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LOCALLY NILPOTENT p-GROUPS WHOSE PROPER SUBGROUPS ARE NC-GROUPS

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Abstract

Let G be a locally nilpotent p-group in which every proper subgroup is an NC-group. It is shown that G is itself an NC-group if either (i) the normal closure of every finite subgroup of G is a Chernikov extension of a CC-group or (ii) every proper normal subgroup of G is the union of an ascending chain of normal CC-subgroups.

1. Introduction

Let G be a group and P be a property of groups. If every proper subgroup of G satisfies the property P but G itself does not satisfy it, then G is called a **minimal non P-group**. For brevity, a group which is a Chernikov (finite) extension of a nilpotent group is called an **NC-group** (**NF-group**). A minimal non NC-group (NF-group) is called an $\overline{NC}$-group ($\overline{NF}$-group). Bruno in [4] studied locally graded $\overline{NF}$-groups. (A group is called **locally graded** if every nontrivial finitely generated subgroup of it has a proper subgroup of finite index.) Later, Otal and Peña in [9] extended the results of [4] to $\overline{NC}$-groups. The study of $\overline{NC}$-groups was continued in [1] and [2].

Of course, nonperfect $\overline{NF} - p$-groups exist. The Heineken-Mohamed group constructed in [6] is the first example of this type. However it is not known yet whether or not perfect $\overline{NF} - p$-groups or $\overline{NC} - p$-groups exist. Even under the imposition of the normalizer condition the problem still remains open (see [2]). Also, the existence problem of minimal non FC-groups and minimal non CC-groups still remain unsolved (see [3], [10] and [12]). The purpose of this work is to study a locally nilpotent p-group in which every proper subgroup is an NC-group under the additional condition that certain subgroups are CC-groups. It was shown in [3] that a locally finite minimal non CC-group cannot contain an element whose centralizer is an NC-group. Here it is shown that in an $\overline{NC} - p$-group normal closures of finite subgroups cannot be Chernikov extensions of CC-groups (Theorem 1). More generally, it is shown that in an $\overline{NC} - p$-group proper normal subgroups cannot be the union of an ascending chain of normal CC-subgroups (Theorem 2). A particular consequence of this work is that if an $NC - p$-group is a

CC-group, then it is an NF-group which generalizes Theorem 2.3 of [5] (See Lemma 2.3).

The definitions of an FC-group, CC-group, FC-element and FC-center are given in [11]. Analogously, the terms CC-element and CC-center are defined. For a group G we denote the FC-center and the CC-center of G by $FC(G)$ and $CC(G)$, respectively. It can be shown as in Lemma 4.31 of [11] that $CC(G)$ is a characteristic subgroup of G. Finally for any $i \geq 1$, $Z_i(G)$ and $K_i(G)$ denote the ith term of the upper central series and the lower central series of G, respectively.

We can now state the main results of this work.

Theorem 1. *Let G be a locally nilpotent p-group such that every proper subgroup of G is an NC-group. If for every finite subgroup F of G, F^G is a Chernikov extension of a CC-group, then G is an NC-group.*

The group in Theorem 1 need not be an NF-group as the Heineken-Mohamed group of [6] shows. However the following holds.

Corollary 1. *Let G be as in Theorem 1. If for every finite subgroup F of G, F^G is a CC-group, then G is an NF-group. Here, if "CC-group" is replaced by "FC-group" then G is nilpotent.*

Corollary 2. ([1], Theorem C). *Let G be as in Theorem 1. If every proper normal subgroup of G is a Chernikov extension of its CC-center, then G is an NC-group.*

The proof of Theorem 1 depends on the following.

Theorem 2. *Let G be a locally nilpotent p-group such that every proper subgroup of G is an NC-group. If every proper normal subgroup of G is the union of an ascending chain of normal CC-subgroups, then G is an NF-group.*

In this theorem "proper" cannot be replaced by "nilpotent" as the following example shows.

Example. Let A and U be two isomorphic copies of C_{p^∞} and let $H = AwrU$ be the restricted wreath product of A by U. Then every normal nilpotent subgroup of H is abelian but H is not an NF-group.

Solution. Let B be the base group of H. Then B is radicable abelian. It sufficies to show that if K is a normal nilpotent subgroup of H, then $K \leq B$. If K is not contained in B, then choose $x \in K \backslash B$. Now $x = bu$ for some $b \in B$ and $u \in U$. Then x^B is nilpotent since $x^B \leq K$ which implies that $Bx^B = Bu^B = Bu$ is nilpotent. However this is impossible since $Awr\langle u \rangle$ is isomorphic to a subgroup of $B\langle u \rangle$ and the former group is not nilpotent by Corollary 3.3 of [7] since A has infinite exponent.

For the convenience of the reader we end this section by stating Theorem A of [2].

Theorem A. *Let G be a locally nilpotent p-group which does not have any proper subgroup of finite index. Suppose that every proper subgroup of G is an NC-group. Then the following hold.*

(i) If $G = G'$, then G is an ascending union of proper normal nilpotent subgroups. Furthermore G has a normal nilpotent subgroup N such that for any normal nilpotent subgroup M containing $N, M/N$ has finite exponent. In particular every proper subgroup X of G has a normal nilpotent subgroup Y with the property that X/Y is Chernikov and YN/N has finite exponent.

(ii) If $G \neq G'$, then G is an NC-group.

2. Proof of Theorem 2.

The following is a direct consequence of the proof of 3.10 Lemma of [4].

Lemma 2.1 (Bruno). Let H be an FC-group and K be a normal nilpotent subgroup of H such that H/K is finite. Let F be a finite subgroup of H such that $H = FK$. If K has nilpotency class c, then $H/Z_c(H)$ is finite.

Proof. See the proof of 3.10 Lemma of [4]. □

Lemma 2.2. Let T be an $NC-p$-group and K be a normal nilpotent subgroup of T such that $T/K \cong C_{p^\infty}^{(n)}$, for some $n \geq 1$. Suppose that for every finite subgroup F of T, F^K is a CC-group. Then T is nilpotent.

Proof. Assume that T is not nilpotent. First assume that K is abelian. Let F be a finite subgroup of T. By hypothesis F^K is a CC-group. Therefore if we let

$$C = C_K(F^{F^K}),$$

then $F^K/C \cap F^K$ is Chernikov. Put $V = [F, K]$. Then

$$F^K = FV \quad \text{and} \quad F^{F^K} = F^{FV} = F^V = F[F, V].$$

Hence

$$C = C_K(F[F, V]) = C_K(FK) = C_K(F),$$

since K is abelian. In particular, C is normal in T since FK is normal in T. Similarly V is normal in T since $V = [FK, K]$. Thus T induces an automorphism group on the Chernikov group $V/V \cap C$, since $V \leq F^K$. By Theorem 3.29.2 of [11] this automorphism group must be trivial since $V \leq K$, K is abelian and T/K is radicable abelian. Hence it follows that

$$[V, T] \leq V \cap C,$$

which means that

$$[K, F, T] = [K, T, F] \leq V \cap C$$

by p.64 of Part II of [11], since T is metabelian. Consequently it follows that

$$[K, T, F, F] = 1$$

for all finite subgroups F of T. Now if F is kept fixed, then for any finite subgroup E of T containing F the last equality yields that

$$[K, T, E, F] = 1$$

which implies that

$$[K, T, T, F] = 1$$

by the choice of E. But it is easy to see that

$$[K, T, T] = [K, T]$$

since $K/[K,T] \leq Z(T/[K,T])$ and $T/K \cong C_{p^\infty}^{(n)}$. Consequently it follows that

$$[K, T, F] = 1.$$

Again since F is any finite subgroup of T, it follows that

$$[K, T, T] = 1 \quad \text{and hence} \quad [K, T] = 1$$

which is a contradiction.

Next, suppose that K is not abelian. Then T/K' is nilpotent by the first part of the proof, but also K is nilpotent by hypothesis which implies that T is nilpotent by Theorom 2.27, of [11] which is another contradiction. □

The following generalizes Theorem 2.3 of [5].

Lemma 2.3. Let H be an $NC-p$-group. Suppose that every proper normal subgroup of H is the union of an ascending chain of normal CC-subgroups. Then H is an NF-group.

Proof. Let K be a normal nilpotent subgroup of H such that H/K is Chernikov. Let T/K be the unique maximal radicable abelian subgroup of H/K. Then H/T is finite and $T/K \cong C_{p^\infty}^{(n)}$ for some $n \geq 0$. Thus to complete the proof it suffices to show that T is nilpotent. If $T = K$ then this obvious. So suppose that $T \neq K$. Let F be a finite subgroup of T. Since T/K is the union of an ascending chain of finite characteristic subgroups of H/K it follows that $F^H K/K$ is finite and hence $F^H K < T \leq H$. Thus $F^H K$ is the union of an ascending chain of normal CC-subgroups by hypothesis and obviously some term of this chain contains F^K and thus makes it a CC-group. Clearly then T must be nilpotent by Lemma 2.2, which was to be shown. □

The group H in the above lemma need not be nilpotent as the infinite locally dihedral 2-group shows.

Lemma 2.4. An $\overline{NC}-p$-group is countably infinite.

Proof. Let G be an $\overline{NC}-p$-group. By Theorem A of [2] G is perfect. Also G is infinite since it is not an NC-group. Now G is not solvable since $G = G'$ but every proper subgroup of it, being an NC-groups is solvable. Therefore for each $n \geq 1$ G contains a finite subgroup F_n such that the derived length of F_n is greater than n. Clearly then $F = \langle F_n : n \geq 1 \rangle$ is a nonsolvable subgroup of G and thus $F = G$, since every proper subgroup of G is solvable. Also F is countable by its construction. □

Proof of Theorem 2. Assume that G is not an NF-group. If G is an NC-group then it is an NF-group by Lemma 2.3 which is a contradiction. Therefore G is an $\overline{NC}-p$-group. Thus in particular G is countable and perfect by Lemma 2.4 and Theorem A of [2]. Also by the same theorem G can be expressed as

$$G = \bigcup_{i=1}^{\infty} N_i, \tag{1}$$

where for each $i \geq 1$, N_i is a normal nilpotent subgroup of G such that $N_i \leq N_{i+1}$. Moreover, by the same theorem G contains a normal nilpotent subgroup N such that N_iN/N has finite exponent for all $i \geq 1$. Since G/N satisfies the hypothesis of the theorem we may, without loss of generality, assume that $N = 1$ and so each N_i has finite exponent.

Choose $a \in G \backslash Z(G)$ and put $C = C_G(a)$. Since $C \neq G$, it contains a normal nilpotent subgroup Y such that C/Y is Chernikov. Let c be the nilpotency class of Y. Without loss of generality $a \in N_1$.

Next choose $i \geq 1$ and put $L = N_i$. By hypothesis

$$L = \bigcup_{j=1}^{\infty} L_j,$$

where for each $j \geq 1$, L_j is a normal CC-subgroup of L. In fact each L_j, being nilpotent, is an FC-group by Theorem 2.3 of [5] (see also Lemma 3.2 of [1]). Let $j \geq 1$. Since $a \in L$, $[L_j : L_j \cap C]$ is finite. Also $L_j \cap C/L_j \cap Y$ is Chernikov. But since L has finite exponent, the group $L_j \cap C/L_j \cap Y$ and hence also the index $[L_j : L_j \cap Y]$ is finite. Therefore L_j contains a normal nilpotent subgroup of finite index whose nilpotency class is at most c. So now applying Lemma 2.1 yields that $L_j/Z_c(L_j)$ is finite. This means that $K_{c+1}(L_j)$ is finite for all $j \geq 1$ by Corollary 2 of Theorem 4.21 of [11]. Consequently it follows that $K_{c+1}(N_i) = K_{c+1}(L)$ is an FC-group since

$$K_{c+1}(L) = \bigcup_{j=1}^{\infty} K_{c+1}(L_j).$$

On the other hand

$$K_{c+1}(G) = \bigcup_{i=1}^{\infty} K_{c+1}(N_i). \tag{2}$$

by (1) and also $G = K_{c+1}(G)$ since G is perfect. So substituting this in (2) and letting $V_i = K_{c+1}(N_i)$ for all $i \geq 1$, we get

$$G = \bigcup_{i=1}^{\infty} V_i,$$

where now for each $i \geq 1$, V_i is a normal FC-subgroup of G such that $V_i \leq V_{i+1}$. Also, each V_i has finite exponent since $V_i \leq N_i$. Therefore we can apply to G and C the same argument which was applied to L and $C \cap L$ above. This yields as before that

$$G = K_{c+1}(G) = \bigcup_{i=1}^{\infty} K_{c+1}(V_i),$$

where for each $i \geq 1$, $K_{c+1}(V_i)$ is a finite normal subgroup of G. This is a contradiction since $G = G'$ and $G \neq 1$. This completes the proof of the theorem. □

3. Proof of Theorem 1.

Lemma 3.1. Let H be a locally nilpotent p-group such that every finite subgroup of H is subnormal. Let X be a subgroup of finite exponent of H such that X^H/K is Chernikov for some normal CC-subgroup K of X^H. Then $K \leq CC(X^H)$.

Proof. Put $L = X^H$ and let T/K be the unique maximal radicable abelian subgroup of L/K. Then $L = ET$ for some finite subgroup E of L. Now $T/K \leq Z(L/K)$ by hypothesis and by Lemma 3.13 of [11]. Let m be the order of E and put

$$D/K = \langle aK : (aK)^m = 1 \rangle.$$

Then D/K is a finite normal subgroup of L/K since T/K is Chernikov and contained in $Z(L/K)$. Also $\frac{L/K}{D/K}$ is radicable abelian and generated by elements of bounded order by definition of L which is possible only if

$$\frac{L/K}{D/K} = 1 \quad \text{and hence} \quad L/K = D/K,$$

that is, L/K is finite.

Let $a \in K$ and put $R = C_K(a^K)$. Then K/R is Chernikov by hypothesis. Next let S be a complete set of right coset representatives for K in L. Then S is finite by the preceding paragraph. Also,

$$M = \bigcap_{x \in L} R^x = \bigcap_{s \in S} R^s,$$

since R is normal in K.

Clearly K/M is Chernikov since S is finite which implies that L/M is Chernikov since L/K is finite. Consequently, it follows that $L/C_L(a^L)$ is Chernikov since $M \leq C_L(a^L)$ and hence $a \in CC(L)$. Since a is any element of K it follows that $K \leq CC(L)$. □

Proof of Theorem 1. Suppose that G is not an NC-group. Then G is an $\overline{NC} - p$-group. Thus G is countable, perfect and every finite subgroup of G is subnormal in G by Lemma 2.4 and Theorem A of [2]. In particular, for every finite subgroup F of G, F^G is a Chernikov extension of a CC-group.

Let E be any finite subgroup of G and put $L = E^G$. By hypothesis L contains a normal CC-subgroup K such that L/K is Chernikov. Now $K \leq CC(L)$ by Lemma 3.1 which implies that $L/CC(L)$ is Chernikov. Since $CC(L)$ is characteristic in L and G is perfect, applying Theorem 3.29 of [11] yields that

$$[L, G] \leq CC(L),$$

that is, $[L, G] = [E^G, G] = [E, G]$ is a CC-group. On the other hand, since G is countable,

$$G = \bigcup_{i=1}^{\infty} F_i,$$

where for each $i \geq 1$, F_i is a finite subgroup of G such that $F_i \leq F_{i+1}$. Hence

$$\begin{aligned} G = [G, G] &= \left[\bigcup_{i=1}^{\infty} F_i^G, G \right] \\ &= \bigcup_{i=1}^{\infty} [F_i^G, G] \\ &= \bigcup_{i=1}^{\infty} [F_i, G]. \end{aligned}$$

Thus G is the union of an ascending chain of normal CC-subgroups. But then G is an NF-group by Theorem 2, which is a contradiction.

Proof of Corollary 1. By Theorem 1 G is an NC-group. Thus G contains a normal nilpotent subgroup K such that G/K is Chernikov. Let T/K be the unique maximal radicable abelian subgroup of G. Then $G = ET$ for some finite subgroup E of G. Thus to complete the proof it suffices to show that T is nilpotent. But since F^T is a CC-group for every finite subgroup F of T by hypothesis, it follows that T is nilpotent by Lemma 2.2.

Next suppose that F^G is an FC-group for every finite subgroup F of G. Then since E^G is a normal FC-subgroup of G it is easy to see that E is subnormal in G and, hence, $G = ET$ is nilpotent by (1) Lemma of [8].

Proof of Corollary 2. Assume that G is an $\overline{NC} - p$-group. By Theorem A of [2], for each finite subgroup F of G, $F^G < G$ and hence $F^G/CC(F^G)$ is Chernikov by hypothesis. But then G is an NC-group by Theorem 1, which is a contradiction. □

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