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ABOUT SOME CLASSICAL FUNCTIONAL EQUATIONS

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Abstract

The purpose of this paper is to give a new method of finding the solution of Lobashevsky's functional equation and those of other classical functional equations. At the beginning we present the properties of solution f, $f \neq 0$, of Lobachevsky's functional equation. Using only the boundedness property on $(-r, r)$, we deduce the continuity, convexity and differentiability properties of the solution.

By establishing the connection between the solution of Lobachevsky's functional equation and the solution of other functional equations we simply, and rigorously deduce their properties in a uniform way.

1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function on $\mathbb{R}$, satisfying Lobachevsky's functional equation [1]

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2, \quad \forall\, x, y \in \mathbb{R}. \tag{1}$$

It is easy to verify that $f(x) = k$ (constant) is a solution of (1) and in what follows we exclude this case.

Lemma 1. Let f be a solution of (1). If there exists an $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$, then $f(x) = 0$, $\forall\, x \in \mathbb{R}$ and if $f(0) \neq 0$, then

$$sgn\ f(x) = sgn\ f(0), \quad \forall\ x \in \mathbb{R}. \tag{2}$$

Proof. From (1) we deduce

$$f(x_0)\ f(2x - x_0) = f(x)^2, \quad \forall\, x \in \mathbb{R}, \ \text{ i.e. } \ f(x) = 0 \ \text{ and } \ \forall\, x \in \mathbb{R}.$$

If $f(0) \neq 0$, then $f(0)f(x) = f(\frac{x}{2})^2$, which implies (2). □

Lemma 2. Let $f, f(0) \neq 0$ be a solution of (1). If f is bounded on a neighbourhood $(-r, r)$, of zero, then f is continuous at zero.

Proof. We consider the case $f(0) > 0$. From (1) we obtain

$$f\left(\frac{x}{2}\right) = (f(x)f(0))^{1/2}, \quad f\left(\frac{x}{2^2}\right) = f(x)^{\frac{1}{2^2}} f(0)^{1-\frac{1}{2^2}}$$

and by induction

$$f\left(\frac{x}{2^n}\right) == f(x)^{\frac{1}{2^n}} f(0)^{1-\frac{1}{2^n}}, \quad \forall\, x \in \mathbb{R},\ \forall\, n \in \mathbb{N}^*. \tag{3}$$

From (3), we have $\lim_{n\to 0} f(\frac{x}{2^n}) = f(0)$. Because f is bounded at zero, it results that $\lim_{y\to 0} f(y) = f(0)$ i.e. the continuity condition of f at zero.

In the case $f(0) < 0$, we have

$$f\left(\frac{x}{2}\right) = -(f(x)f(0))^{1/2}, \quad f\left(\frac{x}{2^2}\right) = -(f(x)f(0))^{\frac{1}{2^2}} \cdot |f(0)|^{1/2}$$

and

$$f\left(\frac{x}{2^n}\right) = -(f(x)f(0))^{\frac{1}{2^n}}, \cdot |f(0)|^{1-\frac{1}{2^{n-1}}}, \quad \forall\, x \in \mathbb{R},\ \forall\, n \in \mathbb{N}^*. \tag{4}$$

Hence

$$\lim_{n\to\infty} f\left(\frac{x}{2^n}\right) = f(0).$$

□

The continuity of f at zero follows as above.

Lemma 3. Let $f, f(0) \neq 0$ be a solution of (1). The function f is continuous on $\mathbb{R}$ if it is continuous at zero.

Proof. Because f is continuous at zero it results that f^2 is continuous at zero. Taking into account (1) and Lemma 1, we have

$$f(x) - f(x_0) = \frac{f\left(\frac{x-x_0}{2}\right)^2 - f(0)^2}{f(-x_0},$$

which implies the continuity of f at $\forall\, x_0 \in \mathbb{R}$. □

Proposition 1. *Let $f, f(0) \neq 0$ be a solution of (1). If f is bounded on $(-r, r)$, then f is continuous on $\mathbb{R}$.*

The proof results from Lemmas 2 and 3. □

Definition. *[4], [6]. The function* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *with* I *the interval, is called strictly increasing (strictly decreasing) at* $x_0 \in I$ *if there exists* $\eta(x_0) > 0$ *so that*

$$sgn\frac{f(x)-f(x_0)}{x-x_0} = 1(-1), \quad \forall\, x \in I \quad \textit{for which} \quad 0 < |x-x_0| < \eta. \tag{5}$$

Lemma 4. [4] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function. It is strictly monotonic on I if and only if it is strictly monotonic at every point of I.

Lemma 5. The solution $f, f(0) \neq 0$ of (1) is strictly monotonic on $\mathbb{R}$ iff it is strictly monotonic at zero.

Proof. Taking into account Definition and Lemma 4 the implication $\Rightarrow$ is obvious. From (1) we get

$$\frac{f(x-x_0)-f(0)}{x-x_0} = \frac{1}{2f(x_0)} \cdot \frac{f\left(\frac{x}{2}\right)^2 - f\left(\frac{x_0}{2}\right)^2}{\frac{x-x_0}{2}}, \quad \forall\, x, x_0 \in \mathbb{R}, \quad x \neq x_0, \quad \text{i.e.}$$

$$sgn\frac{f\left(\frac{x}{2}\right) - f\left(\frac{x_0}{2}\right)}{\frac{x-x_0}{2}} = sgn\frac{f(x-x_0)-f(0)}{x-x_0}, \tag{6}$$

because, according to Lemma 1,

$$sgn\frac{2f(x_0)}{f\left(\frac{x}{2}\right) + f\left(\frac{x_0}{2}\right)} = 1.$$

By the assumption, f is strictly increasing (strictly decreasing) at zero, then there exists a $\eta(0) > 0$ so that

$$sgn\frac{f(x-x_0)-f(0)}{x-x_0)} = 1(-1) \quad \text{for} \quad 0 < |x-x_0| < \eta.$$

$\square$

Proposition 2. *Every solution of (1),* $f(0) \neq 0$*, strictly monotonic at zero has only points of discontinuity of the first kind and the set of discontinuity is at most countable.*

The Proof results from [3, 5] and Lemma 5.

Proposition 3. *If* f *is a solution of (1),* $f(0) \neq 0$*, strictly monotomic at zero, then* f *is differentiable almost everywhere.*

The Proof results from Lebesgue's Theorem [3, 5] and from Lemma 5.

Lemma 6. Every solution $f, f(0) > 0 (< 0)$ is strictly J-convex (strictly J-concave) ([3, 5]; J stands for Jensen).

Proof. From (1) we have

$$f(0)\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right]=\left[f\left(\frac{x}{2}\right)-f\left(\frac{y}{2}\right)\right]^2>0,\quad \text{i.e.}$$

$$sgn\left[f(x)+f(y)-2f\left(\frac{x+y}{2}\right)\right]=sgn f(0);\quad \forall\ x,y\in\mathbb{R}.$$

□

Proposition 4. *Let $f, f(0)\neq 0$ be a solution of (1). If f is bounded on $(-r,r)$, then f is strictly convex (strictly concave) on $\mathbb{R}$ if $f(0)>0(<0)$.*

The Proof results from Lemmas 1 and 6, Proposition 1 and from Theorem [3]: A function which is strictly J-convex (strictly J-concave) and continuous on (a,b) is strictly convex (strictly concave) on (a,b).

Proposition 5. *Let f, for which $f(0)\neq 0$ be a solution of (1). If f is bounded on $(-r,r)$, then f is differentiable at zero.*

Proof. From Proposition 1 results that the function $f(x)$ is continuous on $\mathbb{R}$. Taking into account (3) and

$$\lim_{n\to\infty}\frac{\left(\frac{f(x_0)}{f(0)}\right)^{\frac{1}{2^n}}-1}{\frac{1}{2^n}}=ln\frac{f(x_0)}{f(0)}\tag{7}$$

we have

$$\frac{f\left(\frac{x_0}{2^n}\right)-f(0)}{\frac{x_0}{2^n}}=\frac{f(0)}{x_0}\frac{\frac{f\left(\frac{x_0}{2^n}\right)}{f(0)}-1}{\frac{1}{2^n}}=\frac{f(0)}{x_0}\frac{\left(\frac{f(x_0)}{f(0)}\right)^{\frac{1}{2^n}}-1}{\frac{1}{2^n}}$$

$$\lim_{n\to\infty}\frac{f\left(\frac{x_0}{2^n}\right)-f(0)}{\frac{x_0}{2^n}}=\frac{f(0)}{x_0}ln\frac{f(x_0)}{f(0)}\quad\forall\ x_0\in(-r,r)-\{0\}\quad \text{i.e.}(8)$$

$$f'(0)=\frac{f(0)}{x_0}ln\frac{f(x_0)}{f(0)}\quad\forall\ x_0\in(-r,r)-\{0\}\tag{9}$$

which implies

$$f(x_0)=f(0)e^{\frac{f'(0)}{f(0)}x_0}\tag{10}$$

□

Lemma 7. Let $f, f(0) \neq 0$ be a solution of (1). The function f is differentiable on $\mathbb{R}$ if f is the differentiable at zero and

$$f'(x) = \frac{f'(0)}{f(0)} f(x) = \beta f(x); \quad \forall\, x \in \mathbb{R}, \ \ \beta = \frac{f'(0)}{f(0)}. \tag{11}$$

Proof. We have:

$$\frac{f(x - x_0) - f(0)}{x - x_0} = \frac{1}{f(x_0)} \frac{f\left(\frac{x}{2}\right) + f\left(\frac{x_0}{2}\right)}{2} \cdot \frac{f\left(\frac{x}{2}\right) - f\left(\frac{x_0}{2}\right)}{\frac{x - x_0}{2}}$$

$$\lim_{x \to x_0} \frac{f\left(\frac{x}{2}\right) - f\left(\frac{x_0}{2}\right)}{\frac{x - x_0}{2}} = \frac{f'(0)}{f\left(\frac{x_0}{2}\right)} f(x_0) = \frac{f'(0)}{f(0)} f\left(\frac{x_0}{2}\right), \ \text{ i.e. } \ f'\left(\frac{x_0}{2}\right) = \frac{f'(0)}{f(0)} f\left(\frac{x_0}{2}\right),$$

which implies (11). □

Proposition 6. *Let $f, f(0) \neq 0$ be a solution of (1). If f is bounded on $(-r, r)$, then f is infinitely differentiable often, $f \in C_R^\infty$ and*

$$f(x) = f(0) e^{\frac{f'(0)}{f(0)} x} = \alpha e^{\beta x}; \quad \forall\, x \in \mathbb{R}, \quad \alpha = f(0), \quad \beta = \frac{f'(0)}{f(0)} \tag{12}$$

$$f^{(n)}(x) = \beta^n f(x); \quad \forall\, x \in \mathbb{R}, \ \forall\, n \in \mathbb{N}^* \tag{13}$$

$$sgn\ f^{(n)}(x) = sgn \frac{f'(0)^{(n)}}{f(0)^{n-1}}, \quad \forall\, x \in \mathbb{R}, \ \forall\, n \in N^*. \tag{14}$$

The Proof results from Proposition 5, Lemma 7 and (10).

2. Now let us establish the connections of Lobachevsky's functional equation with some other classical functional equations [1]. It is easy to verify the following Lemmas and Propositions.

Lemma 8. If $f : \mathbb{R} \to \mathbb{R}, \ f(0) \neq 0$ is a solution of (1), then

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = \frac{f(x)}{f(0)} \tag{15}$$

is a solution of Cauchy's additive multiplicative functional equation

$$g(x + y) = g(x) g(y); \quad \forall\ x, y \in \mathbb{R}, \tag{16}$$

and conversely, if $g : \mathbb{R} \to \mathbb{R}$ is a solution of (16), then

$$f(x) = \alpha\, g(x), \quad \forall x \in \mathbb{R}, \quad \alpha = f(0) \neq 0, \tag{17}$$

arbitrary is a solution of (1).

Proposition 7. *Let $f, f(0) \neq 0$ be a solution of (1). If f is bounded on $(-r, r)$ then*

$$g(x) = e^{\beta x} = e^{\frac{f'(0)}{f(0)}x}, \qquad x \in \mathbb{R}.$$

Lemma 9. If $f, f(0) \neq 0$ be a solution of (1), then

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = \ln \frac{f(x)}{f(0)} \tag{18}$$

is a solution of Cauchy's additive functional equation

$$h(x+y) = h(x) + h(y) \quad \forall\, x, y \in \mathbb{R} \tag{19}$$

and conversely, if h is a solution of (9), then $f(x) = \alpha \cdot e^{h(x)}$ is a solution of (1).

Proposition 8. *Let $f, f(0) \neq 0$ be a solution of (1). If f is bounded in $(-r, r)$, then $h(x) = \beta x$ is a solution of (19).*

Lemma 10. If $f : \mathbb{R} \to \mathbb{R},\ f(0) > 0$ is a solution of (1), then

$$\varphi(x) = \ln f(x), \quad \varphi : \mathbb{R} \to \mathbb{R} \tag{20}$$

is a solution of Jensen's functional equation

$$\varphi\left(\frac{x+y}{2}\right) = \frac{\varphi(x) + \varphi(y)}{2}, \quad \forall\, x, y \in \mathbb{R}, \tag{21}$$

and conversely, if $\varphi(x)$ is a solution of (21), then $f(x) = e^{\varphi(x)}$ is a solution of (1).

Proposition 9. *Let $f, f(0) > 0$ be a solution of (1). If f is bounded on $(-r, r)$ then*

$$\varphi(x) = \beta x + \gamma, \quad \gamma = \ln f(0) = \ln \alpha \tag{22}$$

is solution of (21).

Lemma 11. If $f, f(0) \neq 0$ is a solution of (1), then

$$g(x) = \frac{f(x) + f(-x)}{2f(0)}, \quad h(x) = \frac{f(x) - f(-x)}{2f(0)} \tag{23}$$

satisfy

$$g(0) = 1, \ \ h(0) = 0, \ \ g(-x) = g(x), \ \ h(-x) = -h(x) \tag{24}$$

$$g(x)^2 + h(x)^2 = 1 \tag{25}$$

$$g(x)^2 + h(x)^2 = g(2x) \tag{26}$$

$$2h(x)g(x) = h(2x) \tag{27}$$

$$g(x+y) = g(x)g(y) + h(x)h(y) \tag{28}$$

$$h(x+y) = h(x)g(y) + h(y)g(x) \tag{29}$$

$$2g(x)^2 = 1 + g(2x), \quad 2h(x)^2 = g(2x) - 1 \tag{30}$$

$$g(x+y) + g(x-y) = 2g(x)g(y), \;\; g(x+y) - g(x-y) = 2h(x)h(y) \tag{31}$$

and conversely, if $(g(x), h(x))$ is a solution of the system (28)-(29), then

$$f(x) = f(0 = [g(x) + h(x)], \quad f(0) = \alpha, \tag{32}$$

arbitrary is a solution of (1).

Proof. The relations (24)-(27) are obtained from (1) and (23). For (28)-(29) we have

$$g(x+y) = 2g\left(\frac{x+y}{2}\right)^2 - 1, \;\; g(x)g(y) + h(x)h(y) = 2g\left(\frac{x+y}{2}\right)^2 - 1 \tag{33}$$

and

$$h(x+y) = 2g\left(\frac{x+y}{2}\right) \; h\left(\frac{x+y}{2}\right), \;\; g(x)h(y) + g(y)h(x) = 2g\left(\frac{x+y}{2}\right) h\left(\frac{x+y}{2}\right). \tag{34}$$

The relations (30)-(31) are consequences of (28)-(29). Conversely, from (28), (29) and (32) we obtain

$$f(x)f(y) = f(0)^2[g(x) + h(x)]\,[g(y) + h(y)] = f(0)^2[g(x+y) + h(x+y)],$$

i.e.

$$f(x)f(y) = f(0)f(x+y). \tag{35}$$

Now, we demonstrate that

$$f\left(\frac{x+y}{2}\right)^2 = f(0)f(x+y) \tag{36}$$

by applying (30), (32), (33) and (34), we get

$$f\left(\frac{x+y}{2}\right)^2 = f(0)^2\left[g\left(\frac{x+y}{2}\right) + h\left(\frac{x+y}{2}\right)\right]^2 = f(0)^2[g(x+y)+h(x+y)] = f(0)f(x+y).$$

From (35) and (36) results in (1). $\square$

Proposition 10. *Let $f, f(0) = 1$ be a solution of (1). If f is bounded on $(-r, r)$ and $f'(0) = 1$, then*

$$g(x) = \frac{e^x + e^{-x}}{2} = chx, \quad h(x) = \frac{e^x + e^{-x}}{2} = shx, \tag{37}$$

which verify the relations (24)-(31).

The proof results from Lemma 11 and Proposition 6.

Proposition 11. *Let $f, f(0) \neq 0$ be a solution of (1). If f is strictly monotonic in zero, then the solutions of functional equations (16), (18), and of the system (28)-(29) have only points of discontinuity of the first kind. The set of discontinuity is at most countable and the solutions are differentiable almost everywhere. The same assertions are valid for (20) iff, with $f(0) > 0$ is solution of (1). The Proof results from Proposition 2 and 3 and Lemmas 8, 9, and 10.*

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CONTENTS

World renowned Turkish mathematician Cahit Arf passed away on December 26, 1997 in Bebek, İstanbul, at the age of 87. Cahit Arf was born in Selanik (Thessaloniki), which was then a part of the Ottoman Empire. His family migrated to İstanbul with the outbreak of the Balkan War in 1912. The Family finally settled in İzmir where Cahit Arf received his primary education. Upon receiving a scholarship from the Turkish Ministry of Education he continued his education in Paris and graduated from Ecole Normale Superieure. Returning to Turkey he taught mathematics in a high school in İstanbul. In 1933 he joined the Mathematics Department of İstanbul University. In 1937 he went to Gottingen and completed his Ph. D. thesis under the supervision of Helmut Hasse in 1938. He returned to İstanbul University and worked there till 1962 and then joined the Mathematics Department of Robert College in İstanbul. Professor Arf was at the Institute for Advanced Studies in Princeton during 1964-1966. Later he visited University of California, Berkeley for one year. Upon returning to Turkey he joined the Mathematics Department of the Middle East Technical University and in 1980 he retired from this University. Professor Arf received several awards for his contributions to Mathematics, among them are, İnönü Award 1948, TÜBİTAK Science Award 1974, Comandur des Palmes Académiques 1994. Professor Arf Was a member of the Mainz Academy and the Turkish Academy of Sciences. He was the president of the Turkish Mathematical Society from 1985 until 1989.

Professor Arf's influence on Turkish Mathematics was profound. Although he had very few formal students, almost all of the present day active mathematicians of Turkey, at some time of their carrier, had fruitful discussions on their fields of interest with him and had received support and encouragement.

The collected works of Cahit Arf was published, in 1988, by the Turkish Mathematical Society. The first paper of this issue of the Turkish Journal of Mathematics is about the mathematics of Cahit Arf. It was written by Professor M. Ikeda, a good friend and colleague of Cahit Arf, on the occasion of Cahit Arf's receiving the tittle of doctor honoris causa from the Middle East Technical University in 1981. We are grateful to Professor Ikeda for giving us permission to print this article in T.J.M.

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