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ON THE THEORY OF A CERTAIN CLASS OF QUADRATIC PENCILS OF MATRICES AND ITS APPLICATIONS

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Abstract

This paper is devoted to the study of the properties of eigenvalues and eigenvectors of quadratic pencil $\lambda^2 C - \lambda R - J$, where C is a positive diagonal matrix, R is an arbitrary real diagonal matrix, J is a "tridiagonal" real symmetric and positive matrix. The obtained results are then used to solve the corresponding system of differential equations with boundary and initial conditions.

Key words: Quadratic pencils, eigenvalues, eigenvectors.

1. Introduction

Let us consider the system of linear differential equations

$$c_n \frac{d^2 u_n(t)}{dt^2} = r_n \frac{du_n(t)}{dt} + a_{n-1}u_{n-1}(t) + b_n u_n(t) + a_n u_{n+1}(t)$$
$$n = 0, 1, \ldots, N-1; \quad t \geq 0, \tag{1}$$

with the boundary conditions

$$u_{-1}(t) = 0, \quad u_N(t) + h u_{N-1}(t) = 0, \quad t \geq 0, \tag{2}$$

and the initial conditions

$$u_n(0) = f_n, \quad \frac{du_n(0)}{dt} = g_n, \qquad n = 0, 1, \ldots, N-1, \tag{3}$$

where $\{u_n(t)\}_{n=-1}^{N}$ is a desired solution; $f_n, g_n (n = 0, 1, \ldots, N-1)$ are given complex numbers; the coefficients c_n, r_n, a_n, b_n of the equation (1), the number h in the boundary conditions (2) is real, and

$$a_n \neq 0, \quad c_n > 0 \tag{4}$$

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$$
\begin{aligned}
b_0 - |a_0| \geq 0, \qquad & b_{N-1} - h a_{N-1} - |a_{N-2}| \geq 0 \\
b_n - |a_{n-1}| - |a_n| \geq 0, \qquad & n = 1, 2, \ldots, N-2
\end{aligned}
\tag{5}
$$

with strict inequality in at least one relation of (5).

For a possible application of a physical character of problem (1), (2), (3) we refer to [1, pp.16-21] where the case $r_n = 0$ is considered.

If $\{u_n(t)\}_{n=-1}^{N}$ is a solution of the problem (1), (2), (3), then taking the boundary conditions (2) into account, we have

$$
\begin{aligned}
c_0 \frac{d^2 u_0(t)}{dt^2} &= r_0 \frac{du_0(t)}{dt} + b_0 u_0(t) + a_0 u_1(t), \\
c_n \frac{d^2 u_n(t)}{dt^2} &= r_n \frac{du_n(t)}{dt} + a_{n-1} u_{n-1}(t) + b_n u_n(t) + a_n u_{n+1}(t), \\
& \qquad n = 1, 2, \ldots, N-2, \\
c_{N-1} \frac{d^2 u_{N-1}(t)}{dt^2} &= r_{N-1} \frac{du_{N-1}(t)}{dt} + a_{N-2} u_{N-2}(t) + (b_{N-1} - h a_{N-1}) u_{N-1}(t).
\end{aligned}
\tag{6}
$$

Consequently, finding a solution $\{u_n(t)\}_{n=-1}^{N}$ of the problem (1), (2), (3) is equivalent to the problem of finding a solution $\{u_n(t)\}_{n=0}^{N-1}$ of system (6) that satisfies the initial conditions (3).

Setting

$$
u(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix}, \quad f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}, \quad g = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix},
$$

$$
C = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & \cdots & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{bmatrix}, \quad R = \begin{bmatrix} r_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & r_{N-1} \end{bmatrix}
$$

$$
J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} - h a_{N-1} \end{bmatrix},
\tag{7}
$$

we can write problem (6), (3) in the form

$$
C \frac{d^2 u(t)}{dt^2} = R \frac{du(t)}{dt} + J u(t), \quad 0 \leq t < \infty,
\tag{8}
$$

$$u(0) = f, \quad \frac{du(0)}{dt} = g. \tag{9}$$

Thus the initial boundary value problem (1), (2), (3) is equivalent to the initial value problem (8), (9), that is, if $\{u_n(t)\}_{n=-1}^{N}$ is a solution of problem (1), (2), (3), then the vector-function $u(t) = \{u_n(t)\}_{n=0}^{N-1}$ form a solution of the problem (8), (9), and, conversely, if $u(t) = \{u_n(t)\}_{n=0}^{N-1}$ is a solution of (8), (9) then $\{u_n(t)\}_{n=-1}^{N}$, where $u_{-1}(t) = 0$, $u_N(t) = -hu_{N-1}(t)$, form a solution of the problem (1), (2), (3).

It is well known that the problem of kind (8), (9) which can be written in the form of a first-order system (see Section 4 of the present paper) has a unique solution (see, for example, [2, Chapter 1]). Consequently, we can state that the boundary initial value problem (1), (2), (3) has a unique solution $\{u_n(t)\}_{n=-1}^{N}$. Our main problem in this paper is to investigate the structure of the solution, that is, to give an effective formula for it. To this end we shall investigate the solution of the problem (8), (9).

We seek a nontrivial solution of equation (8) which has the form

$$u(t) = e^{\lambda t} \cdot y, \tag{10}$$

where λ is a complex constant, and y is a constant vector (an element of the space $\mathbb{C}^N$) which depends upon λ and which we desire to be nontrivial, that is, not equal to 0, the null vector. Substituting (10) into (8), we obtain

$$(\lambda^2 C - \lambda R - J)y = 0 \tag{11}$$

Definition. *The complex number λ_0 is said to be an eigenvalue of equation (11) (or of quadratic pencil $\lambda^2 C - \lambda R - J$) if there exists a nonzero vector $y \in \mathbb{C}^N$ satisfying the equation (11) for $\lambda = \lambda_0$. This vector y is called an eigenvector of the equation (11) corresponding to the eigenvalue λ_0.*

Thus the vector-function (10) is a nontrival solution of the equation (8) if and only if λ is an eigenvalue and y is the corresponding eigenvector of the equation (11).

Let us denote by $\lambda_1, \ldots, \lambda_m$ all the eigenvalues of equation (11), and by $y^{(1)}, \ldots, y^{(m)}$ the corresponding eigenvectors. Then by the linearity of equation (11) the vector-function

$$u(t) = \sum_{j=1}^{m} \alpha_j e^{\lambda_j t} \cdot y^{(j)} \tag{12}$$

will also be a solution of equation (8), where $\alpha_1, \ldots, \alpha_m$ are arbitrary constants (independent of t). Now we must try to choose the constants α_j so that (12) will also satisfy the initial conditions (9):

$$\sum_{j=1}^{m} \alpha_j \cdot y^{(j)} = f, \quad \sum_{j=1}^{m} \alpha_j \lambda_j y^{(j)} = g. \tag{13}$$

Definition. *If for the arbitrary vectors $f, g \in \mathbb{C}^N$ the unique expansions (13) hold with the same coefficients $\alpha_j (j = 1, \ldots, m)$ in both expansions, then we shall say that the eigenvectors $y^{(1)}, \ldots, y^{(m)}$ of equation (11) form a two-fold basis in $\mathbb{C}^N$. (For a generalization of this definition see [3, Chapter 5, §9].)*

Thus the problem (8), (9) will for $\forall f, g \in \mathbb{C}^N$ have a solution of the form (12), where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues and $y^{(1)}, \ldots, y^{(m)}$ are the corresponding eigenvectors of equation (11), if the vectors $y^{(1)}, \ldots, y^{(m)}$ form a two-fold basis in space $\mathbb{C}^N$.

In the next section we shall show that the eigenvectors of the equation (11) form a two-fold basis in $\mathbb{C}^N$. We shall also find the formulas for the coefficients α_j in (13).

2. Investigation of the Eigenvalue Problem

We consider the eigenvalue problem (11), where the matrices C, R and J have the form (7) and the conditions (4), (5) are satisfied.

We will investigate the equation (11) in the space

$$\mathbb{C}^N = \{y = \{y_n\}_0^{N-1} : y_n \in \mathbb{C}, \quad n = 0, 1, \ldots, N-1\}$$

with the inner product

$$(y, z) = \sum_{n=0}^{N-1} y_n \overline{z}_n, \tag{14}$$

where the bar over a number denotes complex conjugation.

The matrices C, R and J defined by (7) are selfadjoint, that is each of them satisfies the relation:

$$(Ty, z) = (y, Tz), \qquad \forall\ y, z \in \mathbb{C}^N. \tag{15}$$

Moreover, matrices C and J are positive:

$$(Cy, y) > 0, \ \ (Jy, y) > 0, \qquad \forall\ y \in \mathbb{C}^N, \quad y \neq 0. \tag{16}$$

The positiveness of C is obvious. The positiveness of J follows from the conditions (5) by virtue of the following relation: for the any real vector $y = \{y_n\}_0^{N-1} \in \mathbb{R}^N$,

$$\begin{aligned} (Jy, y) \ &= \ (b_0 - |a_0|) y_0^2 + (b_{N-1} - h a_{N-1} - |a_{N-2}|) y_{N-1}^2 \\ &+ \sum_{n=1}^{N-2} (b_n - |a_{n-1}| - |a_n|) y_n^2 + \sum_{n=1}^{N-2} |a_{n-1}| (y_{n-1} \pm y_n)^2, \end{aligned}$$

where the $\pm$ sign in $(y_{n-1} \pm y_n)^2$ is taken to be that of a_{n-1}.

Lemma 1. Each eigenvalue λ of the equation (11) is real, non-zero and have the same sign as

$$2\lambda(Cy, y) - (Ry, y) \neq 0, \tag{17}$$

where y is an eigenvector corresponding to λ.

Proof. Let the complex number λ be an eigenvalue of the equation (11) and $y = \{y_n\}_0^{N-1} \neq 0$ be a corresponding eigenvector. By forming the inner product of both sides of the equation (11) with the vector y, we get

$$\lambda^2(Cy, y) - \lambda(Ry, y) - (Jy, y) = 0. \tag{18}$$

The number (Ry, y) is real by virtue of the property (15) of R, and $(Cy, y) > 0$, $(Jy, y) > 0$ by virtue of the positiveness of matrices C and J. Therefore, $\lambda \neq 0$ and the discriminant of the quadratic equation (18) with respect to λ is positive:

$$(Ry, y)^2 + 4(Cy, y)(Jy, y) > 0.$$

Consequently, the eigenvalue λ as a root of (18) will be real. Besides,

$$\lambda[2\lambda(Cy, y) - (Ry, y)] = \lambda^2(Cy, y) + (Jy, y) > 0$$

and hence follows the last statement of the lemma. □

Lemma 2. The eigenvectors y and z of equation (11) corresponding to the distinct eigenvalues λ and μ respectively satisfy the "orthogonality" relations:

$$(\lambda + \mu)(Cy, z) - (Ry, z) = 0, \tag{19}$$

$$\lambda\mu(Cy, z) + (Jy, z) = 0 \tag{20}$$

$$\lambda\mu(Ry, z) + (\lambda + \mu)(Jy, z) = 0. \tag{21}$$

Proof. Multiplying in the sense of the inner product the first of equalities

$$\lambda^2 Cy - \lambda Ry - Jy = 0, \quad \mu^2 Cz - \mu Rz - Jz = 0$$

from the right by z and the second one from the left by y, and using the reality of eigenvalues and property (15) of the matrices C, R, and J, we get

$$\begin{aligned} \lambda^2(Cy, z) - \lambda(Ry, z) - (Jy, z) &= 0, \\ \mu^2(Cy, z) - \mu(Ry, z) - (Jy, z) &= 0. \end{aligned}$$

Eliminating from these two equalities in turn $(Jy, z), (Ry, z)$, and (Cy, z) we obtain respectively the "orthogonality" relations indicated in the lemma. □

To investigate further properties of the eigenvalues and eigenvectors of the equation (11) we note that the equation (11) is equivalent to the problem of finding a vector $\{y_n\}_{n=-1}^{N}$ that satisfies the boundary value problem

$$(\lambda^2 c_n - \lambda r_n - b_n)y_n - a_{n-1}y_{n-1} - a_n y_{n+1} = 0, \quad n = 0, 1, \ldots, N-1, \tag{22}$$

$$y_{-1} = 0, \quad y_N + hy_{N-1} = 0. \tag{23}$$

We define the solution $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ of the equation (22) satisfying the initial conditions

$$\varphi_{-1}(\lambda) = 0, \quad \varphi_0(\lambda) = 1. \tag{24}$$

Using (24), we can recursively find $\varphi_n(\lambda)$, $n = 1, 2, \ldots, N$, from the equation (22) and $\varphi_n(\lambda)$ will be a polynomial in λ of degree $2n$.

It is easy to see that every solution $\{y_n(\lambda)\}_{n=-1}^{N}$ of the equation (22) satisfying the initial condition $y_{-1} = 0$ is equal to $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ up to a constant factor:

$$y_n(\lambda) = \alpha\varphi_n(\lambda), \qquad n = -1, 0, 1, \ldots, N. \tag{25}$$

Hence

$$y_N(\lambda) + hy_{N-1}(\lambda) = \alpha[\varphi_N(\lambda) + h\varphi_{N-1}(\lambda)].$$

Consequently setting

$$\chi(\lambda) = \varphi_N(\lambda) + h\varphi_{N-1}(\lambda), \tag{26}$$

we have the following lemma.

Lemma 3. Eigenvalues of the equation (11) are roots of a recursively constructed polynomial $\chi(\lambda)$. To each eigenvalue λ_0 corresponds up to a constant factor, a single eigenvector, which can be taken to be the vector $\{\varphi_n(\lambda_0)\}_{n=0}^{N-1}$.

Lemma 4. There exist $2N$ distinct eigenvalues.

Proof. Since $\varphi_n(\lambda)$ for each n is a polynomial of degree $2n$, by (26) $\chi(\lambda)$ will be a polynomial of degree $2N$. Therefore, $\chi(\lambda)$ has $2N$ roots. Now we show that the roots of $\chi(\lambda)$ are simple. Hence the statement of the lemma will follow.

Differentiating the equation

$$(\lambda^2 c_n - \lambda r_n - b_n)\varphi_n(\lambda) - a_{n-1}\varphi_{n-1}(\lambda) - a_n\varphi_{n+1}(\lambda) = 0$$

with respect to λ, we get

$$(2\lambda c_n - r_n)\varphi_n(\lambda) + (\lambda^2 c_n - \lambda r_n - b_n)\dot{\varphi}_n(\lambda) - a_{n-1}\dot{\varphi}_{n-1}(\lambda) - a_n\dot{\varphi}_{n+1}(\lambda) = 0,$$

where the dot over the function indicates the derivative with respect to λ. Multiplying the first equation by $\dot{\varphi}_n(\lambda)$ and the second one by $\varphi_n(\lambda)$, and subtracting the left and right members of the resulting equations, we get

$$(2\lambda c_n - r_n)\varphi_n^2(\lambda) + a_{n-1} \\ [\varphi_{n-1}(\lambda)\dot{\varphi}_n(\lambda) - \dot{\varphi}_{n-1}(\lambda)\varphi_n(\lambda)] - a_n[\varphi_n(\lambda)\dot{\varphi}_{n+1}(\lambda) - \dot{\varphi}_n(\lambda)\varphi_{n+1}(\lambda)] = 0.$$

Summing the last equation for the values $n = 0, 1, \ldots, K$ $(K \leq N-1)$ and using the initial condition (24), we get

$$a_K[\varphi_K(\lambda)\dot{\varphi}_{K+1}(\lambda) - \dot{\varphi}_K(\lambda)\varphi_{K+1}(\lambda)] = \sum_{n=0}^{K}(2\lambda c_n - r_n)\varphi_n^2(\lambda). \tag{27}$$

Let us set $\chi(\lambda_0) = 0$. In particular, setting in (27), $K = N-1$ and $\lambda = \lambda_0$, and using the equality $\varphi_N(\lambda_0) = -h\varphi_{N-1}(\lambda_0)$, which follows from the condition $\chi(\lambda_0) = 0$, we have

$$a_{N-1}\dot{\chi}(\lambda_0)\varphi_{N-1}(\lambda_0) = \sum_{n=0}^{N-1}(2\lambda_0 c_n - r_n)\varphi_n^2(\lambda_0). \tag{28}$$

The right-hand side of (28) is not zero by virtue of Lemma 1. Consequently $\dot{\chi}(\lambda_0) \neq 0$, that is the root λ_0 of the function $\chi(\lambda)$ is simple. □

Lemma 5. Half of the eigenvalues are positive and the other half negative.

Proof. By Lemma 3 the eigenvalues of equation (11) coincide with the roots of the function $\chi(\lambda)$. On the other hand the eigenvalues of the equation (11) coincide with the roots of the polynomial $\det(\lambda^2 C - \lambda R - J)$. Since both $\chi(\lambda)$ and $\det(\lambda^2 C - \lambda R - J)$ are the polynomials in λ of degree $2N$, hence it follows that they differ by at most a constant factor from each other. This factor is easily found. To this end it suffices to compare the coefficients of λ^{2N} in these polynomials. This yields

$$\det(\lambda^2 C - \lambda R - J) = a_0 a_1 \cdots a_{N-1} \cdot \chi(\lambda).$$

Now we consider the auxiliary eigenvalue problem

$$[\lambda^2 C - \lambda\epsilon R - J(\epsilon)]y = 0 \tag{29}$$

depending on parameter $\epsilon \in [0,1]$, where the matrix $J(\epsilon)$ is obtained from the matrix J by means of multiplication its all nondiagonal elements by ϵ. It is obvious that the analog of the conditions (4) and (5) is fulfilled for all $\epsilon \in (0,1]$.

The eigenvalues of the equation (29) are nonzero for all $\epsilon \in [0,1]$ and coincide with the roots of the polynomial

$$\det[\lambda^2 C - \lambda\epsilon R - J(\epsilon)]. \tag{30}$$

For each $\epsilon \in (0,1]$ the roots of the polynomial (30) are distinct by virtue of Lemma 4, being applicable to the equation (29). Denote them by

$$\lambda_1(\epsilon) < \lambda_2(\epsilon) < \cdots < \lambda_{2N}(\epsilon).$$

Since $\lambda_j(\epsilon)$ $(j = 1, \ldots, 2N)$ are the eigenvalues of a matrix of order $2N$ being continuous in $\epsilon \in [0,1]$ (see Section 4 of the present paper) they will be continuous functions of ϵ (see [4, Chapter 2, §5]). Note that at the point $\epsilon = 0$ we do not state that $\lambda_1(\epsilon), \lambda_2(\epsilon), \ldots, \lambda_{2N}(\epsilon)$ are distinct.

Now we show that for all values of $\epsilon \in (0,1]$ half of $\lambda_j(\epsilon)(j = 1, \ldots, 2N)$ are negative and the other half positive:

$$\lambda_j(\epsilon) < 0 \ \ (j = 1, \ldots, N), \quad \lambda_j(\epsilon) > 0 \ \ (j = N+1, \ldots, 2N).$$

Hence, in particular, for $\epsilon = 1$ the statement of the lemma will follow.

Assume the contrary. Let for some value of $\epsilon \in (0,1]$

$$\lambda_j(\epsilon) < 0 \ \ (j = 1, \ldots, K), \quad \lambda_j(\epsilon) > 0 \ \ (j = K+1, \ldots, 2N) \tag{31}$$

where $0 \le K \le 2N$ and $K \ne N$ (for $K = 0$ all the eigenvalues $\lambda_j(\epsilon)$ are understood to be positive, and for $K = 2N$ negative). Since $\lambda_j(\epsilon)$ $(j = 1, \ldots, 2N)$ are different from zero and are distinct and continuous functions for all values of $\epsilon \in (0,1]$, the inequalities (31) will be valid for all values of $\epsilon \in (0,1]$. Passing in (31) to the limit as $\epsilon \to 0$, we get

$$\lambda_j(0) \le 0 \ \ (j = 1, \ldots, K), \quad \lambda_j(0) \ge 0 \ \ (j = K+1, \ldots, 2N).$$

But this is contradiction, since for $\epsilon = 0$ the roots of the polynomial (30) are the numbers

$$\pm\sqrt{\frac{b_j}{c_j}} \ \ (j = 0, 1, \ldots, N-2), \qquad \pm\sqrt{\frac{b_{N-1} - ha_{N-1}}{c_{N-1}}},$$

half of which are negative and the other half positive. Thus the lemma is proved. □

We can summarize the results obtained above in the following theorem:

Theorem 1. *The equation (11) has precisely $2N$ real distinct eigenvalues λ_j $(j = 1, \ldots, 2N)$. These eigenvalues are different from zero; half of them are negative and the other half positive. To each eigenvalue λ_j corresponds, up to constant factor, a single eigenvector which can be taken to be $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$, where $\{\varphi_n(\lambda)\}_{n=-1}^{N}$, is a solution of the equation (22) satisfying the initial conditions (24).*

Now we discuss the basisness of the eigenvectors of equation (11).

Theorem 2. *Eigenvectors of the equation (11) form a two-fold basis in* $\mathbb{C}^N$, *that is if* $\varphi^{(1)},\ldots,\varphi^{(2N)}$ *are eigenvectors, then the vectors* $\phi_j=[\varphi^{(j)},\lambda_j\varphi^{(j)}]\in\mathbb{C}^N\times\mathbb{C}^N$ *form a basis for* $\mathbb{C}^N\times\mathbb{C}^N$.

Proof. Consider the space $\mathbb{C}^N\times\mathbb{C}^N$ of vectors denoted by $[y,z]$, where $y,z\in\mathbb{C}^N$. Define on this space the bilinear form by the formula

$$\langle[y,z],[u,v]\rangle=(Cy,v)+(Cz,u)-(Ry,u), \tag{32}$$

where $(\cdot,\cdot)$ in the right-hand side denotes the inner product in $\mathbb{C}^N$ defined by the formula (14).

Note that the formula (32) does not define an inner product in the space $\mathbb{C}^N\times\mathbb{C}^N$, since for the nonzero vector $[y,z]$ the number $\langle[y,z],[y,z]\rangle$ is not necessarily positive (it may also be zero or negative).

In view of Lemma 2 the vectors

$$\phi_j=[\varphi^{(j)},\lambda_j\varphi^{(j)}],\qquad j=1,\ldots,2N$$

are orthogonal with respect to bilinear form $\langle\cdot,\cdot\rangle$:

$$\langle\phi_j,\phi_\ell\rangle=0,\quad j\neq\ell. \tag{33}$$

Further, it is remarkable that, by virtue of Lemma 1,

$$\rho_j=\langle\phi_j,\phi_j\rangle=2\lambda_j(C\varphi^{(j)},\varphi^{(j)})-(R\varphi^{(j)},\varphi^{(j)})\neq0 \tag{34}$$

and the sign of ρ_j coincide with the sign of λ_j,

$$\text{Sign }\rho_j=\text{ Sign }\lambda_j. \tag{35}$$

From (33) and (34) it follows that the vectors $\phi_1,\ldots,\phi_{2N}$ are linearly independent in space $\mathbb{C}^N\times\mathbb{C}^N$. Since the number of them is equal to $2N$ and $\dim(\mathbb{C}^N\times\mathbb{C}^N)=2N$, they form a basis for the space $\mathbb{C}^N\times\mathbb{C}^N$. The theorem is proved. □

By Theorem 2, for the arbitrary vector $[f,g]$ belonging to $\mathbb{C}^N\times\mathbb{C}^N$ we have the unique expansion

$$[f,g]=\sum_{j=1}^{2N}\alpha_j\phi_j,\quad\text{i.e.,}\quad f=\sum_{j=1}^{2N}\alpha_j\varphi^{(j)},\quad g=\sum_{j=1}^{2N}\alpha_j\lambda_j\varphi^{(j)}, \tag{36}$$

and

$$\alpha_j=\frac{1}{\rho_j}\langle[f,g],\phi_j\rangle=\frac{1}{\rho_j}\left\{\lambda_j(Cf,\varphi^{(j)})+(Cg,\varphi^{(j)})-(Rf,\varphi^{(j)})\right\}, \tag{37}$$

where ρ_j is defined by formula (34).

Remark. To prove Theorem 2 we could also use the orthogonality relation (20) or (21). In the case of (20) we must use on $\mathbb{C}^N \times \mathbb{C}^N$ the bilinear form

$$\langle [y,z],[u,v]\rangle = (Jy,u) + (Cz,v), \tag{38}$$

and in the case of (21)

$$\langle [y,z],[u,v]\rangle = (Jy,v) + (Jz,u) + (Rz,v). \tag{39}$$

The bilinear form (38), in contrast to the bilinear form (32), is an inner product in $\mathbb{C}^N \times \mathbb{C}^N$. But in connection with the formulas (34) and (37) for ρ_j and α_j the bilinear form (32) is more advantages, since both the matrices C and R presented in it are diagonal.

Theorem 3. *Eigenvectors corresponding to negative (or positive) eigenvalues form a basis for* $\mathbb{C}^N$.

Proof. We may assume that

$$\lambda_1 < \cdots < \lambda_N < 0 < \lambda_{N+1} < \cdots < \lambda_{2N}.$$

Let $z = \{z_n\}_0^{N-1} \in \mathbb{C}^N$ and

$$(z, \varphi^{(j)}) = 0, \quad j = 1, \dots, N. \tag{40}$$

It suffices for us to establish that then $z = 0$.

Applying (36) and (37) to the vectors $f = 0$ and $g = C^{-1}z$ we have

$$0 = \sum_{j=1}^{2N} \gamma_j \varphi^{(j)}, \quad C^{-1}z = \sum_{j=1}^{2N} \gamma_j \lambda_j \varphi^{(j)}, \tag{41}$$

where

$$\gamma_j = \frac{1}{\rho_j}(z, \varphi^{(j)}), \quad j = 1, \dots, 2N. \tag{42}$$

From (42), in view of (40), we have $\gamma_j = 0$, $j = 1, \dots, N$, and therefore, (41) takes the form:

$$0 = \sum_{j=N+1}^{2N} \gamma_j \varphi^{(j)}, \quad C^{-1}z = \sum_{j=N+1}^{2N} \gamma_j \lambda_j \varphi^{(j)}.$$

Multiplying the last equalities by z in the sense of inner product in $\mathbb{C}^N$, we get

$$0 = \sum_{j=N+1}^{2N} \gamma_j (\varphi^{(j)}, z) = \sum_{j=N+1}^{2N} \rho_j |\gamma_j|^2, \tag{43}$$

$$(C^{-1}z, z) = \sum_{j=N+1}^{2N} \gamma_j \lambda_j (\varphi^{(j)}, z) = \sum_{j=N+1}^{2N} \rho_j \lambda_j |\gamma_j|^2. \tag{44}$$

By virtue of (35) we have $\rho_j > 0$, $j = N+1, \ldots, 2N$. Consequently from (43) it follows that $\gamma_j = 0$, $j = N+1, \ldots, 2N$, and from (44) we get $(C^{-1}z, z) = 0$. Hence $z = 0$, since

$$(C^{-1}z, z) = \sum_{n=0}^{N-1} \frac{1}{c_n} |z_n|^2.$$

The theorem is proved. □

3. Application

Here we give some applications of the results obtained above.

Theorem 4. *The boundary initial value problem (1), (2), (3) has a unique solution* $\{u_n(t)\}_{n=-1}^{N}$ *that is representable in the form*

$$u_n(t) = \sum_{j=1}^{2N} \alpha_j e^{\lambda_j t} \varphi_n(\lambda_j), \qquad n = -1, 0, 1, \ldots, N, \tag{45}$$

in which $\lambda_1, \ldots, \lambda_{2N}$ *are the eigenvalues of the equation (11), and* $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ *is the solution of equation (22) satisfying the initial conditions (24). Further, the coefficients* $\alpha_1, \ldots, \alpha_{2N}$ *are defined by formulae (37), (34).*

Proof of the theorem follows from the explanation given in the Introduction and from the Theorem 2.

Theorem 5. *For an arbitrary vector* $f = \{f_n\}_0^{N-1} \in \mathbb{C}^N$ *the boundary value problem (1), (2) has a unique solution* $\{u_n(t)\}_{n=-1}^{N}$ *that satisfies conditions*

$$u_n(0) = f_n, \quad \lim_{t \to \infty} u_n(t) = 0, \qquad n = 0, 1, \ldots, N-1. \tag{46}$$

This solution has the form

$$u_n(t) = \sum_{j=1}^{N} \beta_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \ldots, N, \tag{47}$$

where $\lambda_1, \ldots, \lambda_N$ *are the negative eigenvalues of equation (11), and coefficients* $\beta_1, \ldots, \beta_N$ *are defined with the help of expansion*

$$f = \sum_{j=1}^{N} \beta_j \varphi^{(j)}, \tag{48}$$

in which $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$.

Proof. By Theorem 3 the vectors $\varphi^{(1)},\dots,\varphi^{(N)}$ form a basis in $\mathbb{C}^N$. Therefore, for a given vector $f = \{f_n\}_0^{N-1} \in \mathbb{C}^N$ there exist the numbers $\beta_1,\dots,\beta_N$ uniquely determined by the expansion formula (48). Hence we define $u_n(t)$ by the formula (47). Then $\{u_n(t)\}_{n=-1}^{N}$ will be a solution of the problem (1), (2) satisfying the first condition of (46) in view of (48), and the second one in view of that $\lambda_1,\dots,\lambda_N$ are negative.

For proof of uniqueness of solution we note that by virtue of Theorem 4 the general solution of problem (1), (2) has the representation

$$u_n(t) = \sum_{j=1}^{2N} \gamma_j e^{\lambda_j t}\varphi_n(\lambda_j), \qquad n = -1, 0, 1, \dots, N, \tag{49}$$

where $\gamma_1,\dots,\gamma_{2N}$ are arbitrary constant numbers. Let (49) satisfies conditions (46). Then from the second condition of (46) it follows that it must be $\gamma_{N+1} = \cdots = \gamma_{2N} = 0$, since $\lambda_j > 0$ for $j = N+1,\dots,2N$ and are distinct. Further, setting $t = 0$ in (49) and using the first condition of (46) and (48) we get $\gamma_j = \beta_j,\ j = 1,\dots,N$. The theorem is proved. □

Remark. As it follows from (47), for the solution of problem (1), (2), (46) we have

$$u_n(t) = 0(e^{-\delta t}), \qquad n = -1, 0, 1, \dots, N$$

as $t \to \infty$, where $\delta = \min\{|\lambda_1|,\dots,|\lambda_N|\}$.

4. Appendix

The equation (11) is equivalent to the pair of equations

$$\begin{cases} z = \lambda y, \\ C^{-1}Rz + C^{-1}Jy = \lambda z. \end{cases}$$

Therefore, the eigenvalues of the equation (11) coincide with the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ C^{-1}J & C^{-1}R \end{bmatrix}$$

acting in the space $\mathbb{C}^N \times \mathbb{C}^N$. Moreover, the vector

$$x = \begin{bmatrix} y \\ z \end{bmatrix} \quad (y, z \in \mathbb{C}^N)$$

is an eigenvector of the matrix A $(Ax = \lambda x)$ if and only if $z = \lambda y$ and y is an eigenvector of the equation (11) corresponding to the eigenvalue λ. Consequently, the two-fold basisness in $\mathbb{C}^N$ of the eigenvectors of the equation (11) coincides with the ordinary basisness of the eigenvectors of the matrix A in $\mathbb{C}^N \times \mathbb{C}^N$.

Thus the analysis given in Section 2 is, in fact, the analysis of the matrix A.

If we define on $\mathbb{C}^N \times \mathbb{C}^N$ the bilinear form by the formula

$$\langle \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle = (Cy, v) + (Cz, u) - (Ry, u), \tag{50}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{C}^N$ defined by the formula (14), then A becomes a selfadjoint with respect to this bilinear form:

$$\langle A \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle = \langle \begin{bmatrix} y \\ z \end{bmatrix}, A \begin{bmatrix} u \\ v \end{bmatrix} \rangle, \quad \forall \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{C}^N \times \mathbb{C}^N. \tag{51}$$

The bilinear form (50) defines on $\mathbb{C}^N \times \mathbb{C}^N$ only an indefinite inner product: the value

$$\langle \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix} \rangle$$

is not in general positive. But this value is real, and nonzero if $\begin{bmatrix} y \\ z \end{bmatrix}$ is an eigenvector of the matrix A, the sign of this number coincide with the sign of eigenvalue λ (see (35)). Therefore, using (51) we can in the standard way show that the eigenvectors $\begin{bmatrix} y \\ \lambda y \end{bmatrix}$ and $\begin{bmatrix} z \\ \mu z \end{bmatrix}$ of the matrix A corresponding to the eigenvalues $\lambda \neq \mu$ are orthogonal with respect to bilinear form $\langle \cdot,\cdot \rangle$:

$$\langle \begin{bmatrix} y \\ \lambda y \end{bmatrix}, \begin{bmatrix} z \\ \mu z \end{bmatrix} \rangle = 0.$$

It is easily seen that this relation coincide with the relation (19).

Finally, we note that the problem (8), (9) is equivalent to the problem

$$\frac{dw(t)}{dt} = Aw(t), \qquad 0 \le t < \infty, \tag{52}$$

$$w(0) = w^\circ, \tag{53}$$

where

$$w(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, w^0 = \begin{bmatrix} f \\ g \end{bmatrix}.$$

That is if $u(t)$ is a solution of problem (8), (9), then

$$w(t) = \begin{bmatrix} u(t) \\ u'(t) \end{bmatrix}$$

is a solution of the problem (52), (53), and, conversely, if $w(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ is a solution of (52), (53) then $v(t) = u'(t)$ and $u(t)$ is a solution of (8), (9).

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