 generalizes to arbitrary finite groups as follows.

**Theorem 2.3.** *Let $G$ be any finite group acting freely on a $k$ dimensional nonsingular real algebraic set $X$. Let $W$ be the set*

$$\{p \in X_{\mathbb{C}} \mid g(p) = \bar{p}, \text{ for some order two element } g \in G\},$$

*where $\bar{p}$ denotes the complex conjugate of $p$. Now, if $\dim(W) \leq k-1$ then $Y_0 = Nonsing(Y)$ and therefore $Y_0$ is a Zariski open subset of $Y$.*

**Proof.** By Theorem 2.1.b $Y_0$ is union of topological components of $Nonsing(Y)$. Assume that the conclusion of the theorem is not true. So $Nonsing(Y)$ has components other than the ones contained in $Y_0$. In particular, $Y - Y_0$ has real dimension $k = \dim(Y)$. Let $p \in Nonsing(Y) - Y_0$ and $\Delta = (F_{\mathbb{C}})^{-1}(p)$. Since $p$ is a real point and everything is defined over the reals, $\Delta$ is invariant under complex conjugation. Moreover, $\Delta \cap X = \emptyset$ and $G$ acts transitively on $\Delta$. Note that the set $Z$ of points in $X_{\mathbb{C}}$ on which $G$ does not act freely is a proper algebraic subset of $X_{\mathbb{C}}$. So $F_{\mathbb{C}}(Z)$ has complex dimension at most $k-1$ and thus its real part $Y \cap F_{\mathbb{C}}(Z)$ has real dimension not more than $k-1$.

Let $q \in \Delta$, then $\bar{q} \in \Delta$. Assume that $q \notin Z$; then $\Delta \cap Z = \emptyset$. Thus $G$ acts freely and transitively on $\Delta$ and therefore, there exists an element $g \in G$ so that $g(q) = \bar{q}$. Since $g : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ is defined over the reals we get $g(\bar{q}) = q$ and therefore $g^2 = id_G$. Hence, $\Delta \subseteq W$ and we conclude that any fiber $(F_{\mathbb{C}})^{-1}(p)$, for $p \in Nonsing(Y) - Y_0$, is contained in $W \cup Z$. Hence, $Nonsing(Y) - Y_0 \subseteq F_{\mathbb{C}}(W \cup Z)$ which is a contradiction since the real dimension of $Y \cap F_{\mathbb{C}}(W \cup Z)$ is less than $k = \dim(Y)$. □

Let us now consider the entire rational functions on $Y_0$ and $Y$. For any semi-algebraic subset $X$ of $\mathbb{R}^n$ let $\Gamma(X)$ denote the ring of entire rational functions on $X$. Assuming the previous notation we have a $G$ action on $\Gamma(X)$ and the map $F : X \longrightarrow Y_0$ induces a homomorphism $F^* : \Gamma(Y_0) \longrightarrow \Gamma(X)$ via composition by $F$. Moreover, we have the following.

**Proposition 2.4.** $(\Gamma(X))^G = F^*(\Gamma(Y_0))$ *and* $\Gamma(Y) \subseteq \Gamma(Y_0)$, *where the equality holds if and only if* $Y_0 = Y$. *In particular, if* $X$ *and* $Z$ *are irreducible algebraic sets with algebraic* $G$ *actions and if they are* $G$ *equivariantly isomorphic to each other, then their quotients* $X//G$ *and* $Z//G$ *are isomorphic.*

**Proof.** First let us show that $(\Gamma(X))^G = F^*(\Gamma(Y_0))$. Clearly $F^*(\Gamma(Y_0)) \subseteq (\Gamma(X))^G$. Let $f_1/g_1 \in (\Gamma(X))^G$. Then, $l \cdot f_1/g_1 = f_1/g_1 + \cdots + f_l/g_l \in (\Gamma(X))^G$ where the sets $\{f_1, \ldots, f_l\}$ and $\{g_1, \ldots, g_l\}$ are the $G$ orbits of $f_1$ and $g_1$ respectively. Now

$$f_1/g_1 = \frac{h_1 + \cdots + h_l}{l \cdot g_1 \cdots g_l}$$

where $h_i = f_i \cdot g_1 \cdots g_{i-1} \cdot g_{i+1} \cdots g_l$. Note that $h_1 + \cdots + h_l$ and $g_1 \cdots g_l$ are in the invariant subring $T$ and hence $f_1/g_1 \in F^*(\Gamma(Y_0))$.

For the second statement, evidently we have that $\Gamma(Y) \subseteq \Gamma(Y_0)$. If $Y_0 \neq Y$ let $P = (p_1, \ldots, p_m) \in Y - Y_0$ and consider the function

$$\frac{1}{(x_1 - p_1)^2 + \cdots + (x_m - p_m)^2}$$

which is entire rational on $Y_0$ but not on $Y$. So $\Gamma(Y_0) \neq \Gamma(Y)$ and therefore $\Gamma(Y_0) = \Gamma(Y)$ if and only if $Y_0 = Y$. The third statement follows easily. □

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