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THE LINEAR MEAN VALUE OF THE REMAINDER TERM IN THE PROBLEM OF ASYMPTOTIC BEHAVIOR OF EIGENFUNCTIONS OF THE AUTOMORPHIC LAPLACIAN

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Abstract

The purpose of this paper is to obtain the estimate for the average mean value of the remainder term of the asymptotic formula for the quadratic mean value of the Fourier coefficients of the eigenfunctions over the discrete spectrum of the automorphic Laplacian.

1. Introduction

An asymptotic formula for the quadratic mean value of the Fourier coefficients of the eigenfunctions of the discrete spectrum of the Laplace operator which are automorphic for a modular group was found in the paper of N.V Kuznetsov [4].

It can be formulated as follows:

$$\sum_{\kappa_j \leq X} \frac{|\rho_j(n)|^2}{ch\pi\kappa_j} = \frac{1}{\pi^2}X^2 + R'_n(X),$$

where

$$R'_n(X) \ll X lnX + Xn^{\epsilon} + n^{\frac{1}{2}+\epsilon}$$

for any fixed $\epsilon > 0$ and for any $X \geq 2$, $n \geq 1$. The true order of this remainder term is unknown.

The remainder term of the asymptotic formula can be introduced with a small difference from the paper. As well as the Fourier coefficients of the eigenfunctions of discrete spectrum of Laplace operator, the Fourier coefficients of the eigenfunctions of a continuous spectrum are also taken into consideration. So we define the remainder term by the equality:

$$R_n(X) = \sum_{\kappa_j \le X} \frac{|\rho_j(n)|^2}{ch\pi\kappa_j} + \frac{1}{\pi}\int_{-X}^{X} \frac{|\tau_{\frac{1}{2}+ir}(n)|^2}{|\zeta(1+2ir)|^2} dr - \frac{2}{\pi}\int_0^X rth(\pi r)dr,$$

where the $\rho_j(n)$'s, $\tau_{\frac{1}{2}+ir}(n)$'s are the Fourier coefficients of the eigenfunctions of the discrete spectrum and continuous spectrum, correspondingly.

Now we define the mean value of the remainder term by the equality

$$\overline{R}(x;T) = \frac{1}{T}\sum_{n=1}^{\infty} w(\frac{n}{T})R_n(x),$$

where $w(x)$ is infinitely smooth, identically zero outside of the fixed interval [1,2], and is near to 1 inside this interval. And T determines the length of averaging and would be taken sufficiently large.

Our main result is the following assertion:

Theorem 1. *Let $w(x)$ be an infinitely smooth finite function whose support is separated from zero, and $T \gg X^2$. Then*

$$\begin{aligned}\sum_{n=1}^{\infty} w(\frac{n}{T})R_n(X) &= c_1TX\log(2\pi T) + c_2TX - 2c_1TX\log X + 2c_1T(L(1,\chi)+L(1,\chi')) \\ &\quad \hat{w}(1)TX + O(\sqrt{T}X^2 + X\sqrt{\log X}),\end{aligned}$$

where

$$c_1 = \frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)}\ , \quad c_2 = c_1(2 + \frac{\hat{w'}}{w}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}))$$

and

$$\chi(n) = (\frac{-1}{n}), \quad \chi'(n) = (\frac{-3}{n})$$

(Here, $\hat{w}(s)$ is the Mellin transform of $w(x)$).

2. Auxiliary Results

To prove the theorem we use following lemmas:

Lemma 1. Let $h(r)$ be an even function of r, regular in the strip $|Imr| \le \Delta$ for some $\Delta > \frac{1}{2}$, and for some $\rho > 0$, as $|Imr| \to \infty$, $|Imr| \le \Delta$:

$$|h(r)| = O(|r|^{-2-\rho}).$$

Then

$$\int_0^\infty R_n(r)h'(r)dr = \sum_{c\geq 1} \frac{1}{c} S(n,n,c)\varphi(\frac{4\pi n}{c}), \qquad (1)$$

where $S(n,m,c)$ is the classical Kloosterman sums and

$$\varphi(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(x)\frac{r}{ch(\pi r)}h(r)dr$$

with the usual notation for the Bessel function.

Proof. Follows from the main identity [4] after puttin $n = m$. □

Lemma 2. $\sum_{m=1}^{c} S(m,m,c)e^{2\pi i(\frac{nm}{c})} = c\nu_n(c)$, where $\nu_n(c)$ is the number of the solutions of the quadratic congruence $a^2 + na + 1 \equiv 0\ (mod\ c)$.

Proof. It follows directly from the definition of Kloosterman sums.

By multiplying both sides of the equation (1) by $w(\frac{n}{T})$, and taking summing over n, we get

$$\int_0^\infty \overline{R}(r;T)h'(r)dr = \frac{1}{T}\sum_{n=1}^{\infty} w(\frac{n}{T})\sum_{c\geq 1}\frac{1}{c}S(n,n,c)\varphi(\frac{4\pi n}{c}). \qquad (2)$$

□

3. The Dirichlet Series

Now we choose $h(r)$ as

$$h(r) = t_{X,\Delta}(r)\frac{q(r)}{q(r)+M} \quad \text{for} \quad M > 0,$$

where

$$q(r) = \left(r^2 + \frac{1}{4}\right)$$

and

$$t_{X,\Delta} = \int_{-X}^{X} e^{-\frac{(r-\xi)^2}{\Delta^2}}d\xi \left(\int_{-\infty}^{\infty} e^{-\frac{(r-\xi)^2}{\Delta^2}}d\xi\right)^{-1}.$$

Clearly $t_{x,\Delta}$ is almost 1 if r is in the interval $[-X, X]$ and it rapidly decreases outside. Equation (2) equals to

$$\sum_{c\geq 1}\frac{1}{c}\sum_{m=1}^{c} S(m,m,c)\sum_{n_1=0}^{\infty} w\left(\frac{c}{T}(n_1 + \frac{m}{c})\right)\varphi\left(4\pi\left(\frac{m}{c} + n_1\right)\right)$$

since we can replace n by $m + n_1\, c$, where $n_1 = 0, 1, 2..$ and $1 \le m \le c$. Note that $S(n, m, c)$ is a periodic function. Now we consider the following function

$$f(x; c) = \sum_{n_1=0}^{\infty} w\left(\frac{c}{T}(n_1 + x)\right) \varphi(4\pi(n_1 + x)). \tag{3}$$

The series in (3) determines a periodic function of x, with period 1. Hence $f(x; c)$ has the Fourier expansion

$$f(x; c) = \sum_{n=-\infty}^{\infty} e(n\frac{m}{c})\, \phi_n(c),$$

where the coefficients are given by the integrals

$$\phi_n(c) = \int_0^{\infty} e(-nx) w(\frac{c}{T}x) \varphi(4\pi x) dx.$$

Putting the expansion into the equation and by using Lemma 2, (2) equals to

$$\sum_{n=-\infty}^{\infty} \sum_{c \ge 1} \nu_n(c)\ \phi_n(c), \tag{4}$$

and it may be expressed in terms of Dirichlet series; for that we use the Mellin transform of $w(x)$ which is

$$\hat{w}(s) = \int_0^{\infty} w(\frac{c}{T}x) x^{s-1} dx,$$

and the inversion formula

$$w(\frac{c}{T}x) = \frac{1}{2\pi i} \int_{\sigma} \hat{w}(s) c^{-s} x^{-s} T^s ds$$

holds. (Here $\int_\sigma$ means the integration is over the line $Res = \sigma$.) It can be seen from integration by parts that $\hat{w}(s)$ is an rapidly decreasing function. We can arbitrary choose σ since $\hat{w}(s)$ is an entire function. Then (4) equals to

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{(\sigma)} \varphi_n(s) \hat{w}(s) T^s L_n(s) ds, \tag{5}$$

where

$$L_n(s) = \sum_{c \ge 1} \frac{\nu_n(c)}{c^s} \quad \text{for } Res > 1$$

and

$$\varphi_n(s) = \int_0^{\infty} cos(2\pi nx) x^{-s} \varphi(4\pi x) dx\ .$$

(The possibility of changing the order of summation and integration, and two integration will be clear after calculating $\varphi_n(s)$.)

Now two questions arise, of what is the inner integral and what is the analytic continuation of $L_n(s)$? We will solve these questions separately.

Lemma 3. $L_2(s) = \frac{1}{\zeta(2s)}\zeta(s)\zeta(2s-1)$.

Lemma 4. When $n \neq 2$

$$L_n(s) = \frac{\zeta(s)}{\zeta(2s)} \prod_{\substack{p>2 \\ p \nmid n^2-4}} (1 - \frac{1}{p^s}(\frac{n^2-4}{p}))^{-1}\, \eta(s)$$

where

$$\eta(s) = \left(1 + \frac{\nu_n(2)}{2^s} + \frac{\nu_n(4)}{2^{2s}} + \cdots\right) \prod_{\substack{p>2 \\ p | n^2-4}} \left(1 + \frac{\nu_n(p)}{p^s} + \frac{\nu_n(p^2)}{p^{2s}} + \cdots\right) \prod_{\substack{p>2 \\ p|n^2-4}} \left(1 + \frac{1}{p^s}\right).$$

Now we can give the explicit expression for the integrals of $\varphi_n(s)$.

Lemma 5. For any ρ with the condition $0 < \rho < \frac{3}{2}$ we have for $Res < 1 + 2\rho$,

$$\varphi_n(s) = i\pi^{s-2} \int_{Im=-\rho} \psi_n(r,s) r \frac{h(r)}{ch\pi r} dr,$$

where

$$\begin{aligned} \psi_n(r,s) &= 2^{s-1}\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\,\Gamma(\frac{1-s}{2}+ir)}{\Gamma(\frac{1+s}{2}+ir)\Gamma(\frac{s}{2}+ir)\Gamma(\frac{s}{2}-ir)}, \quad \text{if } n=2, \\ &= 2^{s-1}\frac{\Gamma(\frac{1-s}{2}+ir)}{\Gamma(\frac{1+s}{2}+ir)} F(\frac{1-s}{2}+ir, \frac{1-s}{2}-ir, \frac{1}{2}; \frac{n^2}{4}), \quad \text{if } n = 0 \text{ or } 1, \\ &= n^{s-1}\sqrt{\pi}(\frac{2}{n})^{2ir}\frac{\Gamma(\frac{1-s}{2}+ir)}{\Gamma(2ir+1)\Gamma(\frac{s}{2}-ir)}F(\frac{1-s}{2}+ir, 1-\frac{s}{2}+ir, 2ir+1; \frac{4}{n^2}), \\ &\quad \text{if } n > 2. \end{aligned}$$

Here

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(n)\,\Gamma(c+n)}\, z^n$$

is the Gauss hypergeometric function.

Proof. It can be proved by using the discontinuous integral of Weber and Schafheitlin [2]. □

The Principal Term

As a result of Lemma 5, we have for $\rho \in (0, \frac{3}{2})$, $1 < Res < 1 + 2\rho$

$$\int_0^\infty \overline{R}(r,T)h'(r)dr = \sum_{n=0}^\infty \int_{Imr=-\rho} \frac{rh(r)}{ch(\pi r)} \Omega_n(r,T)dr, \tag{6}$$

where

$$\Omega_n(r,T) = \int_{\sigma=Res} \pi^{s-3}\psi_n(r,s)\hat{w}(s)T^s L_n(s)ds.$$

Firstly we consider the case $n = 2$. The integrand of $\Omega_2(r,T)$ is a meromorphic function since $L_2(s)$ has the double pole at s=1, and other multipliers have no singularity for $\frac{1}{2} < Res < 1 + 2\rho$. We move the line of integration to the left and we integrate now on the line $Res = \sigma_1 = \frac{1}{2} + \varepsilon$, $\varepsilon > 0$. Thus

$$\begin{aligned} \Omega_2(r,T) &= \frac{1}{\pi}T\frac{\hat{w}(1)}{\zeta(2)}\frac{ch\pi r}{r}(\log 2\pi T + \frac{\hat{w}'}{w}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2}) \\ & -2\log r - i\frac{\pi}{2}cth\pi r) + \int_{\sigma=\frac{1}{2}+\varepsilon} \pi^{s-3}\psi_2(r,s)\hat{w}(s)T^s L_2(s)ds. \end{aligned}$$

After integrating $\Omega_2(r,T)$ with multiplier $\frac{rh(r)}{ch\pi r}$ on the line $Imr = -\rho$ (here, ρ can be taken as 0) we get the main term of the series in the formula (6), which is : $c_1 TX\log(2\pi T) + c_2 TX - 2c_1 TX \log X + O(\sqrt{T}X^2)$, where

$$c_1 = \frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)} \tag{7}$$

and

$$c_2 = c_1\left(\frac{\hat{w}'}{w}(1) + 3\gamma - 2\frac{\zeta'}{\zeta}(2) + \frac{\Gamma'}{\Gamma}(\frac{1}{2})\right). \tag{8}$$

Here, γ is the Euler constant, $\gamma = -\frac{\Gamma'}{\Gamma}(1)$. And the second integral can be estimated as $O(\sqrt{T}X^2)$.

The Cases $n \neq 2$

For the case $n > 2$, again the integrand of $\Omega_n(r,T)$ is meromorphic function for $L_n(s)$ has a simple pole at $s = 1$. We move the line of integration to the line $Res = \sigma_1 = \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ for $n < T^{\frac{1}{2}}$. We get

$$\Omega_n(r,T) = \frac{2}{\pi} n^{-2ir} \frac{ch\pi r}{r} \hat{w}(1) T B_n + \int_{\sigma_1 = \frac{1}{2}+\varepsilon} \pi^{s-3} \psi_n(r,s) \hat{w}(s) T^s L_n(s) ds \quad n > 2,$$

where B_n is the residue of $L_n(s)$ at $s = 1$. The result follows by the equality

$$\frac{\Gamma(ir)}{\Gamma(2ir+1)\Gamma(\frac{1}{2} - ir)} = \pi^{-\frac{1}{2}} 2^{-2ir} \frac{ch\pi r}{ir},$$

and

$$F(ir, \frac{1}{2} + ir, 1 + 2ir; x)\, x^{ir} = 2^{2ir} e^{-ir\xi}$$

where $x = \frac{4}{n^2}$ and $\xi = \log \frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$ [3].
Here in order to estimate the integral on the line $\sigma_1 = \frac{1}{2} + \varepsilon$, it is necessary to find a bound for $L_n(s)$. To do this we express $L_n(s)$ in terms of the classical Dirichlet's series with the Kronecker symbol,

$$L_n(s) = \sum_{c \geq 1} \left(\frac{n^2 - 4}{c} \right) \frac{1}{c^s}.$$

Since our character is not primitive, we write $n^2 - 4 = k^2 Q$, where Q is square free and $k > 0$. We get

$$L_n(s) = \prod_{p|k} \left(1 - (\frac{Q}{p}) \frac{1}{p^s} \right) L(s, \chi)$$

where $\chi = (\frac{Q}{p})$ is real, primitive character. Then we have the following Lemma.

Lemma 6.

$$L(s,\chi) \ll Q^{\frac{1}{4}+\varepsilon_0} \quad \text{for any} \quad \varepsilon_0 > 0.$$

Proof. Proof can be obtained by expressing $L(s,\chi)$ as a finite sum. We introduce the function $\alpha(x)$ such that $\alpha \in C^\infty(0,\infty)$, $\alpha \equiv 1$ for $x \leq x_0 < 1$ and $\alpha(x) \equiv 0$ if $x > \frac{1}{x_0}$. Then we apply the functional equation for $L(s,\chi)$ (see [1] e.g.).
It is clear that the product $L_n(s) = \prod_{p|k}(1 - (\frac{Q}{p})\frac{1}{p^s})$ is not larger than $\sum_{d|k} d^{-\sigma}$ $(\sigma = Res)$, but is smaller than k^ε for any $\varepsilon > 0$. So we have the estimate

$$|L_n(s)| \ll k^\varepsilon Q^{\frac{1}{4}+\varepsilon} \quad \text{if} \quad n^2 - 4 = k^2 Q.$$

When we integrate $\Omega_n(r,T)$ with multiplier $\dfrac{rh(r)}{ch\pi r}$ on the line $Imr = -\rho$, we get

$$TXn^{-2\rho}B_n\hat{w}(1) + O(T^{\frac{1}{2}}X^{\frac{3}{2}+\varepsilon_0}).$$

In order to obtain the estimate we find the asymptotic expansion of $F(\frac{1-s}{2} + ir,\ 1 - \frac{s}{2} + ir,\ 2ir + 1; \frac{4}{n^2})$ for large values of r and n. We use the standard methods of the asymptotic integration of differential equation with large parameter. We get

$$\begin{aligned} &F(\frac{1-s}{2} + ir, 1 - \frac{s}{2} + ir, 2ir + 1; \frac{4}{n^2}) \\ &= (\frac{4}{n^2})^{-ir}(1 - \frac{4}{n^2})^{\frac{s}{2}-\frac{1}{2}}2^{2ir}e^{-ir\xi}\left(1 + \frac{1}{2ir}\int_\xi^\infty f(\eta)d\eta + O(\frac{e^{-\eta}}{r^2})\right) \end{aligned}$$

where

$$f(\eta) = \frac{-3}{16(ch^2\frac{\eta}{2} - 1)ch^2\frac{\eta}{2}} + \frac{\frac{7}{4} - 2s^2 + 2s}{4ch^2\frac{\eta}{2}} \quad \text{and} \quad \xi \sim 2\log n$$

And for $n > T^{\frac{1}{2}}$, we approximate $\Omega_n(r,T)$ on the line $Res = \sigma_1 = 1 + \varepsilon_0$, and with $\varepsilon > 0$ we get

$$\int_{Imr=-\rho} \Omega_n(r,T)\frac{rh(r)}{ch\pi r}dr \ll T^{1+\varepsilon_0}X^{1-\varepsilon_0}\frac{1}{n^{2\rho}}.$$

So we have in (6)

$$TX\hat{w}(1)\sum_{3\le n\le N}\frac{B_n}{n^{2\rho}} + \sum_{n>N}T^{1+\varepsilon_0}X^{1-\varepsilon_0}\frac{1}{n^{2\rho}} + O(T^{\frac{1}{2}+\varepsilon_0}X^{\frac{3}{2}+\varepsilon_0}) \quad \text{if} \quad T > X,$$

where $N = T^{\frac{1}{2}+\varepsilon_0}$.

The cases $n = 0$ and $n = 1$ are the trivial ones since we have exponentially small function in the integrand. And we get first and second term of the series in the formula (6) as

$$\frac{2}{\pi}\frac{1}{\zeta(2)}T\hat{w}(1)L(1,\chi) + O(T^{\frac{1}{2}})$$

and

$$\frac{2}{\pi}\frac{1}{\zeta(2)}\hat{w}(1)L(1,\chi') + O(T^{\frac{1}{2}}),$$

correspondingly. Here $\chi(n) = (\frac{-1}{n})$ and $\chi'(n) = (\frac{-3}{n})$.

As a result: For $T \gg X^2$,

$$\begin{aligned} &\sum_{n=0}^\infty w(\frac{n}{T})\int_0^\infty R_n(r)h'(r)dr = \\ &\quad c_1TX\log(2\pi T) + c_2TX - 2c_1TX\log X + \hat{w}(1)T(X + 2c_1T(L(1,\chi) + L(1,\chi'))) \\ &\quad +O(\sqrt{T}X^2), \end{aligned}$$

where c_1 and c_2 are as defined in (7) and (8).

By estimating the integral $\int_0^\infty R_n(r)h'(r)dr$ from below by $R_n(X-2\Delta\sqrt{\log X})+O(X\sqrt{\log X})$ and from above by $R_n(X+2\Delta\sqrt{\log X})+O(X\sqrt{\log X})$ we prove the theorem. □

From the theorem it is seen that the average mean value of the remainder term of $R_n(X)$ is positive. As a conclusion: There are infinitely many n's and X's such that for $n\in(T,2T)$ and $T\gg X^2$, for which we have

$$R_n(X)\gg X(\log n)^\alpha,\quad \forall\alpha<1.$$

The mean value of integral with the Fourier coefficients of the eigenfunctions of the continuous spectrum can be computed as

$$\frac{1}{\pi}\sum_{n=1}^{\infty}w(\frac{n}{T})\int_{-\infty}^{\infty}|\tau_{\frac{1}{2}+ir}(n)|^2\frac{h(r)}{|\zeta(1+2ir)|^2}dr$$
$$\frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)}T\log TX+\frac{1}{\pi}\frac{\hat{w}(1)}{\zeta(2)}(\frac{\hat{w}'}{w}(1)-2\frac{\zeta'}{\zeta}(2)+2\gamma)TX+O(T\log X)+O(T^{\frac{1}{2}+\varepsilon_0}).$$

So we can assert that there exist infinitely many n's and X's satisfying

$$R_n'(X)<\frac{-1}{2\pi}\frac{\hat{w}(1)}{\zeta(2)}X\log X$$

for $n\in(T,2T)$ and $T\gg X^2$.

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References

[1] Davenport, H.: Multiplicative number theory, 2nd ed., Springer-Verlag.

[2] Erdelyi, A.: Higher transcendental functions, Vol1, McGraw-hill 1953.

[3] Erdelyi, A.: Higher transcendental functions, Vol2, McGraw-hill 1953.

[4] Kuznetsov, N.V.: Petersson's conjecture for cusp forms of weight zero and Linnik' s conjecture. Sum of Kloosterman sums, Math.USSR Sbornik Vol.39(1981),No.3.

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