Tr. J. of Mathematics
22 (1998) , 61 – 68.

WEIGHTED ERGODIC AVERAGES

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Abstract

Let $(X, \mathcal{F}, \lambda)$ be the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with the usual σ-algebra $\mathcal{F}$ of Lebesgue measurable subsets and the normalized Lebesgue measure λ. Consider a sequence $\nu_n : \mathbb{N} \to \mathbb{R}$, $\nu_n(k) \geq 0$, $\Sigma_{k=1}^{\infty} \nu_n(k) = 1$. For any measure-preserving $\tau : X \to X$, this sequence induces a sequence $(T_n)_1^{\infty}$ of bounded, linear operators on $L^p(X)$, $1 \leq p \leq \infty$, by defining

$$T_n f = \sum_{k=1}^{\infty} \nu_n(k)\, f \circ \tau^k, \quad n = 1, 2, \ldots.$$

We shall prove that under suitable conditions imposed on τ and $(\nu_n)_1^{\infty}$, there exists a large collection of measurable characteristic functions f for which $\limsup_{n\to\infty} T_n f - \liminf_{n\to\infty} T_n f = 1$ a.e on X.

Keywords: Weights, weighted averages, Fourier transforms.

1. Introduction

The classical ergodic theorem of Birkhoff states that if τ is an ergodic transformation on a probability space $(X, \mathcal{F}, \mu)$, then for each $f \in L^1(X)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} f \circ \tau^k = \int_X f\, d\mu$$

almost everywhere (on X). So, the sequence

$$\left\{ \frac{1}{n} \sum_{k=1}^{n} f \circ \tau^k \right\}_{n=1}^{\infty}$$

of averages of f under iterates of τ converges a.e. for all f in $L^1(X)$.

*AMS (MOS) subject classification (1985 revision): Primary: 28D99, Secondary: 60F99

A natural question that arises from the above is that instead of considering the sequence of usual average of f under iterates of τ as above, what would happen if we consider sequence of weighted averages, i.e., averages of the form

$$\sum_{k=1}^{\infty} \nu_n(k) f \circ \tau^k,$$

where $\nu_n(k) \geq 0$ for all n, k and

$$\sum_{k=1}^{\infty} \nu_n(k) = 1, \quad n = 1, 2, \ldots,$$

and investigate its behaviour with respect to convergence. This question has been considered by several authors, including those given in [1] and [4] of the references. For our present purpose, we shall refer to any sequence

$$\nu : \mathbb{N} \to \mathbb{R}, \;\; \nu(k) \geq 0, \quad \sum_{k=1}^{\infty} \nu(k) = 1$$

as weights.

Now, let $(\nu_n)_1^\infty$ be a sequence of weights. For any probability space $(X, \mathcal{F}, \mu)$ and any ergodic $\tau : X \to X$, this sequence of weights induces a sequence of operators

$$(T_n)_1^\infty : L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu),$$

where each T_n is defined by the following formula:

$$T_n f = \sum_{k=1}^{\infty} \nu_n(k) \; f \circ \tau^k, \quad n = 1, 2, \ldots.$$

If we assume that the measure algebra of the space X under consideration is normalized, non-atomic, and separable, it is a standard result in ergodic theory that in the study of such averages, it is enough to consider the special case where $(X, \mathcal{F}, \lambda)$ is the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with the usual σ-algebra $\mathcal{F}$ of Lebesgue measurable subsets and the normalized Lebesgue measure λ, and where τ is an ergodic rotation $\tau(z) = z_* z$ on $\mathbb{S}^1$ for some fixed z_* in $\mathbb{S}^1$.

The purpose of this note is to give an elementary proof of the following result. For many other related results, the readers should consult [1].

Theorem. *Let $(\nu_n)_1^\infty$ and $(T_n)_1^\infty$ be as above. Assume that $z_* \in \mathbb{S}^1$, which defines τ, has the following properties:*
Given any arc J on the unit circle $\mathbb{S}^1$, there exists $\Lambda(J) \subseteq \mathbb{N}$ such that

i) $D = \{z_*^m : m \in \Lambda(J)\}$ *is dense in* $\mathbb{S}^1$

ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} \nu_n(k) z^k$ *exists and belong to* J *for all* z *in* D.

Then, given $\epsilon > 0$, $N \in \mathbb{N}$, *there exists a set* $B \in \mathcal{F}$, $\lambda(B) < \epsilon$ *such that*

$$\lambda\left(\sup_{n\geq N} T_n \chi_B > 1 - \epsilon\right) = 1.$$

Consequently, there exists a dense subset $\mathcal{R}$ *of* $\mathcal{F}$ *such that if* $A \in \mathcal{R}$,

$$\limsup_{n\to\infty} T_n \chi_A = 1 \quad a.e \text{ and } \quad \liminf_{n\to\infty} T_n \chi_A = 0 \quad a.e$$

(Recall that if $(X, \mathcal{F}, \mu)$ is a probability space and if we define $\rho(A, B) = \mu(A\Delta B)$ for A, B in $\mathcal{F}$, then this ρ induces a metric on $\mathcal{F}$ if we identify sets that differ from each other by a set of measure zero.)

2. Some Lemmas

The proof of our theorem follows from the following two simple lemmas. To simplify notation, let us define for $n_j \in \mathbb{N}$, $\ell = 1, 2, \ldots, \infty$, $\omega \in \mathbb{S}^1$,

$$S_{n_j}^{\ell} = \sum_{k=1}^{\ell} \nu_{n_j}(k) \quad \text{and} \quad S_{n_j}^{\ell}(w) = \sum_{k=1}^{\ell} \nu_{n_j}(k) w^k.$$

We also let $\{t\}$ denote the fractional part of a real number t, and $\overline{\gamma}$ is the complex conjugate of $\gamma \in \mathbb{C}$.

Lemma 1. Assume i) and ii) of the previous theorem hold. Let $0 < \epsilon, \eta$ and $N \in \mathbb{N}$ be given. There exists an integer $K > N$ and K points $z_1, \ldots, z_K \in \mathbb{S}^1$, integers $n_1, n_2, \ldots, n_K$, each $n_j > K$, and an integer $m \in \mathbb{N}$ such that

(i) $|S_{n_j}^{\infty}(z_*^m) - z_j| \leq \eta$ for all $j = 1, 2, \ldots, K$

(ii) Any arc of length ϵ contains at least 3 points among $z_1, \ldots, z_K$.

Proof. (i) Let $0 < \epsilon$, η and $N \in \mathbb{N}$ be given. Choose $K > N$, $K \in \mathbb{N}$ such that $\dfrac{8\pi}{K} < \epsilon$. Fix some $\tilde{\eta} > 0$ with $\tilde{\eta} < \min\left(\eta, \frac{1}{2K}\right)$.

Let $I_1 = \{e^{2\pi i x} : x \in (-\tilde{\eta}, \tilde{\eta})\}$ and for $j = 2, 3, \ldots K$, let

$$I_j = \left\{e^{2\pi i x} : x \in \left(\frac{j-1}{K} - \tilde{\eta}, \quad \frac{j-1}{K} + \tilde{\eta}\right)\right\}$$

be K arcs on $\mathbb{S}^1$ of equal arc lengths. By hypothesis, there exists $m_1 \in \mathbb{N}$ and large integers $n_1, h_1 > K$ and some $z_1 \in I_1$ such that

$$|S_{n_1}^{h_1}(z_*^{m_1}) - z_1| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_1}^{h_1} > 1 - \frac{\tilde{\eta}}{2}. \tag{1}$$

Again, by assumption, there exists a dense set $D_2 \subseteq \mathbb{S}^1$ such that $\lim_{n\to\infty} S_n^\infty(z)$ exists and belongs to I_2 for each $z \in D_2$, and moreover, each element of D_2 is of the form z_*^n for some $n \in \mathbb{N}$. Choose $z_*^{m_2} \in D_2$ with $|z_*^{m_2} - z_*^{m_1}|$ so small (so that $|z_*^{m_2 k} - z_*^{m_1 k}|$ is very small for each $k = 1, 2, \cdots h_1$) so that $|S_{n_1}^{h_1}(z_*^{m_2}) - z_1| < \frac{\tilde{\eta}}{2}$. Since $\lim_{n\to\infty} S_n^\infty(z_*^{m_2})$ exists and belongs to I_2, there exists $h_2, n_2 > K$ and $z_2 \in I_2$ with

$$|S_{n_2}^{h_2}(z_*^{m_2}) - z_2| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_2}^{h_2} > 1 - \frac{\tilde{\eta}}{2}. \tag{2}$$

Again, there exists a dense set $D_3 \subseteq \mathbb{S}^1$ such that $\lim_{n\to\infty} S_n^\infty(z)$ exists and belong to I_3 and each element z of D_3 is of the form z_*^n for some $n \in \mathbb{N}$. Choose $z_*^{m_3} \in D_3$ with $|z_*^{m_3} - z_*^{m_j}|$ very small for $j = 1, 2$, so that $|S_{n_1}^{h_1}(z_*^{m_3}) - z_1| < \frac{\tilde{\eta}}{2}$ and $|S_{n_2}^{h_2}(z_*^{m_3}) - z_2| < \frac{\tilde{\eta}}{2}$. Also, because $\lim_{n\to\infty} S_n^\infty(s_*^{m_3})$ exists and belong to I_3, there exists $z_3 \in I_3$ and integers $h_3, n_3 > K$ such that $|S_{n_3}^{h_3}(z_*^{m_3}) - z_3| < \frac{\tilde{\eta}}{2}$ and $S_{n_3}^{h_3} > 1 - \frac{\tilde{\eta}}{2}$. Continuing this process to the K^{th} stage, we obtain an integer $m = m_K$ and $z_1, \ldots, z_K, \quad z_j \in I_j$, integers $h_j,\ n_j > K$ for $j = 1, 2, \ldots, K$ such that

$$|S_{n_j}^{h_j} - z_j| < \frac{\tilde{\eta}}{2} \quad \text{and} \quad S_{n_j}^{h_j} > 1 - \frac{\tilde{\eta}}{2} \quad \text{for all} \quad j = 1, 2, \ldots, K.$$

Hence, it follows that for each $j = 1, 2, \ldots, K$,

$$|S_{n_j}^\infty(z_*^m) - z_j| \le |S_{n_j}^{h_j}(z_*^m) - z_j| + \left| \sum_{k=h_{j+1}}^\infty \nu_{n_j}(k) z_*^{mk} \right| < \frac{\tilde{\eta}}{2} + \frac{\tilde{\eta}}{2} = \tilde{\eta} < \eta.$$

This completes the proof of (i).

(ii) Each arc I_j has arc-length $2\pi(2\tilde{\eta}) = 4\pi\tilde{\eta}$ and the gap between 2 adjacent arcs is $2\pi(\frac{1}{K} - 2\tilde{\eta})$. So any arc of length

$$4(4\pi\tilde{\eta}) + 3\left[2\pi\left(\frac{1}{K} - 2\tilde{\eta}\right)\right] < \pi\left(\frac{2}{K} + \frac{6}{K}\right) = \frac{8\pi}{K} < \epsilon$$

must contain 3 points among those of $z_1, z_2, \ldots, z_K$. □

We shall need the following lemma

Lemma 2. ([1]) Let $\epsilon,\ \rho > 0$ be given. Then there exists $0 < \eta = \eta(\epsilon, \rho)$ satisfying the following.

For any probability measure μ on $\mathbb{R}$,

$$|\hat{\mu}(1)-1|<\eta \Rightarrow \mu(G_\epsilon)<\rho.$$

Here, $G_\epsilon=\cup_{k=-\infty}^{\infty}(\epsilon+k,\ 1-\epsilon+k)$ and $\hat{\mu}:\mathbb{R}\to C$ is the Fourier transform of μ defined by

$$\hat{\mu}(t)=\int_R e^{2\pi ixt}d\mu(x), \quad t\in\mathbb{R}.$$

Using the above, we will show the following:

Lemma 3. Let $\delta,\rho>0$ be given. Then there exists $0<\eta=\eta(\delta,\rho)$ such that

$$\left|\sum_{k=1}^{\infty}\nu(k)z_k-1\right|<\eta \Rightarrow \sum_{|z_k-1|>\delta}\nu(k)<\rho$$

whenever $\nu(k)\ge 0$, $\Sigma_{k=1}^{\infty}\nu(k)=1$ and $(z_k)_1^{\infty}\subseteq\mathbb{S}^1$.

Proof. Let $\delta,\rho>0$ be given. Choose $\epsilon>0$ such that whenever $t\in\mathbb{R}$, $\{t\}\in(\epsilon,1-\epsilon)\Leftrightarrow|e^{2\pi it}-1|>\delta$. Let $0<\eta=\eta(\epsilon,\rho)$ be as in Lemma 2. Note that $\epsilon=\epsilon(\delta)$ so that $\eta=\eta(\delta,\rho)$. Suppose $\nu(k)\ge 0$, $\Sigma_{k=1}^{\infty}\nu(k)=1$ and $(z_k)_1^{\infty}\subseteq\mathbb{S}^1$. Write z_k as $z_k=e^{2\pi it_k}$, $t_k\in\mathbb{R}$, $k=1,2,\ldots$. Define a probability measure μ on $\mathbb{R}$ by $\mu(t_k)=\nu(k)$, so that μ has support contained in $\{t_1,t_2,\ldots\}$. We then have

$$\hat{\mu}(1)=\int_R e^{2\pi ix}d\mu(x)=\sum_{k=1}^{\infty}e^{2\pi it_k}\mu(t_k)=\sum_{k=1}^{\infty}\nu(k)z_k.$$

By Lemma 2. above, we get

$$|\hat{\mu}(1)-1|<\eta \Rightarrow \mu(G_\epsilon)<\rho,$$

and this completes the proof, because

$$\mu(G_\epsilon)=\sum_{k=1}^{\infty}\mu(\epsilon+k,\ 1-\epsilon+k)=\sum_{\{t_k\}\in(\epsilon,1-\epsilon)}\mu(t_k)=\sum_{|z_k-1|>\delta}\nu(k).$$

□

Lemma 4. Given $\epsilon_0>0$, there exists $\eta_0>0$ such that

$$\left|\sum_{k=1}^{\infty}\nu_n(k)\gamma_k-\gamma\right|<\eta_0 \Rightarrow \sum_{|\gamma_k-\gamma|<\epsilon_0}\nu_n(k)>1-\epsilon_0$$

for all sequences $(\nu_n)_1^\infty$ of weights, and all $(\gamma_k)^\infty \subseteq \mathbb{S}^1$, $\gamma \in \mathbb{S}^1$.

Proof. Given $\epsilon_0 > 0$, let $\delta = \rho = \frac{\epsilon_0}{2}$. Choose $\eta = \eta(\delta, \rho)$ as in Lemma 3. Now, suppose $(\nu_n)_1^\infty$ is any sequence of weights and let $(\gamma_k)_1^\infty \subseteq \mathbb{S}^1$, $\gamma \in \mathbb{S}^1$ be arbitrary. Then, by Lemma 3, for every n,

$$\left|\sum_{k=1}^{\infty} \nu_n(k)\gamma_k - \overline{\gamma}\right| < \eta_0 \Rightarrow \sum_{|\gamma_k - \overline{\gamma}| \leq \delta} \nu_n(k) \geq 1 - \rho.$$

Thus, if we let $\eta_0 = \eta$, we then have, for every n,

$$\left|\sum_{k=1}^{\infty} \nu_n(k)\gamma_k - \gamma\right| < \eta_0 \Rightarrow \sum_{|\gamma_k - \gamma| < \epsilon_0} \nu_n(k) > 1 - \epsilon_0.$$

□

3. Proof of the Theorem

Let $\epsilon > 0$, $N \in \mathbb{N}$ be given. Let $\epsilon_0 = \frac{\epsilon}{10}$. Choose $\eta_0 > 0$ as in Lemma 4. Then by Lemma 1, there exists $K > N$ and integers $n_1, \ldots, n_K$, each $n_j > K > N$ and points $z_1, \ldots, z_K$ in $\mathbb{S}^1$ and an integer m satisfying (i) and (ii) with ϵ_0, η_0 in place of ϵ, η there. Let J be any arc on $\mathbb{S}^1$ of length ϵ. Let $B = P_m^{-1}(J)$, where $P_m(z) = z^m$ for all z in $\mathbb{S}^1$. Suppose $z \in \mathbb{S}^1$. Let $J_z = \overline{z}^m J$ so that J_z has arc length $= \epsilon$. Let $\mathbf{p}$ be the middle point of J_z and let $\tilde{J}_z$ be the arc centered at $\mathbf{p}$ with length ϵ_0. By Lemma 1, $\tilde{J}_z$ must contain at least three adjacent points among those of $z_1, \ldots, z_K$. Let z_q be the middle point of these three. Then $|S_{n_q}^\infty(z_*^m) - z_q| < \eta_0$, hence, by Lemma 4

$$\sum_{|z_*^{mk} - z_q| < \epsilon_0} \nu_{n_q}(k) \geq 1 - \epsilon_0.$$

Now, $|z_*^{mk} - z_q| < \epsilon \Rightarrow z_*^{mk} \in J_z$ (since J_z contains $\tilde{J}_z$ in the middle and $\tilde{J}_z$ has arc-length very small compare to that of J_z).

Thus we have

$$\sum_{|z_*^{mk} - z_q| < \epsilon_0} \nu_{n_q}(k) > 1 - \epsilon_0 \Rightarrow \sum_{z_*^{mk} \in J_z} \nu_{n_q}(k) > 1 - \epsilon_0.$$

Now, $z_*^{mk} \in J_z \Leftrightarrow (zz_*^k)^m \in J \Leftrightarrow zz_*^k \in B$, hence,

$$\begin{aligned} \sup_{n \geq N} T_n \chi_B(z) &= \sup_{n \geq N} \sum_{k=1}^{\infty} \nu_n(k) \chi_B(zz_*^k) = \sup_{n \geq N} \sum_{zz_*^k \in B} \nu_n(k) \\ &\geq \sum_{zz_*^k \in B} \nu_{n_q}(k) = \sum_{z_*^{mk} \in J_z} \nu_{n_q}(k) > 1 - \epsilon_0 > 1 - \epsilon. \end{aligned}$$

Since, $z \in \mathbb{S}^1$ was arbitrary, this clearly implies

$$\lambda\left(\left\{\sup_{n\geq N} T_n\chi_B > 1-\epsilon\right\}\right) = 1,$$

as to be shown.

Finally to see the other conclusions of the theorem, we let

$$G_nA = T_n\chi_A, \quad n = 1, 2, \ldots; \;\; A \in \mathcal{M}.$$

Then if A and B are disjoint

$$\begin{aligned} G_n(AUB) &= T_n\chi_{AUB} = \sum_{k=1}^{\infty}\nu_n(k)\chi_{AUB}\circ\tau^k \\ &= \sum_{k=1}^{\infty}\nu_n(k)\chi_A\circ\tau^k + \sum_{k=1}^{\infty}\nu_n(k)\chi_B\circ\tau^k \\ &= T_n\chi_A + T_n\chi_B = G_nA + G_nB. \end{aligned}$$

Also, $(G_n)^{\infty}$ is a sequence of monotone linear maps which are continuous in measure with

$$G_n(\mathbb{S}^1) = \sum_{k=1}^{\infty}\nu_n(k)\chi_{\mathbb{S}^1}\circ\tau^k = \sum_{k=1}^{\infty}\nu_n(k) = 1$$

and such that given $\epsilon > 0$, $N \in \mathbb{N}$, there exists $B \in \mathcal{F}$, $\lambda(B) < \epsilon$ with

$$\lambda\left(\left\{\sup_{n\geq N} G_nB > 1-\epsilon\right\}\right) = \lambda\left(\left\{\sup_{n\geq N} T_n\chi_B > 1-\epsilon\right\}\right) = 1.$$

Hence, by Theorem 1.3 of [3], the conclusion of our theorem follows.

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Received 17.05.1996