Tr. J. of Mathematics
22 (1998) , 93 – 107.

ON DERIVED EQUIVALENCES AND LOCAL STRUCTURE OF BLOCKS OF FINITE GROUPS

Markus Linckelmann

The types of equivalences between $p-$blocks of finite groups usually considered - namely Morita equivalences, derived and stable equivalences, as well as perfect isometries on the character level - all admit refinements to equivalences which in addition take into account the *local structure* of the considered blocks (this is explained in more details in Broué [4]). In particular, the notion of a derived equivalence refines to what we call *splendid derived equivalence* (called "splendid equivalence" in [14] or "Rickard equivalence with groups" in [4]), developed by Rickard in [14]. The desired compatibility property of a splendid Rickard equivalence with the local structure of the considered blocks is, however, essentially proved in [14] for principal blocks only.

Theorem 1.1 below suggests a slight modification of the definition of a splendid equivalence in [14] in order to get a compatibility for arbitrary blocks. We give our statements, explain them in a series of remarks and refer to the sections 2 and 3 for a more detailed description of the notions of Brauer pairs and Brauer homomorphisms that we use (and the proofs of all statements in this section are given in section 4).

Throughout this paper we fix a prime number p and a complete discrete valuation ring $\mathcal{O}$ having a residue field $k = \mathcal{O}/\mathcal{J}(\mathcal{O})$ of characteristic p. By a *block of a finite group* G we mean a primitive idempotent b of the center $Z(\mathcal{O}G)$ of the group algebra $\mathcal{O}G$ of G over $\mathcal{O}$ and call then $\mathcal{O}Gb$ the corresponding *block algebra of* b. Recall that a defect group of such a block b of G is a minimal subgroup P of G such that the obvious map $\mathcal{O}Gb \underset{\mathcal{O}P}{\otimes} \mathcal{O}Gb \longrightarrow \mathcal{O}Gb$ induced by multiplication in $\mathcal{O}Gb$ splits as homomorphism of $\mathcal{O}Gb-\mathcal{O}Gb-$bimodules. The defect groups of b form a unique conjugacy class of $p-$subgroups in G.

Theorem 1.1. *Let G, H be finite groups, b a block of G, c a block of H, having a common defect group P. Let $i \in (\mathcal{O}Gb)^P$ and $j \in (\mathcal{O}Hc)^P$ be primitive idempotents such that $Br^G_P(i) \neq 0$ and $Br^H_P(j) \neq 0$. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, such that $Br^G_Q(i)e_Q \neq 0$ and $Br^H_Q(j)f_Q \neq 0$.*

Assume that for any two subgroups Q, R *of* P *we have* $E_G((Q,e_Q),(R,e_R)) = E_H((Q,f_Q),(R,f_R))$.

Let X *be a Rickard tilting complex of* $\mathcal{O}Gb - \mathcal{O}Hc-$*bimodules. Assume that all terms of* X *are sums of direct summands of the* $\mathcal{O}Gb - \mathcal{O}Hc-$*bimodules* $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, *where* Q *runs over the set of subgroups of* P.

Then for any subgroup Q *of* P *the complex* $e_QX(Q)f_Q$ *is a Rickard tilting complex of* $kC_G(Q)e_Q - kC_H(Q)f_Q-$*bimodules.*

In the case of principal blocks, 1.1 is equivalent to Rickard's theorem [14, 4.1] . Remember that if b is the principal block of the finite group G then $Br_Q(b) = e_Q$ for any subgroup Q of P and e_Q is the principal block of $kC_G(Q)$ (where the notation is as in 1.1; see e.g. [16, (40.17)(a)]). We restate this for completeness:

Corollary 1.2. *(Rickard [14, 4.1]) With the notation and hypotheses of 1.1, suppose that* $Br_Q(b) = e_Q$ *and* $Br_Q(c) = f_Q$ *for any subgroup* Q *of* P.

Let X *be a Rickard tilting complex of* $\mathcal{O}Gb - \mathcal{O}Hc-$*bimodules all of whose terms are sums of direct summands of the bimodules* $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} \mathcal{O}Hc$, *where* Q *runs over the set of subgroups of* P.

Then for any subgroup Q *of* P *the complex* $X(Q)$ *is a Rickard tilting complex of* $kC_G(Q)e_Q - kC_H(Q)f_Q-$ *bimodules.*

The next theorem connects 1.1 to Broué's "tentative de définition" of Morita equivalences compatible with the local structure of the blocks [3, 6.3] via a technique due to Rickard (in [14] applied to $p-$nilpotent groups and in [9] extended to $p-$solvable groups). See also Puig [13, 7.9] for the stability condition occurring in the theorem below.

Theorem 1.3. *Let* G, H *be finite groups,* b *a block of* G *and* c *a block of* H *having a common defect group* P. *Let* $i \in (\mathcal{O}Gb)^P$ *and* $j \in (\mathcal{O}Hc)^P$ *be primitive idempotents such that* $Br_P^G(i) \neq 0$ *and* $Br_P^H(j) \neq 0$. *For any subgroup* Q *of* P *denote by* e_Q *and* f_Q *the unique blocks of* $kC_G(Q)$ *and* $kC_H(Q)$, *respectively, such that* $Br_Q^G(i)e_Q \neq 0$ *and* $Br_P^H(j)f_Q \neq 0$.

Let V *be an endo-permutation* $\mathcal{O}P-$*module. Assume that there is a bounded complex* X_V *of permutation* $\mathcal{O}P-$ *modules such that* X_V *has homology concentrated in degree zero isomorphic to* V *and that the complex* $X_V^* \underset{\mathcal{O}}{\otimes} X_V$ *is split as complex of* $\mathcal{O}P-$*modules with respect to the diagonal action of* P. *Assume finally that* ${}_\varphi X_V \cong Res_Q^P(X_V)$ *for any subgroup* Q *of* P *and any injective group homomorphism* $\varphi : Q \longrightarrow P$ *such that* $\tilde{\varphi} \in E_G((Q,e_Q),(P,e_P)) \cup E_H((Q,f_Q),(P,f_P))$.

If M *is a direct summand of the* $\mathcal{O}Gb - \mathcal{O}Hc-$*bimodule* $\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(V) \underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$ *which induces a Morita equivalence between* $\mathcal{O}Gb$ *and* $\mathcal{O}Hc$ *then there is a direct summand* X *of the complex of* $\mathcal{O}Gb - \mathcal{O}Hc-$*bimodules* $\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(X_V)$

$\underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$ *such that* X *is a Rickard tilting complex having homology concentrated in degree zero isomorphic to* M*, and in particular, all terms of* X *are isomorphic to direct sums of direct summands of the modules* $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$*, where* Q *runs over the set of subgroups of* P*.*

Moreover, for any two subgroups Q*,* R *of* P *we have* $E_G((R, e_R), (Q, e_Q)) = E_H((R, f_R), (Q, f_Q))$*.*

Remarks.

1.4 With the notation of 1.1, the algebras $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$ are *source algebras* of $\mathcal{O}Gb$ and $\mathcal{O}Hc$, a concept due to Puig[10]. They are always considered as *interior* $P-$*algebras*; that is, together with the group homomorphisms $P \longrightarrow A^\times$ and $P \longrightarrow B^\times$ mapping $u \in P$ to ui and uj, respectively. Recall from [10] that $\mathcal{O}Gb$ and A are Morita equivalent via the bimodules $\mathcal{O}Gi$ and $i\mathcal{O}G$. However, $kC_G(Q)e_Q$ and $A(Q)$ need not be Morita equivalent for all subgroups Q of P. A sufficient condition for $kC_G(Q)e_Q$ and $A(Q)$ to be Morita equivalent is that $C_P(Q)$ is a defect group of e_Q; this condition can always be fulfilled by replacing Q by a suitable $G-$conjugate in P (see 3.3 below for details).

1.5 If Q is a subgroup of P, the idempotent i need no longer be primitive in $(\mathcal{O}Gb)^Q$, and hence $Br_Q^G(i)$ need not be a primitive idempotent in $kC_G(Q)$, so the fact that there is a unique block e_Q of $kC_G(Q)$ satisfying $Br_Q^G(i)e_Q \neq 0$ is non trivial; this is due to Broué and Puig [5, 1.8].

1.6 Recall from [3] that for any two symmetric $\mathcal{O}-$algebras A, B, a Rickard tilting complex T of $A-B-$bimodules (called split endomorphism tilting complex in [14]) is a bounded complex of finitely generated $A-B-$bimodules whose terms are projective as left and right modules, such that the total complexes $T^* \otimes_A T$ and $T \otimes_B T^*$ are homotopy equivalent to B and A as complexes of $B-B-$bimodules and $A-A-$bimodules, respectively. Note that in 1.1 for any subgroup Q of P the $\mathcal{O}Gb-\mathcal{O}Hc-$bimodule $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$ is projective as left $\mathcal{O}Gb-$module and as right $\mathcal{O}Hc-$module, so this condition for the terms of X is redundant.

1.7 With the notation of 1.1 , the indecomposable direct summands of $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$ with Q running over the set of subgroups of P , viewed as $\mathcal{O}(G\times H)-$modules, are trivial source modules with vertex contained in $\Delta P = \{(u,u)\}_{u\in P}$; indeed, $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$ is a direct summand of $\mathcal{O}G \underset{\mathcal{O}Q}{\otimes} \mathcal{O}H \cong Ind_{\Delta Q}^{G\times H}(\mathcal{O})$. Similarly, the bimodule M in 1.3, when viewed as $\mathcal{O}(G\times H)-$module, has vertex ΔP and source V (assuming additionally that V is indecomposable).

1.8 If Q, R are subgroups of P, the set $E_G((Q, e_Q), (R, e_R))$ is the set of all group homomorphisms $Q \longrightarrow R$ modulo inner automorphisms of R which are induced by conjugation $u \longrightarrow {}^xu = xux^{-1}$ with an element x of G satisfying ${}^x(Q, e_Q) \subset (R, e_R)$ or equivalently, satisfying $xQx^{-1} \subset R$ and $xe_Qx^{-1} = e_{xQx^{-1}}$. See 3.1 below for more details.

1.9 Recall that an endo-permutation $\mathcal{O}P-$module (cf. Dade [6]) is an $\mathcal{O}-$free $\mathcal{O}P-$module V such that $V^*\underset{\mathcal{O}}{\otimes}V$ is a permutation $\mathcal{O}P-$module (i.e. has a $P-$stable $\mathcal{O}-$basis). The notation $Ind_{\Delta P}^{P\times P}(V)$ means, that we consider first V as $\mathcal{O}\Delta P-$module through the obvious isomorphism $\Delta P=\{(u,u)\}_{u\in P}\cong P$ and second the $\mathcal{O}(P\times P)-$module $Ind_{\Delta P}^{P\times P}(V)$ as $\mathcal{O}P-\mathcal{O}P-$bimodule with $u\in P$ acting on the left as $(u,1)$ and on the right as $(1,u^{-1})$. The complex X_V in 1.3 is called an *endo-split $p-$permutation resolution of V* in [14]. For abelian P, Rickard proved in [14] that any endo-permutation $kP-$module "lifts" to an endo-permutation $\mathcal{O}P-$module having an endo-split $p-$permutation resolution, using Dade's classification in [7] of endo-permutation $\mathcal{O}P-$modules for abelian P.

1.10 A complex X satisfying the hypotheses of 1.1 is going to be called a *splendid tilting complex*, analogously to the terminology in [14] (this is compatible with the terminology in [9]). One also might want to call a Morita equivalence induced by a bimodule M isomorphic to a direct summand of $\mathcal{O}Gi\underset{\mathcal{O}P}{\otimes}j\mathcal{O}H$ a *splendid Morita equivalence*. It has first been observed by L. Scott [15] and independently by L. Puig, that splendid Morita equivalences are precisely those Morita equivalences which arise from an isomorphism of the corresponding source algebras $i\mathcal{O}Gi\cong j\mathcal{O}Hj$ mapping ui to uj for all $u\in P$.

Notation. If not stated otherwise, an $\mathcal{O}-$algebra is supposed to be associative, unitary, $\mathcal{O}-$free of finite rank over $\mathcal{O}$, and a module is a finitely generated left module. For any $\mathcal{O}-$algebra A we denote by $J(A)$ its Jacobson radical, by $A^\times$ its group of invertible elements and by A^0 its opposite algebra. If $u\in A^\times$ and a is an element or subset of A, we write ${}^ua=uau^{-1}$ and $a^u=u^{-1}au$.

For any two $\mathcal{O}-$algebras A, B, an $A-B-$bimodule is a bimodule whose left and right $\mathcal{O}-$module structure coincide and which hence can be considered as $A\underset{\mathcal{O}}{\otimes}B^\circ-$module. If P, Q are finite groups, $\varphi:Q\longrightarrow P$ a group homomorphism and A an interior $P-$algebra (i. e. an $\mathcal{O}-$algebra endowed with a group homomorphism $\sigma:P\longrightarrow A^\times$, see [10]) we may consider any $A-$module U as $\mathcal{O}Q-$module via restriction through σ and φ, usually denoted by ${}_\varphi U$ (if the structural homomorphism σ is clear from the context). If H is a subgroup of a finite group G we denote by $[G/H]$ a system of representatives in G of the set of right transversals of H in G (similarly for double cosets), assuming implicitely, that in all statements where this notation occurs, it is verified that the statement does not depend on the choice of this set of representatives.

2. The Brauer Construction

We recall briefly the definition and some properties of the Brauer construction, due to Brauer [2] in the case of the group algebra, generalized by Broué and Puig [5] to arbitrary $G-$algebras and also to modules in Feit [8,II.3].

2.1 Let P be a finite $p-$group and X an $\mathcal{O}P-\mathcal{O}P-$bimodule. For any subgroup Q of P we set as usual

$$X^Q = \{x \in X | ux = xu \quad \text{for all} \quad u \in Q\}$$

and for any further subgroup R of Q we set

$$X_R^Q = \{ \sum_{u \in [Q/R]} uxu^{-1} | x \in X^R \}.$$

Clearly, X^Q defined above is an $Z(\mathcal{O}Q)-$module and X_R^Q is a submodule of X^Q. We set (cf. [5, 1.2])

$$X(Q) = X^Q/(J(\mathcal{O})X^Q + \sum_{R<Q} X_R^Q)$$

and denote by $Br_Q^X : X^Q \longrightarrow X(Q)$ the canonical surjection.

Clearly a homomorphism of $\mathcal{O}Q-\mathcal{O}Q-$bimodules $X \longrightarrow Y$ induces a homomorphism $X(Q) \longrightarrow Y(Q)$ which makes this construction functorial in an obvious sense.

2.2 We will have to compute $X(Q)$ only in situations where X has an $\mathcal{O}-$basis " $\mathcal{B}$ which is stable with respect to the "diagonal" action of Q on X; that is, which satisfies $u\mathcal{B}u^{-1} = \mathcal{B}$ for all $u \in Q$. In that case, the orbit sums $\sum_{u \in [Q/C_Q(x)]} uxu^{-1}$, where $x \in \mathcal{B}$, form an $\mathcal{O}-$basis of X^Q. Non trivial orbit sums lie clearly in $ker(Br_Q^X)$, and hence $Br_Q^X(\mathcal{B}^Q)$ is a $k-$basis of $X(Q)$. In particular we have $X(Q) = 0$ if $\mathcal{B}$ contains no element of X^Q.

The preceding section applies obviously, if $X = \mathcal{O}G$ for some finite group G containing P; we get then $X(Q) \cong kC_G(Q)$. In that case we write Br_Q^G instead of $Br_Q^{\mathcal{O}G}$ and even sometimes just Br_Q, if no confusion arises. We usually identify $(\mathcal{O}G)(Q)$ to $kC_G(Q)$ and Br_Q to the surjective algebra homomorphism $(\mathcal{O}G)^Q \longrightarrow kC_G(Q)$ induced by the projection mapping $x \in C_G(Q)$ to its image in $kC_G(Q)$ and $x \in G - C_G(Q)$ to 0.

2.3 More generally, if A is an interior $P-$algebra; that is, an $\mathcal{O}-$algebra endowed with a group homomorphism $\sigma : P \longrightarrow A^\times$, we may consider A as $\mathcal{O}P-\mathcal{O}P-$bimodule via σ. Since, for any subgroup Q of P, A^Q is a subalgebra of A and A_R^Q an ideal in A^Q for any subgroup R of Q, the quotient $A(Q)$ becomes naturally a $k-$algebra.

Moreover, if X is an $A-\mathcal{O}Q-$bimodules we may consider X as $\mathcal{O}Q-\mathcal{O}Q-$bimodule with the left action induced by resriction through σ to Q, and then $X(Q)$ becomes naturally an $A(Q)-$module. Observe that if Y is another $A-\mathcal{O}Q-$bimodule, the space $Hom_{A\otimes 1}(X,Y)$ is an $\mathcal{O}Q-\mathcal{O}Q-$bimodule via the right actions of $\mathcal{O}Q$ on X and Y. This gives rise to two bifunctors on the category ${}_AMod_{\mathcal{O}Q}$ of finitely generated $A-\mathcal{O}Q-$bimodules, which turn out to be naturally isomorphic on a suitable subcategory; this is the content of the next proposition (which is crucial for the proof of 1.1 and which is mainly a straightforward generalization of Rickard's lemma [14, 4.2]):

Proposition 2.4. *Let P be a finite $p-$group, A an interior $P-$algebra and Q a subgroup of P. Suppose that A has a $P\times P-$stable $\mathcal{O}-$basis such that A is projective as left and right $\mathcal{O}P-$module.*

Let $\mathcal{M}$ be the full additive subcategory of ${}_AMod_{\mathcal{O}Q}$ generated by the direct summands of the $A-\mathcal{O}Q-$bimodules $A\underset{\mathcal{O}R}{\otimes}({}_\varphi\mathcal{O}Q)$, where R runs over the set of subgroups of P and φ runs over the set of injective group homomorphisms from R to Q, with the additional condition that $R=Q$ and $\varphi=Id_Q$ if φ is an isomorphism.

Then, on $\mathcal{M}$, the bifunctors defined by

$$(X,Y)\longrightarrow (Hom_{A\otimes 1}(X,Y))(Q)$$

and

$$(X,Y)\longrightarrow Hom_{A(Q)}(X(Q),Y(Q))$$

are naturally isomorphic.

Proof. For any two $A-\mathcal{O}Q-$bimodules X, Y we have an obvious natural map

$$(Hom_{A\otimes 1}(X,Y))^Q = Hom_{A\otimes \emptyset Q}(X,Y)\longrightarrow Hom_{A(Q)}(X(Q),Y(Q))$$

which induces a natural map

$$Hom_{A\otimes 1}(X,Y)(Q)\longrightarrow Hom_{A(Q)}(X(Q),Y(Q)),$$

and we have to show that this map is an isomorphism, if X and Y belong to the subcategory $\mathcal{M}$. We clearly may assume that $X=A\underset{\mathcal{O}R}{\otimes}({}_\varphi\mathcal{O}Q)$ and $Y=A\underset{\mathcal{O}S}{\otimes}({}_\psi\mathcal{O}Q)$, where R, S are subgroups of P and $\varphi:R\longrightarrow Q$, $\psi:S\longrightarrow Q$ are injective group homomorphisms.

Note that the standard adjunctions give isomorphisms of $\mathcal{O}Q-\mathcal{O}Q-$bimodules $Hom_{A\otimes 1}(X,Y)\cong Hom_{\mathcal{O}R\otimes 1}({}_\varphi(\mathcal{O}Q),Y)\cong (\mathcal{O}Q)_\varphi\underset{\mathcal{O}R}{\otimes}Y\cong (\mathcal{O}Q)_\varphi\underset{\mathcal{O}R}{\otimes}A\underset{\mathcal{O}S}{\otimes}{}_\psi(\mathcal{O}Q)$.

Using 2.2 above and the hypotheses on A, if R (resp. S) is not isomorphic to Q we have $X(Q)=0$ (resp. $Y(Q)=0$) and also $(Hom_{A\otimes 1}(X,Y))(Q)=0$, so both bifunctors map (X,Y) to zero.

It remains to treat the case $X=Y=A\underset{\mathcal{O}Q}{\otimes}\mathcal{O}Q\cong A$. Since $Hom_{A\otimes 1}(A,A)\cong A^0$ we have natural isomorphisms

$(Hom_{A\otimes 1}(A,A))(Q) \cong A(Q)^0 \cong Hom_{A(Q)}(A(Q),A(Q))$, and the result follows.

□

The Brauer construction gives rise to different characterizations of defect groups which we recall without proof (see e.g. [16, 18.5]):

Proposition 2.5. *Let G be a finite group, b a block of G and P a subgroup of G. The following are equivalent:*

(i) P is a defect group of b.

(ii) P is a minimal subgroup of G such that $b \in (\mathcal{O}G)_P^G$.

(iii) P is a maximal $p-$subgroup of G such that $Br_P^G(b) \neq 0$.

In that case, for any idempotent $e \in (\mathcal{O}Gb)^P$ satisfying $Br_P(e) \neq 0$, the bimodules $\mathcal{O}Ge$ and $e\mathcal{O}G$ induce a Morita equivalence between the algebras $\mathcal{O}Gb$ and $e\mathcal{O}Ge$.

We finally recall a technical result on relative projectivity (which has various generalizations, but we state it in the form we need it):

Lemma 2.6. *Let P be a finite $p-$group, A an interior $P-$algebra, Q a subgroup of P and j a primitive idempotent in A^Q. If $Br_Q^A(j) = 0$ there is a proper subgroup R of Q such that the homomorphism of $A-\mathcal{O}Q-$bimodules*

$$Aj \underset{\mathcal{O}R}{\otimes} \mathcal{O}Q \longrightarrow Aj$$

mapping $aj \otimes u$ to $aju = auj$, where $a \in A$, $u \in Q$, has a section.

Proof. If $Br_Q^A(j) = 0$, by Rosenberg's lemma [16, 4.9] there is a proper subgroup R of Q such that $j \in A_R^Q$. Then if $c \in A^R$ such that $j = \sum_{u\in[Q/R]} ucu^{-1}$, clearly the map sending $a \in Aj$ to $\sum_{u\in[Q/R]} aucj \otimes u^{-1}$ is a section as required. □

3. On Fusion in Block Algebras

A systematic treatment of fusion in block algebras and block source algebras can be found in various works of Alperin-Broué [1], Broué-Puig [5] and Puig [11], for instance. See also [16, chapter 6 and 7].

We give here a very short ad hoc treatment of this subject strictly limited to what we need in this paper.

3.1. Let G be a finite group, b a block of G and P a defect group of b. Recall that for any $p-$subgroup Q of G the map

$$Br_Q : (\mathcal{O}G)^Q \longrightarrow kC_G(Q).$$

is a surjective algebra homomorphism. Hence $Br_Q(b)$ is either zero or an idempotent in $Z(kC_G(Q))$, thus a sum of blocks of $kC_G(Q)$. Following [1], a b-*Brauer pair* is a pair (Q, e) consisting of a $p-$subgroup Q of G and a block e of $kC_G(Q)$ satisfying $Br_Q(b)e = e$. Note that then in particular $Br_Q(b) \neq 0$, hence Q is contained in some defect group of b (or, equivalently, some conjugate of Q is contained in P).

Following [5], there is an inclusion of b-Brauerpairs (Q, e), (R, f) : we define $(R, f) \subset (Q, e)$ and say that (R, f) *is contained in* (Q, e), if $R \subset Q$ and there is a primitive idempotent $j \in (\mathcal{O}Gb)^Q$ such that $Br_Q(j)e \neq 0$ and $Br_R(j)f \neq 0$. In that case, since j is primitive in $(\mathcal{O}Gb)^Q$, its image $Br_Q(j)$ is primitive in $kC_G(Q)$, and hence $Br_Q(j)e = Br_Q(j)$. Even though j need no longer be primitive in $(\mathcal{O}Gb)^R$, it is still true that $Br_R(j)f = Br_R(j)$ and that f does not depend on the choice of j. Thus f is uniquely determined by e (cf. [5, 1.8]). We collect some standard properties of Brauer pairs we need (see [1] and [5]):

3.1.1. *If P is a defect group of b, there is a block e_P of $kC_G(P)$ such that $Br_P(b)e_P = e_P$.*

3.1.2. *For any $b-$Brauer pair (Q, e) there is $x \in G$ such that ${}^x(Q, e) \subset (P, e_P)$; in particular, all maximal $b-$Brauer pairs are conjugate to (P, e_P).*

3.1.3. *For any subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $(Q, e_Q) \subset (P, e_P)$.*

For any two $b-$Brauerpairs (Q, e), (R, f), we denote by $\widetilde{Hom}(Q, R)$ the set of equivalence classes of group homomorphisms from Q to R with respect to the equivalence relation declaring two group homomorphisms φ, ψ from Q to R to be equivalent if there is an inner automorphism τ of R such that $\varphi = \tau\psi$. We denote then by $\widetilde{\varphi}$ the image of φ in $\widetilde{Hom}(Q, R)$ and by $E_G((Q, e), (R, f))$ the image in $\widetilde{Hom}(Q, R)$ of of the set of all group homomorphisms $\varphi : Q \longrightarrow R$ for which there is $x \in G$ satisfying $\varphi(u) = {}^x u$ for all $u \in Q$ and ${}^x(Q, e) \subset (R, f)$.

3.2. Following [10], for any subgroup H of G, a *point of H on $\mathcal{O}Gb$* is an $((\mathcal{O}Gb)^H)^\times-$conjugacy class β of primitive idempotents in $(\mathcal{O}Gb)^H$, and we call then H_β a *pointed group on $\mathcal{O}Gb$*.

For any $p-$subgroup Q of G, a *local point of Q on $\mathcal{O}Gb$* is a point δ of Q on $\mathcal{O}Gb$ such that $Br_Q(\delta) \neq 0$, and we call then Q_δ a *local pointed group on $\mathcal{O}Gb$*. Clearly G acts by conjugation on the set of (local) pointed groups on $\mathcal{O}Gb$.

If Q_δ is a local pointed group on $\mathcal{O}Gb$ then since Br_Q is a surjective algebra homomorphism, $Br_Q(\delta)$ is a conjugacy class of primitive idempotents in $kC_G(Q)Br_Q(b)$ (cf. [16, 3.2]) , hence determines a unique $b-$Brauer pair (Q, e_δ) via the condition $Br_Q(\delta)e_\delta \neq 0$.

If Q_δ, R_ϵ are pointed groups on $\mathcal{O}Gb$ we write $R_\epsilon \subset Q_\delta$ if $R \subset Q$ and for some (and then necessarily any) $j \in \delta$ there is $k \in \epsilon$ such that $k = jk = kj$.

Note that an inclusion of local pointed groups $R_\epsilon \subset Q_\delta$ induces an inclusion of the corresponding Brauer pairs $(R, e_\epsilon) \subset (Q, e_\delta)$. We again recall two basic properties of local pointed groups (cf. [10]):

3.2.1. *If P is a defect group of b , there is a local point γ of P on $\mathcal{O}Gb$.*

3.2.2. *For any local pointed group Q_δ on $\mathcal{O}Gb$ there is $x \in G$ such that ${}^x(Q_\delta) \subset P_\gamma$. In particular, the maximal local pointed groups on $\mathcal{O}Gb$ are all conjugate to P_γ.*

We have no longer a uniqueness of the inclusion as in 3.1.3, since for a subgroup Q of P there may be different local points δ, δ' of Q on $\mathcal{O}Gb$ such that $Q_\delta \subset P_\gamma$, $Q_{\delta'} \subset P_\gamma$. Yet, statement 3.1.3 implies then $e_\delta = e_{\delta'}$.

The next lemmas contain the technical facts about Brauer pairs and local pointed groups required for the proofs of the statements in section 1 (statement 3.3 (ii) is a reformulation of [1, 4.5] and statement 3.3 (v) is based on Puig's idea of relating the bimodule structure of a block algebra to its fusion ; see for instance the proof of [11, 3.1]):

Lemma 3.3. *Let G be a finite group, b a block of G , P_γ a maximal local pointed group on $\mathcal{O}Gb$ and choose $i \in \gamma$. For any subgroup Q of P let e_Q be the unique block of $kC_Q(Q)$ such that $Br_Q(i)e_Q = Br_Q(i)$.*

(i) For any subgroup Q of P, $C_P(Q)$ is contained in a defect group of e_Q.

(ii) For any subgroup Q of P there is $x \in G$ such that ${}^x(Q, e_Q) \subset (P, e_P)$ and such that $C_P({}^xQ)$ is a defect group of ${}^xe_Q = e_{{}^xQ}$.

(iii) Let Q be a subgroup of P such that $C_P(Q)$ is a defect group of e_Q. Then $kC_G(Q)e_Q$ and $Br_Q(i)kC_G(Q)Br_Q(i)$ are Morita equivalent, and for any local point δ of Q on $\mathcal{O}Gb$ such that $Br_Q(\delta)e_Q \neq 0$ we have $Q_\delta \subset P_\gamma$.

(iv) Let Q, R be isomorphic subgroups of P and $x \in G$ such that ${}^x(R, e_R) = (Q, e_Q)$ and such that $C_P(Q)$ is a defect group of e_Q. Then for any local point ϵ of R on $\mathcal{O}Gb$ satisfying $R_\epsilon \subset P_\gamma$ we have ${}^x(R_\epsilon) \subset P_\gamma$.

(v) Let Q, R be isomorphic subgroups of P and $\varphi : R \longrightarrow Q$ a group isomorphism. If ${}_\varphi(\mathcal{O}Q)$ is isomorphic to a direct summand of $i\mathcal{O}Gi$ as $\mathcal{O}R - \mathcal{O}Q-$bimodule, then $\tilde{\varphi} \in E_G((R, e_R), (Q, e_Q))$. The converse holds if moreover $C_P(Q)$ is a defect group of e_Q.

Proof. (i) Since $Br_P(i) \neq 0$ we have $Br_{C_P(Q)}(Br_Q(i)) = Br_{QC_P(Q)}(i) \neq 0$, hence $Br_{C_P(Q)}(e_Q) \neq 0$, and so (i) follows from 2.5.

(ii) Let (R, f) be a maximal $e-$Brauer pair; that is, R is a defect group of e and f is a block of $kC_{C_G(Q)}(R) = kC_G(QR)$ such that $Br_R(e_Q)f = f$. Clearly (QR, f) is then a $b-$Brauer pair such that $(Q, e_Q) \subset (QR, f)$. By 3.1.2 there is $x \in G$ such that ${}^x(Q, e_Q) \subset {}^x(QR, f) \subset (P, e_P)$. Since ${}^xR$ is a defect group of ${}^xe_Q$ contained in $C_P({}^xQ)$, the statement follows from (i).

(iii) Again, as $Br_{C_P(Q)}(Br_Q(i)) = Br_{QC_P(Q)}(i) \neq 0$, it follows from the last statement of 2.5 that the algebras $Br_Q(i)kC_G(Q)Br_Q(i)$ and $kC_G(Q)e_Q$ are Morita equivalent. Thus for every local point δ of Q on $\mathcal{O}Gb$ satisfying $Br_Q(\delta)e_Q = Br_Q(\delta)$ some element of $Br_Q(\delta)$ lies in $Br_Q(i)kC_G(Q)Br_Q(i)$, which is only possible if $Q_\delta \subset P_\gamma$.

(iv) This follows from (iii) applied to to ${}^x(R_\epsilon)$.

(v) Since ${}_\varphi(\mathcal{O}Q)$ is isomorphic to an indecomposable direct summand of $i\mathcal{O}Gi$ as $\mathcal{O}R-\mathcal{O}Q-$bimodule, there are primitive idempotents $j \in (\mathcal{O}Gb)^Q$, $l \in (\mathcal{O}Gb)^R$, such that ${}_\varphi(\mathcal{O}Q)$ is isomorphic to a direct summand of $l\mathcal{O}Gj$ as $\mathcal{O}R-\mathcal{O}Q-$bimodule. Since $i\mathcal{O}Gi$ is a direct summand of $\mathcal{O}G = \bigoplus_{x\in[R\backslash G/Q]}\mathcal{O}[RxQ]$ as $\mathcal{O}R-\mathcal{O}Q-$bimodule, there is $x \in G$ such that ${}_\varphi(\mathcal{O}Q) \cong \mathcal{O}[Rx^{-1}] = \mathcal{O}[x^{-1}Q]$ and $\varphi(u) = {}^xu$ for all $u \in R$. Hence $\mathcal{O}Q$ is isomorphic to a direct summand of ${}^xl\mathcal{O}Gj$ as $\mathcal{O}Q-\mathcal{O}Q-$bimodule. Applying the Brauer construction shows that $kZ(Q)$ is isomorphic to a direct summand of $Br_Q({}^xl)kC_G(Q)Br_Q(j)$; in particular, the latter expression is non zero. Thus $Br_Q({}^xl)$, $Br_Q(j)$ are non zero primitive idempotents in $kC_G(Q)$ belonging to the same block e_Q. As $Br_R(l)$ belongs to e_R and ${}^xBr_R(l) = Br_Q({}^xl)$ belongs to e_Q, it follows that ${}^xe_R = e_Q$, or, equivalently, ${}^x(R, e_R) = (Q, e_Q)$.

In order to prove the converse, assume now that $C_P(Q)$ is a defect group of e_Q. Let $x \in G$ such that ${}^x(R, e_R) = (Q, e_Q)$. Denote by $\varphi : R \longrightarrow Q$ the group isomorphism mapping $u \in R$ to ${}^xu$. Choose any local point ϵ of R on $\mathcal{O}Gb$ such that $R_\epsilon \subset P_\gamma$. By (iii) we have ${}^x(R_\epsilon) \subset P_\gamma$. Thus, if we pick any $l \in \epsilon$, as an $\mathcal{O}R-\mathcal{O}Q-$bimodule, $l\mathcal{O}G{}^xl$ is isomorphic to a direct summand of $i\mathcal{O}Gi$. Since $Br_R(l) \neq 0$, the $\mathcal{O}R-\mathcal{O}R-$bimodule $l\mathcal{O}Gl$ has a direct summand isomorphic to $\mathcal{O}R$. Therefore $k\mathcal{O}G{}^xl$ has a direct summand isomorphic to $\mathcal{O}[Rx^{-1}] = \mathcal{O}[x^{-1}Q] \cong {}_\varphi(\mathcal{O}Q)$, which concludes the proof.

□

Lemma 3.4. *Let G, H be finite groups, b, c be blocks of G, H, respectively, having a common defect group P. Let $i \in (\mathcal{O}Gb)^P$, $j \in (\mathcal{O}Hc)^P$ be primitive idempotents such that $Br_P(i) \neq 0$, $Br_P(j) \neq 0$. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$ satisfying $Br_Q(i)e_Q \neq 0$ and $Br_Q(j)f_Q \neq 0$, respectively. Assume that for any two subgroups Q, R of P we have $E_G((Q, e_Q), (R, e_R)) = E_H((Q, f_Q), (R, f_R))$.*

Then for any subgroup Q of P there are elements $x \in G$ and $y \in H$ such that ${}^x(Q, e_Q) \subset (P, e_P)$, ${}^y(Q, f_Q) \subset (P, f_P)$, ${}^xu = {}^yu$ for all $u \in Q$, and $C_P({}^xQ) = C_P({}^yQ)$ is a defect group of both ${}^xe_Q$, ${}^yf_Q$.

Proof. Let Q be a subgroup of P. By 3.3(ii) there is $x \in G$ such that ${}^x(Q, e_Q) \subset (P, e_P)$ and $C_P({}^xQ)$ is a defect group of ${}^xe_Q$. By the assumptions, there is $y \in H$ such that ${}^y(Q, f_Q) \subset (P, f_P)$ and ${}^yu = {}^xu$ for all $u \in Q$. It therefore suffices to show that assuming that $C_P(Q)$ is a defect group of e_Q, it is a defect group of f_Q, too. For this, choose a defect group R of f_Q in $C_H(Q)$ and a block f of $kC_{C_H(Q)}(R) = kC_H(QR)$ such that (R, f) is an f_Q-Brauer-pair. Then (RQ, f) is a $b-$Brauer pair such that $(Q, f_Q) \subset (QR, f)$. Thus there is $y \in H$ satisfying ${}^y(Q, f_Q) \subset {}^y(QR, f) \subset (P, f_P)$ by 3.1.2. Note that then ${}^yf = f_{{}^y(QR)}$. The assumptions imply that there is $x \in G$ such that ${}^x(Q, e_Q) \subset (P, e_P)$ and ${}^xu = {}^yu$ for all $u \in Q$. In particular, we have ${}^xQ = {}^yQ$ and hence ${}^x(Q, e_Q) \subset ({}^y(QR), e_{{}^y(QR)})$. This in turn means that $({}^yR, e_{{}^y(QR)})$ is an ${}^xe_Q-$Brauer pair, or, equivalently, $({}^{x^{-1}}({}^yR), e_{Q({}^{x^{-1}}({}^yR))})$ is an e-Brauer pair. But then, again by 3.1.2, R is conjugate to a subgroup of $C_P(Q)$. Since R contains $C_P(Q)$ by 3.3 (i), we have in fact the equality $R = C_P(Q)$, and the lemma follows. □

4. Proofs

We keep the notation of 1.1.

Proof of 1.1. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. Since $\mathcal{O}Gb \underset{\mathcal{O}}{\otimes} (\mathcal{O}Hc)^0$ and $A \underset{\mathcal{O}}{\otimes} B^0$ are Morita equivalent (cf. 1.4) clearly $T = iXj$ is a Rickard tilting complex of $A-B-$modules. Let Q be a subgroup of P. We choose first Q in such a way that $C_P(Q)$ is a defect group of both e_Q, f_Q. Let $\mathcal{M}$ be the subcategory of ${}_AMod_{\mathcal{O}Q}$ as defined in 2.4. The main step of the proof is to show, that the terms of the complex T, when restricted to $A - \mathcal{O}Q$, all belong to the category $\mathcal{M}$. By the hypotheses, it suffices to show, that

4.1. for any subgroup R of P, the restriction to $A - \mathcal{O}Q$ of the $A-B-$bimodule $A \underset{\mathcal{O}R}{\otimes} B$ belongs to the category $\mathcal{M}$.

In order to prove 4.1, it suffices to show, that for any indecomposable direct summand W of B as $\mathcal{O}R - \mathcal{O}Q-$bimodule and any primitive idempotent $l \in A^R$ the $A - \mathcal{O}Q-$bimodule $Al \otimes_{\mathcal{O}R} W$ belongs to the category $\mathcal{M}$. Since W is a direct summand of B, hence of $\mathcal{O}H$ as $\mathcal{O}R - \mathcal{O}Q-$bimodule, there is $y \in H$ such that $W \cong \mathcal{O}[RyQ] \cong \mathcal{O}R \underset{\mathcal{O}S}{\otimes}{}_{\varphi}(\mathcal{O}Q)$, where $S = {}^yQ \cap R$ and $\varphi(u) = y^{-1}uy$ for all $u \in S$. Thus $Al \underset{\mathcal{O}R}{\otimes} W \cong Al \underset{\mathcal{O}S}{\otimes}{}_{\varphi}(\mathcal{O}Q)$. Therefore, we may assume that $R = S$. By 2.6, we may also assume that l belongs to a local point ϵ of R on $\mathcal{O}Gb$ such that $R_\epsilon \subset P_\gamma$, where γ is the local point of P on $\mathcal{O}Gb$ containing i. If φ is not an isomorphism, then $Al \underset{\mathcal{O}R}{\otimes}{}_{\varphi}(\mathcal{O}Q)$ belongs to $\mathcal{M}$ by definition. If φ is an isomorphism, we have $\widetilde{\varphi} \in E_G((R, e_R), (Q, e_Q))$ by 3.3 (v). Let $x \in G$ such that ${}^x(R, e_R) \subset (Q, e_Q)$ and $\varphi(u) = {}^xu$ for all $u \in R$. Then by 3.3 (iii) we have ${}^x(R_\epsilon) \subset P_\gamma$.

Thus $Al \underset{\mathcal{O}R}{\otimes}{}_{\varphi}(\mathcal{O}Q) \cong (Al)_{\varphi^{-1}} \cong i\mathcal{O}G^xl$ as $A-\mathcal{O}Q-$bimodules. Since ${}^x(R_\epsilon) \subset P_\gamma$, $i\mathcal{O}G^xl$ is isomorphic to a direct summand of $i\mathcal{O}Gi = A$ and lies therefore in $\mathcal{M}$. This shows 4.1.

The proof of 1.1 concludes now as follows: since, by 4.1, all terms of T are in the category $\mathcal{M}$, we may apply 2.4 and get an isomorphism of complexes

4.2. $(Hom_{A\otimes 1}(T, T))(Q) \cong Hom_{A(Q)}(T(Q), T(Q))$.

Since the terms of T are left and right projective and since A is symmetric (i.e. $A \cong A^*$ as $A - A-$bimodules) we have $Hom_{A\otimes 1}(T, T) \cong T^* \otimes_A T$, and similarly, we have $Hom_{A(Q)}(T(Q),T(Q)) \cong T(Q)^* \otimes_{A(Q)} T(Q)$ as complexes; thus 4.2 translates to an isomorphism of complexes

4.3. $T(Q)^* \otimes_{A(Q)} T(Q) \cong (T^* \otimes_A T)(Q)$.

Now $T^* \otimes_A T$ is homotopic to B, and therefore, $(T^* \otimes_A T)(Q)$ is homotopic to $B(Q)$. This shows, still under the assumtion that $C_P(Q)$ is a defect group of both e_Q, f_Q, that

4.4. the complex $T(Q)$ is a Rickard tilting complex of $A(Q)-B(Q)-$bimodules.

From the Morita equivalences in 3.3(iii) follows therefore that

4.5. the complex $e_Q X(Q) f_Q$ is a Rickard tilting complex of $kC_G(Q)e_Q - kC_H(Q)f_Q$-bimodules.

Clearly the property of $e_Q X(Q) f_Q$ being a Rickard tilting complex is invariant under "simultaneous conjugation" of Q in P by elements of G and H, thus lemma 3.4 implies immediately, that in fact $e_Q X(Q) f_Q$ is a Rickard tilting complex for all subgroups Q of P.

□

Proof of 1.2. Since $Br_P(b) = e_P$ there is a unique local point γ of P on $\mathcal{O}Gb$ (cf. [16, (40.13)(b)]). Similarly, there is a unique local point δ of P on $\mathcal{O}Hc$. Choose $i \in \gamma$ and $j \in \delta$. Let M be an indecomposable $\mathcal{O}Gb - \mathcal{O}Hc-$bimodule such that M is isomorphic to $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} \mathcal{O}Hc$ for some subgroup Q of P, and choose Q to be minimal with this property. By the assumptions, in order to apply 1.1, it suffices to show that in fact M is then isomorphic to a direct summand of $\mathcal{O}Gi \underset{\mathcal{O}R}{\otimes} j\mathcal{O}H$ for some subgroup R of P.

Note first that if $x \in G$, $y \in H$ such that ${}^xQ \subset P$, ${}^yQ \subset P$ and ${}^xu = {}^yu$ for all $u \in Q$, there is an isomorphism of $\mathcal{O}Gb - \mathcal{O}Hc-$bimodules $\mathcal{O}Gb \underset{\mathcal{O}Q}{\otimes} \mathcal{O}Hc \cong \mathcal{O}Gb \underset{\mathcal{O}^xQ}{\otimes} \mathcal{O}Hc$ mapping $a \otimes d$ to $ax^{-1} \otimes yd$ for any $a \in \mathcal{O}Gb$, $d \in \mathcal{O}Hc$. Therefore, by lemma 3.4, we may assume that $C_P(Q)$ is a defect group of e_Q and f_Q.

Observe next that since M is indecomposable, there are primitive idempotents $m \in (\mathcal{O}Gb)^Q$ and $n \in (\mathcal{O}Hc)^Q$ such that M is isomorphic to a direct summand of $\mathcal{O}Gm \underset{\mathcal{O}Q}{\otimes} n\mathcal{O}H$. By the minimality of Q, the idempotents m and n belong to local points μ and ν of Q on $\mathcal{O}Gb$ and $\mathcal{O}Hc$, respectively (cf. 2.6). By 3.3(iii) we have $Q_\mu \subset P_\gamma$ and $Q_\nu \subset P_\delta$. Consequently, the bimodule $\mathcal{O}Gm \underset{\mathcal{O}Q}{\otimes} n\mathcal{O}H$ and hence M is isomorphic to a direct summand of $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$.

Thus all terms of X are indeed isomorphic to sums of direct summands of the bimodules $\mathcal{O}Gi \underset{\mathcal{O}Q}{\otimes} j\mathcal{O}H$, where Q runs over the set of subgroups of P, hence 1.1 applies and the proof of 1.2 is complete.

□

Proof of 1.3. Set $Y = \mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P \times P}(X_V) \underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$ and

$U = \mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(V) \underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$. Since $\mathcal{O}Gb$ is symmetric and all terms of Y are projective as left $\mathcal{O}Gb-$modules, we have an isomorphism of the total complexes of $\mathcal{O}Hc - \mathcal{O}Hc-$bimodules

4.6. $Hom_{\mathcal{O}Gb\otimes 1}(Y,Y) \cong Y^* \underset{\mathcal{O}Gb}{\otimes} Y$

and our first step is to show that

4.7. the complex $Y^* \underset{\mathcal{O}Gb}{\otimes} Y$ is split.

As $Y^* \underset{\mathcal{O}Gb}{\otimes} Y$ is isomorphic to the complex

$\mathcal{O}H\, j \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(X_V^*) \underset{\mathcal{O}P}{\otimes} i\mathcal{O}Gi \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(X_V) \underset{\mathcal{O}P}{\otimes} j\mathcal{O}H$, it suffices to show that for any direct summand W of $i\mathcal{O}Gi$ as $\mathcal{O}P - \mathcal{O}P-$bimodule

4.8. the complex of $\mathcal{O}P - \mathcal{O}P-$bimodules $Ind_{\Delta P}^{P\times P}(X_V^*) \underset{\mathcal{O}P}{\otimes} W \underset{\mathcal{O}P}{\otimes} Ind_{\Delta P}^{P\times P}(X_V)$ is split.

By 3.3 (v) we have $W \cong \mathcal{O}P \underset{\mathcal{O}Q}{\otimes} {}_{\varphi}(\mathcal{O}P)$ for some subgroup Q of P and an injective group homomorphism $\varphi : Q \longrightarrow P$ such that $\widetilde{\varphi} \in E_G((R,e_R),(Q,e_Q))$. Thus the complex in 4.8 is isomorphic to

4.9. $Ind_{\Delta P}^{P\times P}(X_V^*) \underset{\mathcal{O}Q}{\otimes} {}_{\varphi} Ind_{\Delta P}^{P\times P}(X_V) \cong Ind_{\Delta Q}^{P\times P}(X_V^* \underset{\mathcal{O}}{\otimes} {}_{\varphi} X_V)$

and the latter is split since ${}_{\varphi}X_V \cong Res_Q^P(X_V)$ and $X_V^* \underset{\mathcal{O}}{\otimes} X_V$ is split by the assumptions.

We use now an argument of Rickard (occurring in the proof of [14, 7.5]) to show that an isomorphism between the degree zero homology of X_V and V induces an algebra isomorphism

4.10. $End_{K^b(\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0)}(Y) \cong End_{\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0}(U)$.

Indeed, since the complex $Hom_{\mathcal{O}Gb\otimes 1}(Y,Y)$ is split by 4.6 and 4.7, taking homology commutes to taking $H-$fixpoints. If we take first $H-$fixpoints, we obtain the complex $Hom_{\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0}(Y,Y)$, and its degree zero homology is well-known to be isomorphic to $End_{K^b(\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0)}(Y)$. As X_V has homology concentrated in degree zero isomorphic to V, the complex Y has homology concentrated in degree zero isomorphic to U. It follows that the complex $Hom_{\mathcal{O}Gb\otimes 1}(Y,Y)$ has homology concentrated in degree zero isomorphic to $End_{\mathcal{O}Gb\otimes 1}(U)$. Taking $H-$fixpoints yields $End_{\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0}(U)$, which implies 4.10.

Now M is isomorphic to a direct summand of U, thus corresponds to a primitive idempotent in $End_{\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0}(U)$. This corresponds to a primitive idempotent in

$End_{K^b(\mathcal{O}Gb\underset{\mathcal{O}}{\otimes}(\mathcal{O}Hc)^0)}(Y)$ through the algebra isomorphism 4.10, and yields hence a direct summand X of Y. Clearly X has the properties stated in 1.3. The last statement is a corollary to Puig's work [12, 5.3], [11, section 3] on fusion.
□

Acknowledgements

The author wishes to thank K.W. Roggenkamp and the members of the Institute of Mathematics at the University of Stuttgart for the invitation to Stuttgart and their hospitality, as well as the Deutsche Forschungsgemeinschaft for their support during the writing of this paper.

References

[1] J.L. Alperin, M. Broué, Local Methods in Block Theory, *Ann. of Math.* **110** 1979 143–157.

[2] R.Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung, *Math. Z.* **63**, 1956, 406–444.

[3] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, *Astérisque*, **181-182**, 1990, 61–92.

[4] M. Broué, Equivalences of Blocks of Group Algebras *Proceedings of the Nato Advanced Research Workshop on Representations of Algebras and Related Topics*, Kluwer Academic Publishers Dordrecht, 1994 1–26.

[5] M.Broué, L. Puig, Characters and Local Structure in G-Algebras, *J. Algebra*, **63**, 1980, 306–317.

[6] E.C. Dade, Endo-permutation modules for p-groups I *Ann. of Math.*, **107**, 1978, 459–494.

[7] E.C. Dade, Endo-permutation modules for p-groups II, *Ann. of Math.*, **108**, 1978, 317–346.

[8] W. Feit, *The representation theory of finite groups*, North Holland, 1982.

[9] M. E. Harris, M. Linckelmann, Splendid Rickard equivalences for blocks of p-solvable groups, Preprint 1996, to appear in J. London Math. Soc.

[10] L. Puig, Pointed groups and construction of characters, *Math. Z.*, **176**, 1981, 265–292.

[11] L. Puig, Local fusion in block source algebras *J. Algebra*, **104**, 1986, 358–369.

[12] L.Puig, Nilpotent blocks and their source algebras, *Invent. Math.*, **93**, 1986, 77–129.

[13] L. Puig, Unpublished manuscript 1995.

[14] J. Rickard, Splendid equivalences: derived categories and permutation modules, *Proc. London Math. Soc.*, **72**, 1996, 331–358.

[15] L. L. Scott, Unpublished notes 1990.

[16] J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications, Clarendon Press Oxford, 1995.

Markus LINCKELMANN
CNRS, Université Paris 7
UFR Mathématiques
2, place Jussieu
75251 Paris Cedex 05 - FRANCE

Received 04.11.1996