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SOME COMMUTATIVITY PROPERTIES FOR RINGS WITH UNITY

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Abstract

In this paper, we prove the commutativity of a ring R with unity satisfying one of the following ring properties:

(P_1) For each x, y in R, $\{1 - h(yx^r)\}[x, yx^r - f(yx^r)]\{1 - g(yx^r)\} = 0$ for some $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$.

(P_2) Given x, y in R, $\{1 - h(yx^r)\}[x, yx^r - f(x^r y)]\{1 - g(yx^r)\} = 0$ and $\{1 - \tilde{h}(xy^r)\}[y, y^r x - \tilde{f}(xy^r)]\{1 - \tilde{g}(xy^r)\} = 0$ for some $f(X), \tilde{f}(X) \in X^2\mathbb{Z}[X]$ and $g(X), \tilde{g}(X), h(X), \tilde{h}(X) \in X\mathbb{Z}[X]$.

(P_3) For each $x, y \in R$, $[x, yx^r - x^s f(y)x^t] = 0$ for some $f(X) \in X^2\mathbb{Z}[X]$.

Introduction

Throughout this paper R will represent a ring with unity 1, $N(R)$ the set of nilpotent elements in $R, N'(R)$ the subset of $N(R)$ consisting of all elements $a \in R$ with $a^2 = 0$ and $U(R)$ the group of units in R. For $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$. Let $\mathbb{Z}$ be the ring of integers and let r, s and t be non-negative integers.

In this paper we consider the following properties:

(P_1) For each x, y in R, $\{1 - h(yx^r)\}[x, yx^r - f(yx^r)]\{1 - g(yx^r)\} = 0$ for some $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$.

(P_2) Given x, y in R, $\{1-h(yx^r)\}[x, yx^r - f(x^r y)]\{1-g(yx^r)\} = 0$ and $\{1-\tilde{h}(xy^r)\}[y, y^r x - \tilde{f}(xy^r)]\{1-\tilde{g}(xy^r)\} = 0$ for some $f(X), \tilde{f}(X) \in X^2\mathbb{Z}[X]$ and $g(X), \tilde{g}, h(X), \tilde{h}(X) \in X\mathbb{Z}[X]$.

(P_3) For each $x, y \in R$, $[x, yx^r - x^s f(y)x^t] = 0$ for some $f(X) \in X^2\mathbb{Z}[X]$.

Main Results

The main results of this paper are stated as follows:

Theorem 1. *Let R be a ring with unity 1. If R satisfies (P_1), then R is commutative.*

Theorem 2. *Let R be a ring with unity 1. If R satisfies (P_2), then R is commutative.*

Theorem 3. *Let R be a ring with unity 1. If R satisfies (P_3), then R is commutative.*

As is easily seen from the proof of [5, Korollar (1)], if R is a non-commutative ring, then there exists a factor subring of R which is of type (a), (b), (c), (d) or (e):

(a) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(b) $M_\sigma(\mathbb{F}) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \middle| \alpha, \beta \in \mathbb{F} \right\}$, where $\mathbb{F}$ is a finite field with a non-trivial automorphism σ.

(c) A non-commutative division ring.

(d) A domain $S = \langle 1 \rangle + T, T$ is a simple radical subring of S.

(e) $S = \langle 1 \rangle + T, T$ is a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

The following result plays an essential role in our subsequent study:

Meta Theorem. *Let P be a ring property which is inherited by factor subrings. If no rings of type (a), (b), (c), (d), or (e) satisfy P, then every ring with 1 and satisfying P is commutative.*

Proof of Theorem 1. In view of Meta Theorem, it suffices to show that R cannot be of type (a), (b), (c), (d) or (e). For each $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$, we set

$$x = e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad y = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in the hypothesis to get

$$\{1 - h(e_{21}e_{11}^r)\}[e_{11}, e_{21}e_{11}^r - f(e_{21}e_{11}^r)]\{1 - g(e_{21}e_{11}^r)\} = e_{21} \neq 0.$$

This is a contradiction.

Suppose that $R = M_\sigma(\mathbb{F})$, and put

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, \quad (\sigma(\alpha) \neq \alpha) \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_{21}.$$

Then for each $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ we have

$$\{1 - h(yx^r)\}[x, yx^r - f(yx^r)]\{1 - g(yx^r)\} = e_{21}(\alpha^r(\alpha - \sigma(\alpha)) \neq 0,$$

which is a contradiction.

Suppose that R is a division ring. For any $x \neq 0$ and y in R, there exist $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ such that

$$\{1 - h(yx^r x^{-r})\}[x, yx^r x^{-r} - f(yx^r x^{-r})]\{1 - g(yx^r x^{-r})\} = 0.$$

Thus

$$\{1 - h(y)\}[x, y - f(y)]\{1 - g(y)\} = 0.$$

Then either $[x, y - f(y)] = 0$, $y - yg(y) = 0$ or $y - yh(y) = 0$. Therefore R is commutative by [3, Theorem 3].

Let $v, w \in T$. Then $u = 1 + v \in U(R)$, and there exist $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ such that

$$0 = \{1 - h(wu^r u^{-r})\}[u, wu^r u^{-r} - f(wu^r u^{-r})]\{1 - g(wu^r u^{-r})\}.$$

Thus

$$0 = \{1 - h(w)\}[u, w - f(w)]\{1 - g(w)\}.$$

Then, either $[u, w - f(w)] = 0$, $w - wg(w) = 0$ or $w - wh(w) = 0$. Hence, T is commutative by [3, Theorem 3]. But this is a contradiction.

Finally, suppose that R is of type (e). Let $v, w \in T$. Then $u = 1 + v \in U(R)$ and there exist $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$. In the hypothesis, replace x by $1 + v$ and y by w, to get

$$0 = \{1 - h(w(1 + v)^r)\}[(1 + v), w(1 + v)^r - f(w(1 + v)^r)]\{1 - g(w(1 + v))\}.$$

Thus

$$0 = \{1 - h(w(1 + v)^r)\}(1 + v)^r(1 + v)^r[v, w]\{1 - g(w(1 + v)^r)\}.$$

This implies that $[v, w] = 0$. Therefore T is commutative. This is a contradiction.

Proof of Theorem 2. Let $x, y \in R$ and let $f(X) \in X^2\mathbb{Z}[X]$ and $g(X), h(X) \in X\mathbb{Z}[X]$ such that $\{1 - h(yx^r)\}[x, yx^r - f(x^r y)]\{1 - g(yx^r)\} = 0$. Let $z \in R$ such that $f(x^r y) = x^r z$ and $f(yx^r) = zx^r$. Now we choose $\tilde{f}(X) \in X^2\mathbb{Z}[X]$ and $\tilde{g}(X)\tilde{f}(X) \in X\mathbb{Z}[X]$ such that

$$0 = \{1 - \tilde{h}(f(yx^r))\}[x, f(x^r y) - \tilde{f}(f(yx^r))]\{1 - \tilde{g}(f(yx^r))\} \tag{1}$$

and

$$0 = \{1 - \tilde{h}(zx^r)\}[x, x^r z - \tilde{f}(zx^r)]\{1 - \tilde{g}(zx^r)\}. \tag{2}$$

Now, combining (1) and (2) gives

$$0 = \{1 - \tilde{h}(f(yx^r))\}\{1 - h(yx^r)\}[x, yx^r - \tilde{f}(f(yx^r))]\{1 - g(yx^r)\}\{1 - \tilde{g}(f(yx^r))\}.$$

This implies that

$$0 = \{1 - h(yx^r)\}[x, yx^r - f(yx^r)]\{1 - g(yx^r)\}.$$

By using the same line of the proof of Theorem 1, we prove the commutativity of R.

In preparation for proving Theorem 3, we first establish the following Lemmas:

Lemma 1. If R satisfies (P_3) and x is in $U(R)$, then for each $y \in R$, there exists $h(X) \in X^2\mathbb{Z}[X]$ such that $[x, y - h(y)] = 0$.

Proof. Let $x \in U(R)$ and $y \in R$. Then we choose $f(X) \in X^2\mathbb{Z}[X]$ such that $[x^{-1}, yx^{-r} - x^{-s}f(y)x^{-t}] = 0$. Hence $[x, yx^{-r} - x^{-s}f(y)x^{-t}] = 0$ and $[x, y]x^{-r} = x^{-s}[x, f(y)]x^{-t}$. Thus

$$x^s[x, y]x^t = [x, f(y)]x^r. \tag{3}$$

Now let $g(X) \in X^2\mathbb{Z}[X]$ such that $[x, f(y)x^r - x^s g(f(y))x^t] = 0$. Since $g(f(X)) \in X^2\mathbb{Z}[X]$ and

$$[x, f(y)]x^r = x^s[x, h(y)]x^t. \tag{4}$$

Combining equation (3) and (4) gives $x^s[x, y]x^t = x^s[x, h(y)]x^t$ and so $[x, y] = [x, h(y)]$. Therefore, $[x, y - h(y)] = 0$. □

Lemma 2. If R satisfies (P_3) and R is a division ring, then R is commutative.

Proof. Let R be a division ring. By Lemma 1, for each $x, y \in R$, there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[x, y - f(y)] = 0$. Thus R is commutative by [3, Theorem 3]. □

Lemma 3. If R satisfies (P_3) and $R = \langle 1 \rangle + T$, T is a radical subring of R, then R is commutative.

Proof. Let $v, w \in T$. Then $1 - v$ is a unit and by Lemma 1, there exists $f(X) \in X^2\mathbb{Z}[X]$ such that

$$[v, w - f(w)] = [1 - v, w - f(w)] = [-v, w - f(w)] = -[1 - v, w - f(w)] = 0.$$

Thus R is commutative by [3, Theorem 3]. □

Proof of Theorem 3. In view of Lemma 2 and Lemma 3, no rings of type (c) or (d) satisfy (P_3). In $R = M_2(GF(p))$, where $GF(p)$ is the Galois field over a prime p, we see that $[e_{11}, e_{21}e_{11}^r - e_{11}^s f(e_{21})e_{11}^t] = e_{21} \neq 0$, for every $f(X) \in X^2\mathbb{Z}[X]$. Thus we have a contradiction. Hence, no rings of type (a) satisfy (P_3).

Next consider the ring $M_\sigma(\mathbb{F})$. Let

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, \quad (\sigma(\alpha)) \neq \alpha) \quad \text{and} \quad y = e_{21}.$$

Then

$$\begin{aligned} [x, yx^r - x^s f(y)x^t] &= [x,y]x^t \\ &= (\sigma(\alpha) - \alpha)y\alpha^r \\ &= (\alpha - \sigma(\alpha))\alpha^r y \neq 0 \end{aligned}$$

for every $f(X) \in X^2\mathbb{Z}[X]$.

Finally, suppose that R is of type (e). For each $v, w \in T$, there exists $f(X) \in X^2\mathbb{Z}[X]$ such that

$$[v,w] = [v,w](v+1)^r - (v+1)^s[v, f(w)](v+1)^t = 0.$$

This is a contradiction.

We have thus seen that no rings (a), (b), (c), (d) or (e) satisfy (P_3). Hence R is commutative by Meta Theorem.

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