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CAHİT ARF'S CONTRIBUTION TO ALGEBRAIC NUMBER THEORY AND RELATED FIELDS

by
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Dedicated to the memory of Professor Cahit ARF

1. Introduction

Since 1939 Cahit Arf has published number of papers on various subjects in Pure Mathematics. In this note I shall try to give a brief survey of his works in Algebraic Number Theory and related fields. The papers covered in this survey are: two papers on the structure of local fields ([1], [5]), a paper on the Riemann-Roch theorem for algebraic number fields ([6]), two papers on quadratic forms ([2], [3]), and a paper on multiplicity sequences of algebraic braches ([4]). The number of the papers cited above is rather few, but the results obtained in these papers as well as his ideas and methods developed in them constitute really essential contributions to the fields. I am firmly convinced that these papers still contain many valuable suggestions and hidden possibilities for further investigations on these subjects. I believe, it is the task of Turkish mathematicians working in these fields to undertake and continue further study along the lines developed by Cahit Arf who doubtless is the most gifted mathematician Turkey ever had.

In doing this survey, I did not follow the chronological order of the publications, but rather divided them into four groups according to the subjects: Structure of local fields, Riemann-Roch theorem for algebraic number fields, quadratic forms, and multiplicity sequences of algebraic braches. My aim is not to reproduce every technical detail of Cahit Arf's works, but to make his idea in each work as clear as possible, and at the same time to locate them in the main stream of Mathematics.

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2. Algebraic Number Theory in 1930's

In order to locate Cahit Arf's works, on Algebraic Number Theory and related fields, in the main stream of Mathematics properly, we need some knowledge about the history of Algebraic Number Theory. Besides Cahit Arf completed his thesis in Göttingen in 1938, so we also have to know what the mathematical atmosphere in Germany was like in 1930's. I think therefore, it is not meaningless to begin this note with talking how Algebraic Number Theory developed.

As is known, the Göttingen school of Mathematics has had a long glorious tradition beginning with C.F. Gauss followed by B. Riemann and others. Algebraic Number Theory was born in Göttingen by Gauss, and was raised to a gigantic theory, Class Field Theory, by mathematicians in or closely related to Göttingen. Algebraic Number Theory founded by Gauss at the beginning of the 19th century, however, was merely the theory of quadratic extensions of $\mathbb{Q}$, the field of rational numbers, in which the central theorem was the so-called reciprocity law of quadratic residue symbol. The general development of Algebra in the 19th century, in particular the invention of the duality by E. Galois, enabled algebraic number theorists at that time to tackle more general extensions of $\mathbb{Q}$ than quadratic ones. In 1890's D. Hilbert, one of the representative figures of the Göttingen school then, continued further study on cyclotomic fields, i.e. number fields of the form $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive n-th root of unity, following the investigations initiated by Kronecker. Hilbert then realized that one can completely describe the decomposition law of primes in $\mathbb{Q}(\zeta_n)$ by means of the multiplicative group $(Z/n)^x$. This observation led him to formulate - though in a rather vague fashion - one of the most important problems in Algebraic Number Theory. The problem is this: How can one completely describe the decomposition law of primes in a finite Galois extension $F/\mathbb{Q}$ in terms of some (canonically definable) object in $\mathbb{Q}$? Of course this object must coincide with $(Z/n)^x$ for the cyclotomic case. The same problem can be asked for finite Galois (relative galoisien) extensions of any algebraic number field of a finite degree over $\mathbb{Q}$. The year 1920 may be considered as a turning point of Algebraic Number Theory. Namely T. Takagi, an ex-student at Göttingen but an unknown mathematician then, succeeded to solve the problem for the case of relative abelian extensions. After seven years, the theory of Takagi was perfected in a surprisingly beautiful way by the discovery of the general reciprocity law by E. Artin in Hamburg; this gives an explicit description of the decomposition law via the Artin symbol. By the way, it is very remarkable that this fundamental discovery was in fact a by-product of another investigation on general L-functions defined by him (Artin L-functions). In trying to show that his general L-functions coincide with the classical L-functions for abelian extensions, E. Artin was led to conjecture the reciprocity mentioned above. In this way, the problem raised by Hilbert at the beginning of this century was completely solved by Takagi-Artin for relative abelian extensions; today their theory is know as the "global" class field theory.

The valuation theory founded by K. Hensel in Marburg was intensively and successful applied by his pupil H. Hasse both in Number Theory and Algebraic Function

Theory; roughly speaking, Hasse's idea consists in the deduction of a statement about an algebraic number (or function) field from its counter-part for all local completions of the field (Hasse's principle). So it was quite natural to look for the counter-part of the "global" class field theory for local number fields. The theory thus obtained is the "local" class field theory. It is the theory of relative abelian extensions of local number fields. The local class field theory was established by applying the structure theory of central simple algebras over local and/or global number fields which was intensively studied for its own sake by many mathematicians in Göttingen, such as R. Brauer, H. Hasse and E. Noether. By the way, this approach is in fact the origin of the cohomological approach in class field theory initiated by J. Tate in 1950's and accepted as standard approach today. Because local number fields are of much more simple structure than global ones, the local class field theory is less complicated than global one. It is therefore natural to try to construct the "global" theory upon the local ones. In fact, this was done by C. Chevalley in 1940 by using the concept of ideles.

Now in his thesis in 1923, Artin close hyperelliptic function fields over finite constant fields as analogues of quadratic number fields, and defined the zeta-function on such fields. Further he pointed out that the analogue of the famous Riemann hypothesis seemed to be true for these zeta-functions. This was the origin of the problem known as the Riemann hypothesis for congruence zeta-functions; the problem engaged enthusiastic interest in 1930-41, and was solved for algebraic function field of genus 1 by H. Hasse in 1936, and for general case by A. Weil in 1941. Under these circumstances there revived the approach based on tracking down the parallelism between theories of algebraic numbers and algebraic functions. This approach was not new, for instance when Hilbert and others were trying to establish the class field theory, the model they had in mind was Riemann surfaces. In any case, this approach stimulated investigations in the general theory of algebraic function fields over arbitrary constant fields. A typical example for this, is the Riemann-Roch theorem for algebraic function fields over arbitrary constant fields obtained by A. Weil in 1938. In this work Weil successfully utilized adeles, analogues of Chevalley's ideles. The approach cited above also intensified the attempts making analogy of known facts in number fields for function fields and vice versa. This in turn stimulated the study of the structure of local fields. Everything known for local number fields was re-examined from the point of view in what extent it remains valid for local fields. This was mainly done by young mathematicians in Göttingen in 1930's, such as O. Teichmüller and E. Witt.

It should not be forgotten that in 1930's a radical political change took place in Germany. In the middle of 1930's the traditional autonomy of German Universities finally gave way to the political pressure, as a result, many brilliant mathematicians of non-German origin, E. Artin among others, were compelled to leave the country. Thus the golden age of the German school came to a sudden end. But there remained still number of brilliant mathematicians in Göttingen, such as H. Hasse and E. Witt. It was like that in Göttingen when Cahit Arf arrived there in 1937.

3. Structure of Local Fields

(a) The paper [1] appeared in the Crelle Journal in 1939 was the thesis of Cahit Arf, and was accepted by the University of Göttingen in June 1938, soon after his arrival in Göttingen. This fact shows not only that Cahit Arf was highly gifted in Mathematics, but that he was then already mathematically mature enough. In order to talk about the content of this paper, we have to mention the work [7] by E. Artin. In the course of his study on the functional equation for his L-functions Artin was led to the concept of "non-abelian" conductor (Artin conductor), which was defined and studied in [7]. The main result of [7] is a generalization of the so-called conductor-discriminant theorem. To be explicit, we adopt the usual notations: K, a Galois extension of an algebraic number field k of a finite degree over $\mathbb{Q}$; G stands for the Galois group of K/k; for any prime ideal $\underline{p}$ of k and for any prime factor $\underline{P}$ of $\underline{p}$ in K, the inertia group of $\underline{P}$ is denoted by $T(=V_0)$, and the ramification groups by $V_1, V_2, \ldots$; $\underline{x}$ denotes any character of G, and

$$\underline{x}(T) = \sum_{\sigma \in T} \underline{x}(\sigma), \qquad \underline{x}(V_1) = \sum_{\sigma \in V_1} \underline{x}(\sigma)$$

etc.
The Artin conductor belonging to $\underline{x}$ is then defined by

$$f(\underline{x}, K/k) = \prod_{\underline{p}} \underline{p}^{N(\underline{x},p)}$$

where

$$N(\underline{x}, \underline{p}) = \frac{1}{|V_0|} \sum_{i=0}^{\infty} (|V_i|\underline{x}(1) - \underline{x}(V_i))\,.$$

$N(\underline{x}, \underline{P})$ is independent of the choice of $\underline{P}$. Now the main results of Artin can be summarized as follows:

1. $f(\underline{x}, K/k)$ is an integral ideal of k (i.e. $N(\underline{x}, \underline{p})$ is a non-negative integer);
2. if K/k is abelian, $f(\underline{x}, K/k)$ coincides with the conductor in the class field theory;
3. some formulas for $f(\underline{x}, K/k)$;
4. $\{\underline{x}_v\}$ being the totality of irreducible characters of G, the discriminant $D_{K/k}$ is equal to $\prod_v f(\underline{x}_v, K/k)^{\underline{x}_v(1)}$.

As Cahit Arf pointed out in [1], the statements above, except (1) and (2), are formal consequences of the definition of $f(\underline{x}, K/k)$. In fact, one of the main problems in [7] was the proof of the integrality of $N(\underline{x}, \underline{p})$. But $N(\underline{x}, \underline{p})$ is the number defined in the completion of K at $\underline{p}$. So Cahit Arf aimed in [1] to show the integrality of $N(\underline{x}, \underline{p})$

for any Galois extension K of an arbitrary complete field k with respect to a discrete valuation, so that the result of Artin, except (2), will be valid for such K/k. This was the motivation of [1]. A typical example of one of the approach already mentioned in the formed section. Cahit Arf, however, did in [1] much more than that. Namely he started with the question how one can construct a finite extension of a complete field with respect to a discrete valuation. A very fundamental question. What Cahit Arf did in §1-3 in [1] can be considered as the ramification theory in non-normal totally ramified extensions of a complete field with respect to a discrete valuation. (For the sake of simplicity, I call such a field local). Usually the ramification theory is done for Galois extensions, so that Galois groups are available, and every thing can be neatly done by using ramification groups. But for non-normal extensions, the corresponding theory is difficult to formulate because of the lack of automorphisms. A naive idea for this will be the imbedding of the extension into its Galois closure. But Cahit Arf developed a quite different and very original approach to this. I have strong feeling that §1-3 of [1] still contain many useful informations about the structure of local fields. Now let us look at the contents of [1] more closely.

The sections 1-3 are devoted, as said above, to obtain invariants of a totally ramified extension K of a local field k. For this purpose, he chooses a fixed multiplicatively closed system of representatives, say R, of the residue class field $\overline{k}$, and consider the expressions of the form

$$S_r(x) = (f_1(x))^{p^r} + (f_2(x))^{p^r} + \cdots + (f_3(x))^{p^r}$$

where p is the characteristic of $\overline{k}$, and $f_i(x)$'s are formal power series in the variable x with coefficients in R. If $\{f_i(x)\}$ is a sequence of formal power series with the initial terms of orders tending to infinity, then the infinite sum of the form

$$S_r(x) = \sum_{i=1}^{\infty} (f_1(x))^{p^r}$$

will be taken into consideration (Look at $S_{r_h}(x)$ in the following discussion).

Let Π be a prime element of K, then for any element

$$\theta = \Pi^n \sum_{i=0}^{\infty} \xi_i \, \Pi^i \quad (\xi_i \in R).$$

Can be written in the form

$$\theta = \Pi^n \sum_{\ell=0}^{h} \Pi^{i_\ell} S_{r_\ell}(\Pi).$$

The whole theory developed in [1] is based on this rather strange looking decomposition. Of course the decomposition is not unique; after some discussions about connections between different decompositions, the concept of "minimal decomposition" is introduced:

$$\theta = \Pi^n \sum_{\ell=0}^{\ell} \Pi^{i_\ell} S_{r_\ell}(\Pi)$$

is said to a minimal decomposition if $\{i_0, i_1, \ldots, i_\ell\}$ is the greatest in the sense of lexicographical order. It is shown that $\{i_1, \ldots, i_h\}$ depends only on the order of θ. Furthermore it is shown that the ration

$$\theta : \Pi^n S_{r_0}(\pi) : \cdots : \Pi^{n+i_h} S_{r_h}(\pi) \bmod {}^x\Pi$$

depends only on the order of θ. Taking a prime element π in place of θ, we can speak of the sequence $\{i_1, \ldots, i_h\}$ and the ratio belonging to π; they are the invariants of K/k. In fact they determine K/k up to isomorphism. By the way, a necessary and sufficient condition for K/k to be Galois is given in terms of these invariants. Now in §2, the most important part of [1], the ramification theory is developed. To reproduce the technical details here is almost impossible, but, roughly speaking, starting from a minimal decomposition of any element θ in K, a number sequence and a function are defined which turn out respectively to be the sequence of jump indices of ramification groups and the Herbrand function if K/k is Galois. Really the process in defining them is wonderful. §4 is devoted to the imbedding of a totally ramified Galois extension into a central simple algebra; maybe this part can be interpreted in cohomological language, and maybe simplified in some extent. Finally in §5 the proof of the integrality of $N(\underline{x}, \underline{p})$ is given. This is done by proving a more general statement concerning geneating elements of maximal (totally ramified) subfields of a central simple algebra. By the way, the proof of the integrality of $N(\underline{x}, \underline{p})$ is reduced first to the case of abelian extensions, and the statement reduced to the latter case is usually referred as the Hasse-Arf theorem.

(b) The paper [5] appeared in the Abhandlungen aus dem mathematischen Seminar der Universität Hamburg in 1955. In this remarkable paper Cahit Arf gave an explicit construction of the separable closure of the field of formal Laurent series over a finite field (i.e. a local field with a finite residue class field of the equi-characteristic case). The method of construction consists in symbolizing the process of taking the successive extensions of Artin-Schreier type. If k is a field of characteristic p, the extension defined by the equation $x^p - x = \omega$ with $\omega \in k$ in said to be of Artin-Schreier type. This type of extensions play the counter-part of the role Kummer extensions play in the zero characteristic case. It may be worthwhile to mention that the extensions of Artin-Schreier type are always normal, so that the construction problem of Galois extensions by the characteristic p case is rather easy compared with the zero characteristic case.

Not let P be the prime field of characteristic p, k be the algebraic closure of P, and $L = P\langle t\rangle$ be the field of formal Laurent series over P. Further let K be the field of formal Laurent series such that each series in K has coefficients in a finite extensions of P. Clearly K is a separable algebraic extension of L. Let Ω be the union of all extensions of the form $K(\sqrt[v]{t})$ where v ranges over all positive integers not divisible by p. Then we consider the vector space S over Ω spanned by symbols

$$\begin{pmatrix} \xi_1, \ldots, \xi_n \\ \nu_1, \ldots, \nu_n \end{pmatrix}$$

where n is any positive integer, the ξ_i 's are elements of Ω, and the v_i 's are positive p-integers. The addition and multiplication of vectors are defined. The addition is simple, but the multiplication is rather complicated and defined by recursion on n. The meaning of the symbols

$$\begin{pmatrix} \xi_1, \ldots, \xi_n \\ \nu_1, \ldots, \nu_n \end{pmatrix}$$

can be seen from the relations:

$$\begin{pmatrix} \xi \\ \nu \end{pmatrix}^p - \begin{pmatrix} \xi \\ \nu \end{pmatrix} = \frac{\xi}{t^\nu} \text{ and } \begin{pmatrix} \xi_1, \ldots, \xi_n \\ \nu_1, \ldots, \nu_n \end{pmatrix}^p - \begin{pmatrix} \xi_1, \ldots, \xi_n \\ \nu_1, \ldots, \nu_n \end{pmatrix} = \begin{pmatrix} \xi_1, \ldots, \xi_{n-1} \\ \nu_1, \ldots, \nu_{n-1} \end{pmatrix} \frac{\xi_n}{t^{\nu_n}}.$$

It is shown that S is the desired closure of L, then the action of elements in the Galois group of S/L is explicitly given by using these symbols.

It must be noted that this short but elegant paper is the starting point of Cahit Arf's further study on the structure of S/L in which he attempts to establish the "non-abelian" local class field theory for L. So far as I remember, his idea in doing this is to put Artin representations into connection. More precisely, if one can construct a canonical $Gal(S/L)$-module M in such a way that the Artin representation belonging to any finite Galois subextension F/L of S/L is afforded by a factor module (or a submodule) of M, then this module M may serve as the object in the formation of the "non-abelian" local class field. Cahit Arf always says that this will be one of his life work. Unfortunately the most part of his recent works in this line, including his painstaking research done during his stay in the States in 1963-65, has never been published.

4. Riemann-Roch Theorem for Algebraic Number Fields

As was already mentioned in Section 1, the classical theorem of Riemann-Roch, one of the most fundamental theorems in Complex Analysis, was first generalized by A. Weil to the case of algebraic function fields over arbitrary constant fields. In his work, Weil introduced the concept of adele groups in analogy of Chevalley's idele group. The latter was successfully utilized by Chevalley in establishing the global class field theory without using zeta-functions. On the other hand, as already mentioned in Section 1, it has long been recognized that there is a certain parallelism between the theories of algebraic numbers and algebraic functions in one variable. These theories have developed and are still developing stimulating each other. Hence it is quite natural to ask if there is an analogue of Riemann-Roch theorem in Weil's version for algebraic number fields. This is the motivation of this paper.

Before entering into detail, it will be worthwhile to outline the Riemann-Roch theorem in Weil's version.

Let K be an algebraic function in one variable over the constant field k, and A_K be the adele ring of K, i.e. the subset of the Cartesian product of the completions K_p, where P ranges over all prime divisor of K, such that $\prod_p x_p \in A_K$ iff $v_p(x_p) \geq 0$ for almost all P. The field K is imbedded into A_K by the diagonal map, and A_K is a k-algebra under the componentwise operations. Further A_K is a topological algebra under a topology defined as follows. A divisor a of K is a finite formal product of prime divisors of K. Now $\underline{a} = \prod_p p^{v_p(\underline{a})}$ being a divisor of $K, N(\underline{a})$ denotes the k-subspace consisting of all adeles $\prod_p x_p$ satisfying the condition $v_p(x_p) \geq -v_p(\underline{a})$ for all P. The topology on A_K is then defined by taking $\{N(\underline{a})\}$ ($\underline{a}$ ranging over all divisors of K) as a base of open sets around the zero. Since each $N(\underline{a})$ is a k-subspace of A_K, the topology thus defined is a linear topology, and each $N(\underline{a})$ is open and closed. "Linear compactness" and "linear local compactness" are defined in analogous ways to the usual compactness and local compactness. It turns out that A_K is a linear locally compact space over k. On the other hand, Pontrjagin's duality holds for any linear locally compact space X over k, where the dual $\hat{X}$ of X is the space of all k-linear continuous maps from X in k, being considered as a discrete space. As is the usual Pontrjagin duality for locally compact abelian groups, $\hat{X}$ is topologically isomorphic to X; if X is discrete, $\hat{X}$ is linear compact. For any subspace Y of X, the annihilator $Y^{\perp}$ is defined, and $Y^{\perp} = (\hat{X/Y})$. Now returning to A_K, it can be shown that K is a discrete subspace and A_K/K linear compact. Furthermore A_K is self-dual, i.e. $\hat{A}_K \cong A_K$. Choosing a fixed element $\delta_0 \in \hat{A}_K$ such that $\delta_0(K) = 0$, one can give an isomorphism between A_K and $\hat{A}_K$ by means of this pairing, we get $K = K^{\perp}$. On the other hand, for each prime divisor P one defines $\deg P$ as the degree of extension of the residue class field at P over k; further if $\underline{a} = \prod_p p^{v_p(\underline{a})}$ is a divisor, $\deg \underline{a}$ is defined by linearity. It can be shown that $N(\underline{a}) \cap K$ is a finite dimensional subspace (i.e. a discrete and linear compact space!), and its dimension is defined to be $\dim a$. One can find a divisor D such that $N(\underline{a})^{\perp} = N(\underline{a}^{-1}D)$ for all $\underline{a}$. Now simple computations show the invariance of the number $\dim \underline{a} - \deg \underline{a} - \dim(\underline{a}^{-1}D)$ which is usually denoted by $1 - g$ (g is the genus of K):

$$\dim \underline{a} - \deg \underline{a} - \dim(\underline{a}^{-1}D) = 1 - g \ (*).$$

This is the Riemann-Roch theorem in Weil's version. The basic property of $N(\underline{a})$ used in the computation above is the finite dimensionality of

$$N(\underline{a})/N(\underline{a}) \cap N(\underline{a})^{\perp} \quad \text{and} \quad N(\underline{a})^{\perp}/N(\underline{a}) \cap N(\underline{a})^{\perp}.$$

The basic idea in Cahit Arf's [6] consists in looking at (*) as a relation satisfied by certain closed subspaces of adele ring having the property above. Hence to obtain an analogue of Riemann-Roch theorem for algebraic number fields, one first looks at such closed subspaces satisfying the corresponding property of finiteness, and, defining the functions deg and dim in a suitable way, one proves invariance of a relation including these functions. Cahit Arf considers first the case of $\mathbb{Q}$. The topological part of the discussion is easily settled. Compared with function field case, the topology here is the usual topology; $A_{\mathbb{Q}}$ is a locally compact abelian group, by choosing a suitable pairing, $A_{\mathbb{Q}}$ is identified with its dual $\hat{A}_{\mathbb{Q}}$ so that $\mathbb{Q}^{\perp} = \mathbb{Q}$ discrete and $A_{\mathbb{Q}}/\mathbb{Q}$ is compact. Then we restrict ourselves only to such closed subgroups F that $F/F \cap F^{\perp}$ and $F^{\perp}/F \cap F^{\perp}$ are both finite. Such F is said to be almost isotropic. Now the main difficulty in this work lies in the definition of the numerical functions $\dim F$ and $\deg F$. To do this, Cahit Arf looks at almost isotropic subgroups more closely. By an integral subgroup G, he understands a closed subgroup of $A_{\mathbb{Q}}$ such that $G^{\perp} \supset G$ and $[G^{\perp} : G]$ finite. It is then shown that for any almost isotropic subgroup F there is an integral subgroup G satisfying $G^{\perp} \supset F \supset G$, that is to say that F is approximated by an integral group up to finite factor group. By using these facts almost isotropic subgroups are explicitly described in terms of certain subgroups which are neighbourhoods of zero. Then the function dim is defined from the explicit description of almost isotropic subgroups. Now almost isotropic subgroups are divided into families by the following relations: F and F' are said to be related it $F/F \cap F'$ and $F'/F \cap F'$ are both finite. Then each family is subdivided into orbits with respect to the automorphisms of $A_{\mathbb{Q}}$. Then the function deg is defined by choosing a fixed almost isotropic F_0. Then the invariance of the expression $\dim F - \deg F - \dim F^{\perp}$ is proved. Because of the dependence of deg on the choice of F_0, this constant denoted by $\log g_{\lambda}$ depends on F_0. But it can be shown that g_{λ} depends only on the orbit F_{λ} containing F_0. That is why the constant is denoted by g_{λ}. Then turning to general case, a similar formula is obtained.

This work may be considered as a detailed study of almost isotropic subgroups of the adele group over an algebraic number field. The question naturally arises is this: Has the constant g_{λ} any significant meaning in connection with the structure of the field. As is known, the genus of an algebraic function field governs the structure of the field very strongly. But this kind of analogy may be meaningless.

In 1950's there were two works done by K. Iwasawa and J. Tate on adele rings of algebraic number fields. The work by Tate can be found as appendix of Cassels-Fröhlich's book. The topological part of these works are almost the same as that in this work. But after that, these works depart from this work significantly. This is because of the difference of aims. Still you can find an Riemann-Roch theorem in Tate's work. I do not know whether this has something to do with the Riemann-Roch theorem in this work.

5. Quadratic Forms

Because quadratic forms appear as fundamental forms of metric in Geometry, they are of fundamental importance in studying local isometries. Hence it occupies one of the main research fields in Mathematics. On the other hand, since Gauss formulated his theory of quadratic fields in terms of binary quadratic forms, many algebraic number theorists are still interested in quadratic forms over algebraic number fields, while the later development of Algebraic Number Theory departed from the theory of quadratic forms rather significantly.

The main problem in the theory of quadratic forms over a field k (or a ring R) consists in finding out the complete system of invariants in the sense that if these invariants of two quadratic forms coincide then these forms are equivalent (i.e. they go over via linear transformations with coefficients in k or R) and vice versa. Now the nature of the theory is strongly influenced by the structure of the ground field or the ground ring. The theory of quadratic forms over an arbitrary field was rather recently established. The first step in this line was taken by E. Witt [10]. Because the characteristic two case adds some complications, one usually assumes that the characteristic of the ground field is different from 2. Witt showed that, under this assumption on the ground field, every quadratic form F can be diagonalized, and, for

$$F = \sum_{i=1}^{n} a_i x_i^2,$$

the number of variables n, the Clifford algebra $\prod_{i \leq j}(a_i, a_j)$ and the class $\prod_{i=1}^{n} a_i$ mod $(k^x)^2$ form a complete system of invariants.

Although there is a series of works on quadratic forms over the ring of (algebraic) integers by Gauss, Minkowski and Hasse, there is still no general treatment of quadratic forms over an arbitrary (commutative) ring which can be compared with Witt's work. This is because the structure of rings is much more complicated than that of fields.

Now the work [2] by Cahit Arf was done to fill up the gap in Witt's paper [10]. That is, the aim of [2] is to give a complete system of invariants of quadratic forms over a field of characteristic 2. Because of the parity business in Geometry, the Arf invariants obtained in [2] are of importance in recent researches in Geometry.

The work [3] deals with the quadratic forms over the ring of formal pover series (in one variable) over a perfect field of characteristic 2. (a) In the paper [2] done in 1940, Cahit Arf considers quadratic forms over an arbitrary field k of characteristic 2. Now if a quadratic form $F(x) = \sum a_{ij} x_i x_j$ is given, it defines a metric (without triangle relation) on $V = k^n$, the n-dimensional space over $k : |x|^2 = F(x)$. V together with F is called a quadratic space. The equivalence between two quadratic forms can be interpreted as an isometry. By $|x+y|^2 = |x|^2 + |y|^2 + x \cdot y$ the inner product $x \cdot y$ is defined on V. As usual, orthogonality is defined; the radical V^* of V is defined to be the subspace consisting of all vectors orthogonal to V. If $V^* = \{0\}, V$ is said to be completely regular. Then it can be shown that V is an orthogonal sum of 2-dimensional completely regular subspaces and its radical. This implies that every quadratic form can be transformed into

$$\sum_{i=1}^{m}(a_i x_i^2 + b_i x_i y_i + c_i y_i^2) + \sum_{i=1}^{\ell} d_i^2 z_i^2$$

with $b_i \neq 0$. The latter form is said to be quasi-diagonal; this is the counter-part of diagonal forms in Witt [10]. The subspace of V^* consisting of all vectors of length 0 is denoted by V_0^*; if $V_0^* = \{0\}, V$ is said to be regular. Regular spaces are the counter-part of non-degenerate forms in [10]. Because of the assumption on the characteristic, k^2 is a subfield of k. Now if V is regular, F can be transformed into a quasi-diagonal form

$$\sum_{i=1}^{n}(a_i x_i^2 + b_i x_i y_i + c_i y_i^2) + \sum_{i=1}^{\ell} d_i z_i^2$$

with $d_i \neq 0$, further $\{d_i\}$ forms a k^2-basis of the space $\sum_{i=1}^{\ell} d_i k^2$. Then, as in [10], the Clifford algebra $C(F)$ of $F = \Sigma a_{ij} x_i x_j$ is introduced:

$$C(F) = \Sigma k u_1`\xi_1 \cdots u_m^{\xi_m} \quad \text{with} \quad u_i^2 = a_{ii} \ \text{ and } \ u_i u_j + u_j u_i = a_{ij} \ \ (i < j).$$

Now if

$$\sum_{i=1}^{n} f_i + F^*$$

is a quasi-diagonal decomposition of F, where f_i stands for binary forms and F^* for the radical part, then $C(F)$ can be decomposed into $C(f_i)$ and $C(F^*)$:

$$C(F) = \prod_{i=1}^{n} C(f_i) \cdot C(F^*).$$

Each $C(f_i)$ is a central simple algebra, and $C(F^*)$ is the centre of $C(F)$. Ignoring the radical of $C(F^*)$, we then get an algebra

$$C_0(F) = \prod_{i=1}^{n} C(f_i) \cdot K,$$

where K is a purely inseparable extension of k of degree $2^{\ell-1}$. After the discussion on the equivalence of binary forms, it is shown that if F is completely regular, the number of variables $2n$ (in f_i 's), the Clifford algebra $C(F)$ and the class

$$\Delta(F) \equiv \sum_{i=1}^{n} \frac{a_i c_i}{b_i^2} \bmod pk$$

form a complete system of invariants, where $\underline{p}$ stands for the Artin-Schreier operation: $\underline{p}x = x^2 + x$. To obtain further invariants for regular forms, more detailed discussions are needed. For later applications in [3], invariants for binary, ternary and quarternary forms are obtained. At the end of the paper there are some remarks about invariants under some additional assumption on k.

(b) Cahit Arf has already given systems of invariants for quadratic forms over the field of characteristic 2. The main problem in [3] is to give a complete system of invariants for quadratic forms over the ring of formal power series $k\{t\}$ over a perfect field k of characteristic 2. Clearly the invariants for quadratic forms over the field of fractions of $k\{t\}$, i.e. the field of formal Laurent series, are invariants in the sense of this work. But they are not complete; some additional invariants are needed. The invariants obtained (in the paper) are of complicated nature even for binary forms. Because of these circumstances Cahit Arf restricted his consideration only to the cases of binary and ternary forms. They are (besides invariants already obtained in [2]) either integers chosen among orders of the coefficients of the form, or certain residue classes represented by the coefficients.

6. Multiplicity Sequences of Algebraic Branches

The work [4] was done upon a question raised by P. Du Val [8] on the multiplicity sequence of an algebraic branch.

An algebraic branch may roughly be thought as a point on an algebraic hypersurface with coordinates belonging to a ring of formal power series in one variable. Let $\alpha = (Y_i(t))$ be an algebraic branch, then the point $(Y_i(0))$ is called the centre of α. If α is an algebraic branch, and D an algebraic hypersurface defined by $G = 0$. Then the order of the element $G(Y_i(t))$ in $k\{t\}$ the ring of formal power series containing $Y_i(t)$, is called the order (or intersection number) of D on α. This number is important, because the intersection multiplicity of algebraic hypersurfaces C and D at a point P can be determined by the order of C on β and the orders of D on α, where α and β range respectively over the algebraic branches of C and that of D with the centre at P. In this way, the set of orders of elements in $k[Y_1(t), \ldots, Y_n(t)]$ is nothing but the set of all possible orders taken by hypersurfaces on $\alpha = (Y_i(t))$. This set forms an additive semi-group of non-negative integers, and is called the characteristic semi-group of the branch α, or the multiplicity sequence of α. Now Du Val showed that if α is canonical (see below), the set of minimal generators of the characteristic module can be determined by means of an algorithm. The aim of [4] is to give an algebraic interpretation of Du Val's result.

The paper begins with an interesting lemma: For any semi-group $\{i_v\}$ of non-negative integers there is an index r such that $\{i_r, i_{r+1}, \ldots\}$ is an arithmetic progression with the common difference equal to the greatest common divisor of i_r 's. Then an arbitrary subring H of the ring $k\{t\}$ of formal power series over an arbitrary field k is considered. As usual H is assumed to contain the identity of $k\{t\}$. The set of orders of elements in H is shown by $W(H)$. Further let $\{S_{i_h}\}$ be any set of elements such that S_{i_h} is of order i_h, and I_h be the ideal of elements whose order is not smaller than h. Then the concept of canonical rings is introduced: H is said to be canonical if the ring generated by I_h coincides with I_h/S_h for every $h \in W(H)$. This is the algebraic interpretation of canonical branches. It can then be shown that for any subring H of $k\{t\}$ there is the smallest canonical ring containing H. This is the canonical closure of H, and is denoted by H^*. On the other hand, a semi-group $\{i_v\}$ of non-negative integers is said to be canonical if $i_h - i_h = 0$, $i_{h'+1} = i_{h+1} - i_h$, $i_{h'+2} = i_{h+2} - i_h, \ldots$ form a semi-group for every h. For any semi-group g of non-negative integers, there is the smallest canonical semi-group of non-negative integers. This is the canonical closure of g, and is denoted by g^*. Note that if $H = H^*$, i.e., if H is canonical, then $W(H) = W(H)^*$, i.e. $W(H)$ is canonical. But the converse is not always true. Now for any semi-group g of non-negative integers there is the smallest sub semi-group g_x such that $g_x^* = g^*$. It is shown that g_x is of the form

$$\left(\sum_{i=1}^{n} \alpha_i \lambda_i \mid \alpha_i \geq 0 \right)$$

with non-negative integers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Hence any canonical semi-group of non-negative integers is the canonical closure of finitely generated semi-group. Further it is shown that for a finitely generated semi-group g of the form

$$\left(\sum_{i=1}^{n} \alpha_i \lambda_i \mid \alpha_i \geq 0 \right)$$

g^* is equal to $\{0, \nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \cdots + \nu_{N-1} + \nu Z\}$ with suitable integers $\nu_1, \ldots, \nu_{N-1}$ and ν which can be determined from $\lambda_1, \ldots, \lambda_n$ by the algorithm of Du Val. This is the algebraic interpretation of Du Val's result. Further the methods determining generators of canonical rings are discussed with many numerical examples.

It should be noted that some twenty years after [4] appeared O. Zariski and J. Lipman [9] have defined Arf rings and studied their structure. The definition of Arf rings is highly technical, but it is an abstraction of canonical rings cited above by means of language of Commutative Algebra, and the Arf closure is that of the canonical closure.

7. Conclusion

Though the survey above is utterly incomplete and may be full of mistakes, I have tried to summarize the works of Cahit Arf in Algebraic Number Theory and related fields. During the preparation of this note, I have reexamined his papers, and realized once more his greatness. He is a man of ideas and he is full of energy. To every problem he has his own idea of approach. The characteristic of his approach is "thoroughness"; he always seeks invariants, and prefers explicit constructions rather than combinations of existing theories. Once he determines his approach to a problem, he energetically tackles the problem, and never gives up until he achieves his aim. If one studies Cahit Arf's works, which are full of originality and full of painstaking computations, one will surely wonder where Cahit Arf gets his inspirations, and how he gets insight to most complicated computations. As one sees from the survey above, each work of Cahit Arf is fundamental and deep; it is often referred to in later researches. This means that Cahit Arf's works are full of suggestions and full of ideas. To my regret, however, Cahit Arf never had pupils (in true sense) in Turkey; it might be because he is too great, or because his works are too hard for common mathematicians. In any case, it is a pity not for Cahit Arf but for Turkish Mathematics. I really do not see why Arf rings are studied by American mathematicians but not by Turkish mathematicians, and why his wonderful thesis is intensively re-examined by German mathematicians but not by Turkish mathematicians. I am sure that the growth of Mathematics in a country is, as history shows, only possible if the mathematicians in that country mathematically understand and stimulate each other. So I should like to emphasize again that it is the task of young Turkish mathematicians working in these fields first to learn by heart what Cahit Arf did, and then to continue further study along the lines indicated by him.

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