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FINITE DIRECT SUMS OF (D1)-MODULES

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Abstract

In this paper we give necessary conditions for a finite direct sum of $(D1)$–modules to be a $(D1)$–module.

1. Introduction

Let R be a ring and $M = M_1 \oplus M_2$ a decomposition of a right R–module M. We are interested in conditions on M_1, M_2 which make M a $(D1)$–module. If M is a $(D1)$–module it is well-known that M_1 and M_2 are both $(D1)$–modules. In this paper, we prove that if M_1 and M_2 are relatively projective, quasi-projective and $(D1)$–modules then M is a $(D1)$–module. Let $M = \oplus_{i \in I} M_i$ be a decomposition that complements direct summands. We prove that M is (quasi-) discrete if and only if (i) for every $i \in I, M(I-i)$ is (quasi-) discrete, (ii) for every $i \in I, M_i$ and $M(I-i)$ are relatively projective modules.

Throughout, all rings will have identities and all modules will be unital right modules.

Let R be a ring and M an R–module. Let A and L be submodules of M. L is called a *supplement* of A in M if it is minimal with respect to the property $M = A + L$. A submodule K of M is called a *supplement* (in M) if K is a supplement of some submodule of M. It is easy to check that L is a supplement of A in M if and only if $M = A + L$ and $A \cap L$ is small in L.

Let R be a ring and M an R–module. We consider

$(D1)$ For every submodule A of M there exists a direct summand M_1 of M such that $M = M_1 \oplus M_2$ and $M_1 \leq A$, $A \cap M_2$ is small in M_2.

$(D2)$ For any submodule A of M for which M/A is isomorphic to a direct summand of M then A is a direct summand of M.

$(D3)$ If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

M is said to *have* (Di) (or to be a (Di)*–module*) if it satisfies (Di) $(i = 1, 2, 3)$. M is called a *(quasi-) discrete* module if it has $((D1)$ and $(D3))$ $(D1)$ and $(D2)$.

Lemma 1. *Let A and B be modules with local endomorphism rings such that $M = A \oplus B$ has $(D1)$. Let C be a submodule of A and let $f : B \to A/C$ be a homomorphism. Then the following hold.*

(i) If f cannot be lifted to a homomorphism from B to A, then f is an epimorphism and there exists an epimorphism from A to B.

(ii) If any epimorphism from A to B is an isomorphism, then B is A–projective.

(iii) If there is no epimorphism from A to B, then B is A–projective.

Proof. (i). Let $f : B \to A/C$, and suppose f cannot be lifted to a homomorphism from B to A. Consider the canonical epimorphism $\pi : A \to A/C$. Set $U = \{ a + b : a \in A, b \in B, f(b) = -\pi(a) \}$. Then $M = U + A$. By Proposition 4.8 in [2] there exists a supplement U^* of A in M with $U^* \leq U$ and U^* is a direct summand of M. By the Krull-Schmidt-Azumaya Theorem [1, Corollary 12.7], $M = U^* \oplus A$ or $M = U^* \oplus B$. Assume $M = U^* \oplus A$. Let α denote the canonical projection of $M = U^* \oplus A$ onto A. Let $\alpha|_B$ denote the resriction of α to B. It is easily checked that $\pi\alpha|_B = f$. This is a contradiction, for f cannot be lifted to a homomorphism from B to A. Hence $M = U^* \oplus B$. We prove that f is epic. Indeed, if $a + C \in A/C$ then we write $a = u^* + b = a_1 + b_1 + b$ where $u^* \in U^*$, $u^* = a_1 + b_1$, $f(b_1) = -\pi(a_1)$, $a_1 \in A$ and $b, b_1 \in B$. Hence $a = a_1$, $b = -b_1$ and $f(b) = a + C$. Thus f is epic. Now let $\beta|_A$ denote the resriction of the canonical projection $\beta : U^* \oplus B \to B$ to A. Since $M = U^* \oplus B = U^* + A$ then $\beta|_A(A) = B$.

(ii). Suppose any epimorphism from A to B is an isomorphism. Let C be a submodule of A and $f : B \to A/C$ any homomorphism. As in the proof of (i), if $M = U^* \oplus A$ then f can be lifted to a homomorphism from B to A. Assume $M = U^* \oplus B$. Let ψ denote the canonical projection of $M = U^* \oplus B$ onto B and $\psi|_A$ the restriction of ψ to A. Then $\psi|_A$ is an epimorphism from A onto B and then, by assumption, $\psi |_A$ is an isomorphism. It follows easily that $M = U^* \oplus A$.

(iii). This is clear from (i). □

Corollary 2. *Let M be a uniserial module with unique composition series $M \supset U \supset V \supset 0$. Then $M \oplus (U/V)$ does not have $(D1)$.*

Proof. Assume M is uniserial with unique composition series $M \supset U \supset V \supset 0$. Clearly M and U/V have local endomorphism rings. Suppose $M \oplus (U/V)$ has $(D1)$. Let f denote the inclusion map from U/V to M/V. Then f is not an epimorphism. By Lemma 1(i), f can be lifted to a homomorphism g from U/V to M. Note that g is not epic. Hence $Img = U$ or $Img = V$. Each case leads to a contradiction. □

Remark. Let M be a uniform module and N a non-zero module isomorphic to L/K for some submodules $K < L$ of M. Then N is not M–projective by [2, Lemma 4.30 and Proposition 4.31]. Therefore in Corollary 2, U/V is not an M–projective module.

Lemma 3. *Let M_1 be a simple module and M_2 a uniserial module with unique composition series $M_2 \supset U \supset 0$. Then $M = M_1 \oplus M_2$ has $(D1)$.*

Proof. Let L be a non-zero submodule of M. We show that there exists a submodule K of M such that $M = K \oplus K'$, $K \leq L$ and $L \cap K'$ is small in K' for some submodule K' of M. If $M_1 \cap (L + M_2) = 0$ then $L \leq M_2$. Hence L is a small submodule or direct summand of M. Assume $M_1 \cap (L + M_2) \neq 0$. Then $M_1 \leq L + M_2$ and $M = L + M_2$. If $L \cap M_2 = M_2$ or $L \cap M_2 = 0$ or $L \cap M_2 = U$ and $L \cap M_1 = M_1$ we are done. Assume $L \cap M_2 = U$ and $L \cap M_1 = 0$. Then $U \leq L$. Hence $M = L \oplus M_1$. Thus M has $(D1)$. □

Example 4. Let p be a prime integer and M denote the $\mathbb{Z}$–module, $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^2\mathbb{Z})$. Then M has $(D1)$ and $\mathbb{Z}/p\mathbb{Z}$ is not $\mathbb{Z}/p^2\mathbb{Z}$–projective.

Proof. By Lemma 3 and Remark. □

Lemma 5. *The following statements are equivalent for a module $M = M_1 \oplus M_2$.*

(i) *M_2 is M_1–projective.*

(ii) *For each submodule N of M with $M = M_1 + N$ there exists a submodule M' of N such that $M = M_1 \oplus M'$.*

Proof. The proof is in [3, 41.14, (3) ⇔ (4)]. A proof of $(i) \Rightarrow (ii)$ can also be found in [2, Lemma 4.47].

Consider the $\mathbb{Z}$–module $M = \mathbb{Z} \oplus \mathbb{Z}(p^\infty)$ where $\mathbb{Z}(p^\infty)$ denotes the Prufer p–group. Then it is well-known that $\mathbb{Z}$ and $\mathbb{Z}(p^\infty)$ are relatively projective, M does not have $(D1)$ and $\mathbb{Z}(p^\infty)$ has $(D1)$. Also, $\mathbb{Z}$ is not semisimple. In this vein we prove the following theorem. □

Theorem 6. *Let the module $M = M_1 \oplus M_2$ be a direct sum of relatively projective modules M_1, M_2, such that M_1 is semisimple and M_2 has $(D1)$. Then M has $(D1)$.*

Proof. Let L be a non-zero submodule of M.

Case 1. $K = M_1 \cap (L + M_2) \neq 0$. Then $M_1 = K \oplus K'$ for some submodule K' of M_1 and hence $M = K \oplus K' \oplus M_2 = L + (M_2 \oplus K')$. By [2, Prop. 4.31, Prop. 4.32 and Prop. 4.33], K is $M_2 \oplus K'$–projective. By Lemma 5, there exists a submodule L' of L

such that $M = L' \oplus (M_2 \oplus K')$. Assume $L \cap (M_2 \oplus K') \neq 0$. Let X be any submodule of M_2. Since $L \cap (X + K') \leq X \cap (L + K') + K' \cap (L + X)$ and $K' \cap (L + X) = 0$, then $L \cap (X + K') \leq X \cap (L + K')$. In the same way, $X \cap (L + K') \leq L \cap (X + K')$. So $L \cap (X + K') = X \cap (L + K')$ for every submodule X of M_2. Since M_2 has $(D1)$, there exists a submodule A_1 of $M_2 \cap (L + K') = L \cap (M_2 \oplus K')$ such that $M_2 = A_1 \oplus A_2$ and $A_2 \cap (L + K')$ is small in A_2 for some submodule A_2 of M_2. Thus $M = (L' \oplus A_1) \oplus (A_2 \oplus K')$, $(L' \oplus A_1) \leq L$ and $L \cap (A_2 \oplus K') = A_2 \cap (L + K')$ is small in $A_2 \oplus K'$.

Case 2. $M_1 \cap (L + M_2) = 0$. This implies $L \leq M_2$. Since M_2 has $(D1)$, there exists a submodule B_1 of L such that $M_2 = B_1 \oplus B_2$ and $L \cap B_2$ is small in B_2 for some submodule B_2 of M_2. Hence $M = B_1 \oplus (M_1 \oplus B_2)$ and $L \cap (M_1 \oplus B_2) = L \cap B_2$ is small in $M_1 \oplus B_2$. It follows that M has $(D1)$. □

Let Rad$M$ denote the Jacobson radical of any R–module M.

Corollary 7. *Let M_1 be a semisimple module and M_2 a module such that* RadM_2 $= M_2$. *Then $M = M_1 \oplus M_2$ has $(D1)$ if and only if M_2 has $(D1)$ and M_1 and M_2 are relatively projective.*

Proof. Sufficiency is clear from Theorem 6. Conversely assume $M = M_1 \oplus M_2$ has $(D1)$. It is well-known that M_2 has $(D1)$ by [2, Lemma 4.7]. Since M_1 is semisimple, M_2 is M_1–projective. We prove that M_1 is M_2–projective. Let N be a submodule of M with $M = N + M_2$. By Proposition 4.8 of [2] there exists a submodule K of N such that $M = K + M_2 = K \oplus K'$ and $K \cap M_2$ is small in K for some submodule K' of M. It follows easily that Rad$K = K \cap M_2$. Since Rad$M$ = Rad$K \oplus$ Rad$K' = M_2$, then $K \cap M_2$ is a direct summand of K. Hence $M = K \oplus M_2$. Thus M_1 is M_2– projective by Lemma 5. □

Theorem 8. *Let the module $M = M_1 \oplus M_2$ be a direct sum of relatively projective modules M_1, M_2 such that M_1 and M_2 are quasi–discrete modules. Then M has $(D1)$.*

Proof. Let L be a non-zero submodule of M.

Case 1. $M_1 \cap (L + M_2) \neq 0$. Since M_1 has $(D1)$, there exists a submodule A_1 of $M_1 \cap (L + M_2)$ such that $M_1 = A_1 \oplus A_2$ and $A_2 \cap (L + M_2)$ is small in A_2 for some submodule A_2 of M_1. Then $M = L + (A_2 \oplus M_2)$. If $M_2 \cap (L + A_2) = 0$ then by [2, Lemma 4.7], $A_2 = C_1 \oplus C_2$ and $L \cap C_2$ is small in C_2 for some submodules C_1 and C_2 in A_2 with $C_1 \leq (L \cap A_2)$. Hence $M = L + (C_2 \oplus M_2) = (A_1 \oplus C_1) \oplus (C_2 \oplus M_2)$. Since M_1 and A_2 are quasi–discrete and M_1 is M_2–projective, then $A_1 \oplus C_1$ is $C_2 \oplus M_2$–projective from [2, Lemma 4.23, Prop. 4.31, Prop. 4.32 and Prop. 4.33]. Hence there exists a submodule L' of L such that $M = L' \oplus C_2 \oplus M_2$ by Lemma 5. Note that $L \cap (C_2 \oplus M_2) \leq C_2 \cap (L + M_2) = L \cap C_2$. Therefore $L \cap (C_2 \oplus M_2)$ is small in $C_2 \oplus M_2$, because $L \cap C_2$ is small in C_2. Assume $M_2 \cap (L + A_2) \neq 0$. Since M_2 has $(D1)$, there exists a submodule B_1 of $M_2 \cap (L+A_2)$ such that $M_2 = B_1 \oplus B_2$ and $B_2 \cap (L+A_2)$ is small in B_2 for some submodule B_2 of M_2. Then $M = L+(A_2 \oplus B_2) = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$ and $L \cap (A_2 \oplus B_2)$ is small in $A_2 \oplus B_2$ because $A_2 \cap (L+B_2)$ is small in A_2 and $B_2 \cap (L+A_2)$ is small in B_2. Since $A_1 \oplus B_1$ is $A_2 \oplus B_2$–projective, there exists a submodule L' of L such that $M = L' \oplus A_2 \oplus B_2$ by Lemma 5. This completes the proof in this case.

Case 2. $M_1 \cap (L + M_2) = 0$. The proof of this case is the same as that of case 2 of Theorem 6. □

Theorem 9. *Let the module $M = M_1 \oplus M_2$ be a direct sum of relatively projective modules M_1, M_2 such that M_1 and M_2 have $(D1)$. Suppose further that M_1 and M_2 are quasi–projective modules. Then M has $(D1)$.*

Proof. By [2, Lemma 4.6 and Prop.4.38], M_1 and M_2 are quasi–discrete. Hence M has $(D1)$ by Theorem 8. □

Example 10. For any non-zero positive integer a, $\mathbb{Z}/a\mathbb{Z}$ is quasi-projective by [1, Exer. 16.14]. Let p be any prime integer. Then the $\mathbb{Z}$–module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$ does not have $(D1)$ and $\mathbb{Z}/p\mathbb{Z}$ is not $\mathbb{Z}/p^3\mathbb{Z}$–projective.

Proof. By Corollary 2 and Remark. □

Theorem 11. *Let M be a $(D1)$–module.Then the following statements are equivalent.*

(i) M has $(D3)$.

(ii) Whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_1 and M_2 are relatively projective.

Proof. $(i) \Rightarrow (ii)$. By [2, Lemma 4.23].

$(ii) \Rightarrow (i)$. By [3, 41.14. (4) ⇒ (6)]. □

Proposition 12. *Let the module $M = M_1 \oplus M_2$ be a direct sum of relatively projective modules M_1, M_2 such that M_2 is quasi-discrete. Let K, L be direct summands of M such that $M = K + L$. Suppose further that $M = K + M_2$. Then $K \cap L$ is a direct summand of M.*

Proof. Assume $M = K + M_2$. By Lemma 5, there exists a submodule K' of K such that $M = K' \oplus M_2$. Without loss of generality we may assume $K' = M_1$ so that M_1 is a submodule of K. Then $K \cap M_2$ is a direct summand of M_2. We write $M_2 = T \oplus (K \cap M_2)$

for some submodule T of M_2. By Theorem 11, $K \cap M_2$ and T are relatively projective. Note that $K = M_1 \oplus (K \cap M_2)$. By [2, Prop. 4.32 and Prop. 4.33], T is K–projective. Then by Lemma 5, $M = K \oplus L'$ for some submodule L' of L. Hence $L = L' \oplus (K \cap L)$. Thus $K \cap L$ is a direct summand of M. □

Let $M_1, \ldots, M_t$ be hollow and relatively projective modules. Then $M_1 \oplus \cdots \oplus M_t$ complements direct summands [2, Corollary 4.50]. Therefore we have the following corollary, which is also given in [2, Corollary 4.50].

Corollary 13. *Let M be a module such that $M = M_1 \oplus \cdots \oplus M_t$ is a finite direct sum of hollow modules M_i $(1 \leq i \leq t)$. Then M is quasi-discrete if and only if $M_1, \ldots, M_t$ are relatively projective.*

Proof. The necessity is clear. Conversely suppose that $M = M_1 \oplus M_2$ and M_1, M_2 are relatively projective hollow modules. Since M_1 and M_2 are quasi–discrete, M has $(D1)$ by Theorem 8. Let K and L be direct summands of M with $M = K + L$. Since M complements direct summands, either $M = K \oplus M_1$ or $M = K \oplus M_2$. Hence by Proposition 12, $K \cap L$ is a direct summand of M. Thus M has $(D3)$. The proof is completed by induction on t. □

Let I be any index set. In the next two theorems we use $M(J)$ to denote $\oplus_{j \in J} M_j$ for $J \subseteq I$ and $M(I - i)$ to denote $M(I - \{i\})$ for $i \in I$.

Theorem 14. *Let $M = \oplus_{i \in I} M_i$ be a decomposition that complements direct summands. Then M is quasi-discrete if and only if*

(i) $M(I - i)$ is quasi-discrete for every $i \in I$, and

(ii) M_i and $M(I - i)$ are relatively projective for every $i \in I$.

Proof. The necessity follows by Theorem 11 and [2, Lemma 4.7]. Conversely assume the conditions hold. Since $M = M_i \oplus M(I - i)$, by Theorem 8, M has $(D1)$. We prove that M has $(D3)$. Let A and B be submodules of M such that $M = A \oplus B$. Then $M = A \oplus M(J)$ for some subset J of I. It follows by (ii) and [2, Prop. 4.31 and Prop. 4.32] that A and B are relatively projective. By Theorem 11, M has $(D3)$. Hence M is quasi-discrete. □

Note that Corollary 13 may also be obtained using Theorem 14.

Theorem 15. *Let $M = \oplus_{i \in I} M_i$ be a decomposition that complements direct summands.Then M is discrete if and only if*

(i) $M(I - i)$ is discrete for every $i \in I$, and

(ii) M_i *and* $M(I-i)$ *are relatively projective for every* $i \in I$.

Proof. The necessity follows by Theorem 14 and [2, Lemma 4.7]. Conversely suppose that (i) and (ii) hold for M. Then by Theorem 14, M is quasi-discrete, and since M_i $(i \in I)$ is discrete, then by [2, Theorem 4.15], M_i $(i \in I)$ is a direct sum of hollow modules and each hollow summand of M_i $(i \in I)$ is discrete. Thus M is a direct sum of hollow modules each of which is discrete. By Theorem 5.2 of [2], M is discrete. □

Corollary 16. *Let* M *be a module such that* $M = M_1 \oplus \cdots \oplus M_t$ *is a finite direct sum of hollow modules* M_i, $(1 \leq i \leq t)$. *Then* M *is discrete if and only if* $M_1, \ldots, M_t$ *are relatively projective discrete modules.*

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References

[1] Anderson, F. W. and Fuller, K. R. *Rings and categories of modules* (Springer–Verlag, 1974).

[2] Mohamed, S. H. and Muller, B. J. *Continuous and discrete modules*, London Math. Soc. LNS 147 (Cambridge Univ. Press, Cambridge, 1990).

[3] Wisbauer, R. *Foundations of module and ring theory* (Gordon and Breach, Philadelphia, 1991).

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