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CROSSED ℕ-CUBES AND n-CROSSED COMPLEXES OF COMMUTITIVE ALGEBRAS

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Abstract

In this paper we will define crossed ℕ-cubes and n-crossed complexes of commutative algebras and construct a functor from the category of simplicial algebras to that of n-crossed complexes.

Introduction

The definition of a crossed complex was first introduced under the name 'group system' by Blaker and systematically used by Whitehead, [15]. More recently Brown and Higgins have studied over a groupoid (cf. [7]). Crossed complexes give useful information on the homotopy type, but crossed n-cubes (of groups) defined by [13] give complete information up to dimension n. Crossed n-complex and crossed ℕ-cubes were introduced by the author, A.Mutlu and T.Porter in [6].

It is obvious that one should be able to develop an analogous and theory of n-crossed complexes and Crossed ℕ-cubes for other algebraic structures such as Lie Algebras or Commutative Algebras. In this article we have chosen to work with commutative algebras. Many of the results in here are analogous of known group theoretic results. We will show how the $C_{\alpha,\beta}$ maps which are defined in [4] fit in proofs of some results.

1. Preliminaries

Let $\mathbf{k}$ be a fixed commutative ring with $1 \neq 0$. All of the $\mathbf{k}$-algebras discussed herein are assumed to be commutative and associative but we will want to consider ideals and modules to be algebras and so will not be requiring *algebras* to have unit elements. The category of commutative algebras will be denoted by **Alg**.

1.1. Simplicial Algebras

A simplicial (commutative) algebra $\mathbf{E}$ consists of a family of algebras $\{E_n\}$ together with face and degeneracy maps $d_i = d_i^n : E_n \to E_{n-1}, \quad 0 \leq i \leq n, \quad (n \neq 0)$ and $s_i = s_i^n : E_n \to E_{n+1}, \quad 0 \leq i \leq n$, satisfying the usual simplicial identities given in André

[1]. It can be completely described as a functor $\mathbf{E}$: $\Delta^{op} \to \mathbf{CommAlg}_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < \cdots < n\}$ and increasing maps.

The Moore complex and the homotopy module of a simplicial algebra

Recall that given a simplicial algebra $\mathbf{E}$, *the Moore complex* $(\mathbf{NE}, \partial)$ *of* $\mathbf{E}$ is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(\mathbf{E})$ of $\mathbf{E}$ is the nth homology of the Moore complex of $\mathbf{E}$, i.e.,

$$\begin{aligned} \pi_n(\mathbf{E}) &\cong H_n(\mathbf{NE}, \partial) \\ &= \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}\Big(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}\Big). \end{aligned}$$

1.2. Crossed modules and crossed complexes

Whitehead (1949) [15] used crossed modules in various contexts especially in his investigations into the algebraic structure of relative homotopy groups. In this section, we recall the definition and elementary theory of crossed modules of commutative algebras given by T.Porter [14].

Throughout this paper we denote an action of $r \in R$ on $m \in M$ by $r \cdot m$.

Let R be a **k**-algebra with identity. *A pre-crossed module of commutative algebras* is an R-algebra C, together with an R-algebra morphism

$$\partial : C \longrightarrow R,$$

such that for all $c \in C, r \in R$

$CM1)$ $\quad \partial(r \cdot c) = r\partial c.$

This is *a crossed module* if in addition, for all $c, c' \in C$,

$CM2)$ $\quad \partial c \cdot c' = cc'.$

This second condition is called *the Peiffer identity.* We denote such a crossed module by (C, R, ∂). Clearly any crossed module is a pre-crossed module.

A standart example of a crossed module is any ideal I in R giving an inclusion map the image $I = \partial C$ of C is an ideal in R.

A morphism of crossed modules from (C, R, ∂) to (C', R', ∂') is a pair of **k**-algebra morphisms,

$$\theta : C \longrightarrow C', \quad \psi : R \longrightarrow R',$$

such that

$$\theta(r \cdot c) = \psi(r) \cdot \theta(c) \quad \text{and} \quad \partial'\theta(c) = \psi\partial(c).$$

In this case, we will say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity.

A crossed complex of $\mathbf{k}$*-algebras* is a sequence of $\mathbf{k}$-algebras

$$\mathcal{C}: \qquad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

i) (C_1, C_0, ∂_1) is a crossed module,

ii) for $i > 1$, C_i is an C_0-module on which $\partial_1 C_1$ operates trivially and each ∂_i is an C_0-module morphism,

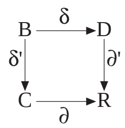
iii) for $i \geq 1, \partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way.

2. Crossed Complexes as crossed N-cubes

The following definition is due to Ellis [12]. More details can also be found in [2].

A crossed square of commutative algebras is a commutative diagram of commutative algebras.



together with an action of R on B, C and D (there are thus actions of C on B and D via ∂, and of D on B and C via ∂') and a function $h : C \times D \to B$ such that, for all $c, c' \in C,\ d, d' \in D,\ r \in R,\ b \in B,\ k \in \mathbf{k}$;

1. each of the maps $\delta, \delta', \partial, \partial'$ and the composite $\partial'\delta = \partial\delta'$ are crossed modules
2. the maps δ, δ' preserve the action of R
3. $kh(c, d) = h(kc, d) = h(c, kd)$
4. $h(c + c', d) = h(c, d) + h(c', d)$
5. $h(c, d + d') = h(c, d) + h(c, d')$
6. $r \cdot h(c, d) = h(r \cdot c, d) = h(c, r \cdot d)$
7. $\delta h(c, d) = c \cdot d$
8. $\delta' h(c, d) = -d \cdot c$
9. $h(c, \delta b) = c \cdot b$

10. $h(\delta' b, d) = -d \cdot b$.

Example 1. Suppose

$$\begin{array}{ccc} C_2 & \xrightarrow{\partial_2} & C_1 \\ \downarrow & & \downarrow \partial_1 \\ 0 & \longrightarrow & C_0 \end{array}$$

is a crossed square, then it is easily shown that $\partial_1\partial_2 = 0$. Indeed, $h : 0 \times C_1 \to C_2$ and from axiom 7, we have $\partial_2 h(0, c_1) = 0c_1 = 0$, this implies

$$\partial_2(C_2) = 0 \ \Rightarrow \partial_1(\partial_2(c_2)) = \partial_1(0) = 0.$$

The only other non-trivial h-map is that giving the action of C_0 on C_2. It is easy to check that

$$C_2 \ \overrightarrow{\partial_2} \ C_1 \ \overrightarrow{\partial_1} \ C_0$$

is a truncated crossed complex.

This example enable us to define an infinite dimensional crossed *or* crossed **N**-cubes. Thus *all* crossed complexes will be special cases of crossed **N**-cubes.

We consider the set $\mathbf{N} = \{1, 2, \ldots\}$ of positive natural numbers with its usual ordering. For any $m, \langle m \rangle = \{1, 2, \ldots, m\}$ both as a subset of **N** and, in discussion of crossed m-cubes, as a set in its own right. We say a subset $B \subseteq \mathbf{N}$ is a *down segment* if $B = \{k : k \leq m\}$ for some m and of dourse B is then equal to $\langle m \rangle$. The order on **N** is important but on the subsets $\langle m \rangle$ tends to play less of a role.

Definition 2.1. A crossed **N**-cube of commutative algebras *is a family of commutative algebras,*

$$\{M_A : A \subseteq \mathbf{N}, \ A \text{ finite}\}$$

together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ *for* $i \in \mathbf{N}$ *and for* $A, B \subseteq \mathbf{N}$*, functions*

$$h : M_A \times M_B \longrightarrow M_{A\cup B}$$

such that for all $k \in \mathbf{k},\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in \mathbf{N}$ *and* $A \subseteq B$

$$\begin{array}{rrcl}
1) & \mu_i a & = & a \quad \text{if } i \notin A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a,b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a,b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a,a') & = & aa' \\
6) & h(a,b) & = & h(b,a) \\
7) & h(a+a',b) & = & h(a,b) + h(a',b) \\
8) & h(a,b+b') & = & h(a,b) + h(a,b') \\
9) & k \cdot h(a,b) & = & h(k \cdot a, b) = h(a, k \cdot b) \\
10) & h(h(a,b),c) & = & h(a, h(b,c)) = h(b, h(b,c)).
\end{array}$$

A morphism of crossed **N**-cubes *is defined in the obvious way: It is a family of commutative algebra homomorphisms, for* $A \subseteq \mathbf{N}$ $f_A : M_A \longrightarrow M'_A$ *commuting with the* μ_i*'s and* h*'s.*

We thus obtain a category of crossed **N**-cubes denoted by $\mathbf{Crs^N}$.

If the subsets A are restricted to be subsets of $< n >= \{1, \ldots, n\}$, then we have the corresponding definition of a crossed n-cube of algebras due to Ellis [12]. This suggests giving the first example of a crossed **N**-cube.

Example 2. Any crossed n-cube determines a crossed **N**-cube satisfying $M_A = 0$ unless $A \subseteq < n >$.

A neat example is the following:

A 1-crossed complex is a crossed **N**-cube verifying $M_A = 0$ if A is not of the form $< n >$ for some n, where $< 0 >= \emptyset$ by convention.

Proposition 2.2. *Let* $\mathcal{M}$ *be a 1-crossed complex and write* $C_n = M_{<n>}, \partial_n : C_n \to C_{n-1}$ *for* μ_n*. Then* $(\mathbf{C}, \partial)$ *is a crossed complex.*

Conversely if we are given any crossed complex $(\mathbf{C}, \partial)$*, then the crossed* **N***-cube defined by* $M_{<n>} = C_n, \mu_n = \partial_n$ *with* $n \in \mathbf{N}$ *and* $M_A = 0$ *otherwise is a 1-crossed complex in which the action of* C_0 *on* C_n*, for the various* n*, give the only non trivial* h*-maps.*

Proof. i) Take the h-map

$$\begin{array}{rccl}
h & M_\emptyset \times M_{<n>} & \longrightarrow & M_{<n>} \\
 & (a,b) & \longmapsto & h(a,b) = a \cdot b
\end{array}$$

for $a \in M_{\emptyset} = R$ & $b \in C_n = M_{<n>}$. As

$$\begin{aligned} \partial_n(a \cdot b) &= \partial_n(h(a,b)); \\ &= h(\partial_n a, \partial_n b) \quad \text{by axiom 3;} \\ &= h(a, \partial_n b) \quad \text{by } \partial_n a = a \text{ (as axiom 1);} \\ &= a \cdot \partial_n(b), \end{aligned}$$

$\partial_1 : C_1 \to C_2$ is a crossed module.

ii) If $a \in M_{<1>},\ b \in M_{<n>}$, $n \geq 2$, then

$$\begin{aligned} \partial_1 a \cdot b &= h(\partial_1 a, b) \\ &= h(\mu_1 a, b) \\ &= h(a, \mu_1 b) \\ &= h(a, 0) = 0 \quad \text{by } 1 \in <1> \cap <n> \end{aligned}$$

So $\partial_1 C_1$ acts trivially on C_n for $n \geq 2$.

Conversely we will determine maps h. be. Suppose one of A, B is not of the form $< n >$ for some n. Then $h : M_A \times M_B \to M_{A \cup B}$ must be trivial as $M_A = 0$. If $A =< m >, B =< n >$ and $1 \leq m \leq n$, then any suitable : $M_A \times M_B \to M_{A \cup B}$ satisfy $h(a,b) = h(\mu_1 a, b) = h(0,b) = 0$. This leaves us only to see what

$$h : C_0 \times C_n \to C_n \quad \& \quad h : C_1 \times C_n \to C_n$$

should be. If $a \in C_0$, $b \in C_n$, then we take $h(a,b) = ab$, the multiplication of a and b. If $a \in C_1,\ b \in C_n$, then $1 \in < n > \cap < 1 >$and thus

$$\begin{aligned} h(a,b) &= h(\mu_1 a, b) \\ &= \mu_1 a \cdot b. \end{aligned}$$

If $n = 0$ or 1, these giving equation can be easily defined as $\partial_1 : C_1 \to C_0$ is a crossed module. As for all informations given above, there is only left to check the ten axioms of a crossed **N**-cube. We leave it to the reader as an exercise. □

Example 3. Let R be an algebra and given a family $\{I_i : i \in < n >\}$ of ideals of R, then for $A \subseteq < n >$

$$M_A = \bigcap_{i \in A} I_i \quad \text{and} \quad M_{\emptyset} = R$$

with $\mu_i : M_A \to M_{A - \{i\}}$ defined by inclusion and

$$h : M_A \times M_B \longrightarrow M_{A \cup B}$$

given by $h(a,b) = ab$. Then
$$\{M_A : A \subseteq< n >, \mu_i, h\}$$
is a crossed n-cube, called *the inclusion crossed n-cube* given by the ideal $(n+1)$-ad of algebras $(R; I_1, \ldots, I_n)$.

The following result is due to [5].

Proposition 2.3. *Let $(E; I_1, \ldots, I_n)$ be a simplicial ideal $(n+1)$-ad of algebras and define for $A \subseteq< n >$*
$$M_A = \pi_0(\bigcap_{i\in A} I_i)$$
with homomorphism $\mu_i : M_A \to M_{A-\{i\}}$ and h-maps induced by corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous example to each level. Then
$$\{M_A :\ A \subseteq< n >,\ \mu_i, h\}$$
is a crossed n-cube.

Example 4. Let $\mathbf{E}$ be any simplicial algebra and let Dec$\mathbf{E}$ denote the décalé $\mathbf{G}$, i.e., the simplicial algebra with
$$(\mathrm{Dec}\mathbf{E})_n = E_{n+1}$$
and all the face and degeneracy maps of $\mathbf{E}$ except the last ones at each level. The functor Dec comes with a natural transformation δ :Dec $\to Id$. Iterating Dec n-times and taking the n resulting transformation δ_i :Dec^n $\to$Dec^{n-1} gives n ideals $\{\mathrm{Ker}\delta_i\}$ of Dec$^n\mathbf{E}$. In [5] we defined a functor
$$M(-,n) : \mathbf{SimpAlg} \longrightarrow \mathbf{Crs^n}$$
by considering π_0 of the resulting simplicial crossed n-cube. For details, see [2]. This gives
$$M(\mathbf{E},1) \cong (\frac{NE_1}{d_2(NE_2)} \longrightarrow E_0)$$
which is a crossed module and
$$M(\mathbf{E},2) \cong \begin{pmatrix} NE_2/\partial_3(NE_3) & \longrightarrow & \mathrm{Ker}\, d_0 \\ \downarrow & & \downarrow \\ \mathrm{Ker}\, d_1 & \longrightarrow & E_1 \end{pmatrix}$$

which is a crossed squre and so on.

3. n-Crossed Complexes

We start by recalling the result from [3].

Proposition 3.1. *Let* $\mathbf{E}$ *be a simplicial algebra, then defining*

$$C_n(\mathbf{E}) = \frac{NE_n}{(NE_n + D_n) + d_{n+1}(NE_{n+1} + D_{n+1})}$$

with $\partial_n(z) = \overline{d_n z}$ *yields a crossed complex* $C(\mathbf{E})$ *over an algebra.* □

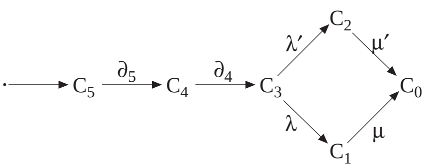
The motivation for the way that will be followed here is that the crossed complex $(C(\mathbf{E}), \partial)$ associated to a simplicial algebra E by giving the previous result has at its base the crossed module

$$\frac{NE_1}{d_2(NE_2 \cap D_2)} \longrightarrow E_0.$$

This is precisely $M(\mathbf{sk_1E}, 1)$ where $\mathbf{sk_1E}$ is the 1 skeleton of a simplicial algebra $\mathbf{E}$. To define n-crossed complexes, they have amongst them objects with $M(\mathbf{sk_nE}, n)$ at their base with a chain complex somehow attached to the 'top left hand corner' of the crossed n-cube. We therefore will make the following definition:

Definition 3.2. *An* n-crossed complex $\mathcal{M}$ *of commutative algebras is a crossed* $\mathbf{N}$*-cube in which if* $A \subseteq \mathbf{N}$ *is finite,* $M_A = 0$ *unless* $A \subseteq < n >$ *or if* $A =< m >$*, for some* m*.*

Example 5. A 2-crossed complex consists of a diagram of algebra homomorphisms



together with an action of C_0 on C_3, C_2, C_1 and C_n for $n \geq 4$, and a function $h : C_2 \times C_1 \rightarrow C_3$. The ten axioms are trivially satisfied.

Proposition 3.3. *Let* $\mathcal{M}$ *be a n-crossed complex. Then*

(i) if $m > n$, $\mu_m \mu_{m+1} = 0$,

(ii) the algebra $M_\emptyset = R$ *acts on all the* $M_{<m>}, m > n$*, in such a way that each of the ideal* $\mu_i M_{\{i\}}$ *acts trivially on* $M_{<m>}$*, if* $1 \leq i \leq n$*. No* M_A *with* $A \subseteq < n >, A = \emptyset$ *acts non trivially on* $M_{<m>}$ *for* $m > n$*.*

Proof. The proof of (i) is shown in the previous proposition

To show (ii), it is clear that the h-map

$$\begin{array}{rccc} h: & M_{\emptyset}\times M_{<m>} & \longrightarrow & M_{<m>} \\ & (a,b) & \longmapsto & a\cdot b \end{array}$$

can be defined as before. On the other hand, if $\mu_i a' = a$, then

$$\begin{aligned} a\cdot b &= h(\mu_i a', b) \\ &= h(a', \mu_i b) \\ &= 0 \end{aligned}$$

as $\mu_i : M_{<m>} \to 0$. Likewise if $a \in M_A,\ b \in M_{<m>}$ and $i \in A \subseteq < n >$, then again $\mu_i b = 0$ implies $h(a,b) = h(a, \mu_i b) = 0$. □

Remark. Consider

$$M_{<0>}/\sum_{i=1}^{n} \mathrm{Im}\mu_i = Q(\mathcal{M}).$$

which is a sort of 'total quotient' of a crossed n-cube. It acts as $\pi_0(\mathcal{M})$. The above proposition shows that

$$\to M_{<m>} \to M_{<m-1>} \to \cdots \to M_{<n+1>}$$

is a chain complex of $Q(\mathcal{M}')$-modules, where $\mathcal{M}'$ is the bottom crossed n-cubes of $\mathcal{M}$. It should also be clear that the category n-**CrsComp** of n-crossed complexes is a full subcategory of $\mathbf{Crs}^{\mathbf{N}}$, the category of crossed **N**-cubes in such a way that if $m > n$,

$$n - \mathbf{CrsComp} \subset m - \mathbf{CrsComp} \subset \mathbf{Crs}^{\mathbf{N}}.$$

Each subcategory is determined merely by specifying that certain position of a **N**-cube are trivial. In each case we have a variety in the category of crossed **N**-cubes as this latter category is a category of algebras for a many sorted theory. This implies in particularly that the inclusion

$$n\text{-}\mathbf{CrsComp} \hookrightarrow (n+1)\text{-}\mathbf{CrsComp}$$

should have a left adjoint L_n^{n+1}. We now give this left adjoint in the following.

4. The Construction of the Functor $\mathbf{L}_n^{n+1}$from (n+1)-CrsComp to n-CrsComp

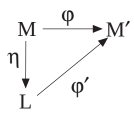
Let $\mathcal{M} = (M_A)$ be an $n+1$-crossed complex and will write $L_n^{n+1}(\mathcal{M}) = \mathcal{L} = (L_A)$ for the corresponding n-crossed complex. The form of the two structures makes it clear how to define L_A for most A.

- If $A \not\subseteq < n >$ and for any $m < n \in \mathbf{N}$, A is not in $< m >$, $L_A = 0$,
- If $A =< m >, m > n+1, L_A = M_A$,
- If $A \subset < n >, A \neq < n >$, then $L_A = M_A/\mathrm{Im}\mu_{n+1}$,
- For $< n >$, let $I_{<n>}$ be the ideal of $M_{<n>}$ generated by all the elements $\mu_{n+1}h(a,b)$ where $\{A,B\}$ forms a non-trivial partition of $< n+1 >, B \neq \emptyset, n+1 \in A$, then $L_{<n>} = M_{<n>}/I_{<n>}$.
- For $< n+1 >$ let $I_{<n+1>}$ be the ideal of $M_{<n+1>}$ generated by all elements, $h(a,b),\ a \in M_A,\ b \in M_B$ where $\{A,B\}$ is as before (above), then

$$L_{<n+1>} = M_{<n+1>}/I_{<n+1>}.$$

It is now obvious to check that definitions of μ_i and h maps give $\mathcal{L}$ the structure of an n-crossed complex.

Proposition 4.1. *The structure $\mathcal{L}$, an n-crossed complex has the following universal strcture:*



*with $\varphi'\eta = \varphi$, where the natural quotient $\eta : \mathcal{M} \to \mathcal{L}$ a map of $(n+1)$-crossed complexes and $\mathcal{M}'$ is in n-**CrsComp**. Thus L_n^{n+1} is left adjoint to the inclusion of n-**CrsComp** into $(n+1)$-**CrsComp**.*

Remark. If we take the $(n+1)$-cube determined by $\{L_A : A \subseteq < n+1 >\}$, then the cokernel of $\mu_{n+1}$is exactly the same as that for $\{M_A : A \subseteq < n+1 >\}$. In fact the only case that is not immediate is that of the cokernel of

$$\mu_{n+1} : L_{<n+1>} \longrightarrow L_{<n>}.$$

This is induced by $\mu_{n+1}$of M on the quotients

$$\frac{M_{<n+1>}}{I_{<n+1>}} \longrightarrow \frac{M_{<n>}}{I_{<n>}}$$

but $\mu_{n+1}I_{<n+1>} = I_{<n>}$, so the quotient is $M_{<n>}/\mathrm{Im}\mu_{n+1}$as promised.

The remark will be significant to form functors from simplicial algebras to that of n-crossed complexes.

We omit the proof of the above proposition which can be obtained by changing slightly the corresponding result in [11].

5. From Simplicial Algebras to n-Crossed Complexes

We are expecting n-crossed complexes look like $C(\mathbf{E}_m)$ for $m > n$ and $M(\mathbf{sk_n E}, n)$ at the base, i.e. for the bottom crossed n-cube. To form these structures, we look at a family of algebras

$$\{M_A : A \text{ finite, } A \subseteq \mathbf{N}\}$$

together with:

- if $A \supset < n >$, then $M_A = 0$,
- if $A =< m >$ with $m > n$, $M_A = C(\mathbf{E})_m$,
- if $A \subseteq < n >$, then $M_A = M(\mathbf{sk_n E}, n)_A$.

Here we recall the formula for $M(E, n)_A$ which is defined in [5]. In the following the $C_{\alpha,\beta}$ maps given in [4].

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various σ_i^n by composition. The composition of these generating maps is subject to the following rule $\sigma_j \sigma_i = \sigma_{i-1}\sigma_j,\ j < i$. This implies that every element $\sigma \in S(n, n-r)$ has a unique expression as $\sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \ldots \circ \sigma_{i_r}$ with $0 \leq i_1 < i_2 < \ldots < i_r \leq n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \sigma(i) = \sigma(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) \ : \ 0 \leq i_1 < i_2 < \ldots < i_r \leq n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

Let $P(n)$ be a set consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$, where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, ..., j_1) \in S(n)$. We write $\#\alpha = r$, i.e. the length of the string α. The **k**-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \longrightarrow NE_n : (\alpha, \beta) \in P(n), \quad n \geq 0\}$$

are given as composites $C_{\alpha,\beta} = p\mu(s_\alpha \otimes s_\beta)$ where

$$s_\alpha = s_{i_r} \ldots s_{i_1} : NE_{n-\#\alpha} \longrightarrow E_n \ , \quad s_\beta = s_{j_s} \ldots s_{j_1} : NE_{n-\#\beta} \longrightarrow E_n,$$

$p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1} \ldots p_0$, where $p_j = 1 - s_j d_j$ with $j = 0, 1, \ldots n-1$ and we denote the multiplication by $\mu : E_n \otimes E_n \to E_n$. Thus

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta) = (1 - s_{n-1}d_{n-1}) \ldots (1 - s_0 d_0)(s_\alpha(x_\alpha) s_\beta(y_\beta)).$$

If $A \subseteq <n>$, then

$$M(\mathbf{sk_n E}, n)_A = \frac{\bigcap\limits_{i\in A} \mathrm{Ker} d_i^n}{d_{n+1}(\mathrm{Ker} d_0 \cap \bigcap\limits_{i\in A} \mathrm{Ker} d_{i+1}^{n+1} \cap D_{n+1})}$$

We need to define μ_i and h-maps relative to these algebras. We note that

$$M_{<n+1>} = \frac{NE_{n+1}}{(NE_{n+1} \cap E_{n+1}) + d_{n+2}(NE_{n+2} \cap E_{n+2})}$$

and

$$M_{<n>} = \frac{NE_n}{d_{n+1}(NE_{n+1} \cap D_{n+1})},$$

so defining μ_{n+1} to be induced by d_{n+1} is a reasonable choice. For $m > n+1$,

$$\mu_m : M_{<m>} \longrightarrow M_{<m-1>}$$

is the crossed complex part of $\mathcal{M}$, i.e. the boundary induced by d_m have thus stopped with the μ_i.

We have specified the h-maps within the bottom crossed n-cube (cf. [2]).

Suppose if $A = <m>$ and $B = <l>$ with say $l > m > n$ then for any $i \in <m>$ for $\mathcal{M}$ to be a crossed n-complex we must have for $a \in M_{<l>}$ and $b \in M_{<m>}$,

$$h(a,b) = h(\mu_i a, b) = h(0,b) = 0.$$

Likewise if $A = <m>,\ m > n$ and $B \subseteq <n>$ but $B \neq \emptyset$ then there is some $i \in B$ and again for $a \in M_A,\ b \in M_B$, this implies that

$$\begin{aligned} h(a,b) &= h(\mu_i a, b) \\ &= h(a, \mu_i b) \\ &= 0 \qquad \text{as } M_B = 0 \text{ for } B \subseteq <n>. \end{aligned}$$

Thus the following h-maps are needed to specified

$$h : M_\emptyset \times M_{<m>} \longrightarrow M_{<m>}.$$

These as in a crossed complex are defined by using the $C_{\alpha,\beta}$ maps, i.e., for $x \in NE_{n-1}$ and $y \in NE_n$ by taking $\beta = (m, m-1, \ldots, n), \alpha = (n-1)$, it follows that

$$C_{(n-1)(m,m-1,\ldots,n)}(x \otimes y) = s_n \ldots s_m(x)[\sum_{k=0}^{m-n} (-1)^k s_{n-1+k}(y)].$$

So

$$\begin{aligned} d_{m+1}C_{\alpha,\beta}(x\otimes y) &= s_n \dots s_{m-1}(x)[\sum_{k=0}^{m-n}(-1)^k s_{n-1+k}(y)+y] \\ &= s_n^{(m-n)}(x)[\sum_{k=0}^{m-n}(-1)^k s_{n-1+k}d_m(y)+y] \end{aligned}$$

and this shows us

$$s_n^{(m-n)}(x)y \in [(NE_m \cap D_m)+d_{m+1}(NE_{m+1}\cap D_{m+1})]$$

that implies the actions of NE_n and NE_m defined by a multiplication

$$x \cdot y = s_n^{(m-n)}(x)y$$

via degeneracies, are trivial if $n \geq 1$. Thus if

$$a = x + \partial(NE_1) \qquad \& \quad b = y + [(NE_m \cap D_m) + d_{m+1}(NE_{m+1}\cap D_{m+1})],$$

then

$$h(a,b) = s_n^{(m-n)}(x)y + [(NE_m \cap D_m) + d_{m+1}(NE_{m+1}\cap D_{m+1})].$$

The h-map is well defined, as $C_{\alpha,\beta}$ is. Of course $h : M_{<m>} \times M_{\emptyset} \to M_{<m>}$ is then given by axiom 6.

Now we have specified the μ and h-maps for $\mathcal{M}$, it remains only to check the ten axioms. This is not too long since they are all satisfied within $M(\mathbf{sk_n E}, n)$ and most of the h-maps outside that n-cube are trivial. This leaves us some special cases still to check. The axioms involving, μ's alone trivially checked. They are true by our specification of these homomorphisms.

Given the link both with crossed complex $C(\mathbf{E})$ of $\mathbf{E}$ and the crossed n-cube construction $M(\mathbf{E}, n)$, it is appropriate to allocate the notation $C(\mathbf{E}, n)$ to these n-crossed associated to E. Of course $C(\mathbf{E}, 1)$ and $C(\mathbf{E})$ are identical.

Another useful point to note is that considering

$$\mu_{n+1} : C(\mathbf{E})_{n+1} \longrightarrow \frac{NE_n}{d_{n+1}(NE_{n+1}\cap D_{n+1})}$$

as part of a crossed $(n+1)$-cube, we can form the quotient crossed n-cube and we find it is $M(\mathbf{E}, n)$. In fact it is only at this 'highest' corner that $M(\mathbf{E}, n)$ and $M(\mathbf{sk_n E}, n)$ different, since all other comes have the form $\cap \mathrm{Ker} d_i^{n-1}$ up to isomorphism. The image of μ_{n+1} is of course $d_{n+1}(NE_{n+1})$ so the quotient of the above map is $M(\mathbf{E}, n)_{<n>}$. As all the other images of μ_{n+1} is trivial this proves the claimed result.

This shows that $C(\mathbf{E},n)$ contains the information on the n-type of $\mathbf{E}$. We still have to check that it contains the information on $C(\mathbf{E},1)$, i.e., on the crossed complex of a simplicial algebra $\mathbf{E}$.

6. From n-Crossed Complexes to (n-1)-Crossed Complexes

We now have a functor

$$C(-,n) : \mathbf{SimpAlg} \longrightarrow \mathbf{n\text{-}CrsComp}$$

for each n and also functors

$$L^n_{n-1} : \mathbf{n\text{-}CrsComp} \longrightarrow \mathbf{(n-1)\text{-}CrsComp}.$$

We have expressed the hope that $C(\mathbf{E},n)$ contains not only the information on $M(\mathbf{E},n)$ and thus on the n-type of $\mathbf{E}$ but also on $C(\mathbf{E})$ the crossed complex associated to $\mathbf{E}$ earlier (see also [10]). One way in which this can happen is if

$$L^n_{n-1}C(\mathbf{E},n) \cong C(\mathbf{E},n-1)$$

as then repeated use of the L^k_{k-1} functor will get form $C(\mathbf{E},n)$ to $C(\mathbf{E},1)$ i.e. to $C(\mathbf{E})$. It is this isomorphism that we set out to prove in this section. Before proving this isomorphism, we need to recall the following lemma due to the first author (cf. [2]).

Lemma 6.1. If, for $n \geq 2$ $x \in NE_{n-1}$ and $y \in NE_n$, then

$$s_{n-1}(x)y \in [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

Proposition 6.2. *There is a natural isomorphism*

$$L^n_{n-1}C(\mathbf{E},n) \cong C(\mathbf{E},n-1).$$

Proof. We begin by analyzing the $< n >$-position. In this position the result is equivalent to checking that

$$I_{<n>} \cong \frac{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}{d_{n+1}(NE_{n+1} \cap D_{n+1})}$$

where $I_{<n>}$ is the ideal generated by all $h(a,b), a \in M_{\ A}, b \in M_B$ for $A \cup B =< n >$ $A \cap B = \emptyset, B \neq \emptyset$ and $n \in A$. (As $h(a,b) = h(b,a)$, we could equally well required that $A \neq \emptyset$ and $n \in B$).

Each such $h(a,b)$ in this case is given by a 'multiplication coset' i.e. if $a = \overline{x},\ b = \overline{y}$, the cosets represented by x and y respectively then $h(a,b) = \overline{xy}$.

Take

$$x \in \bigcap_{i\in A} Kerd_i \qquad \text{and} \qquad y \in \bigcap_{j\in B} Kerd_j$$

and suppose $1 \in B$ and $k \in B$ but $k+1 \in A$. (If $1 \in A$ the roles of x and y in what follows must be reversed.)

Consider the $C_{\alpha,\beta}$, for $\alpha = (k-1), \beta = (k)$,

$$C_{(k-1)(k)}(x \otimes y) = (-1)^k s_k(x)[s_k(y) - s_{k-1}(x)] + \sum_{i=0}^{k+1}(-1)^i s_i(x)s_i(y).$$

On the other hand, for $\alpha = (n-2), \beta = (n-1)$,

$$C_{\alpha,\beta}(x \otimes y) = s_{n-1}(x)[s_{n-2}(y) - s_{n-1}(y)]$$

and then

$$d_{n+1}C_{\alpha,\beta}(x \otimes y) = s_{n-1}d_n(x)[s_{n-2}d_n(y) - s_{n-1}d_n(y)] = a.$$

We calculate $d_{n+1}C_{(k-1)(k)}(x \otimes y)$ and find it has to form $xy - a$ so we have

$$xy = a + d_{n+1}C_{(k-1)(k)}(x \otimes y).$$

If $0 \leq j \leq n$, one can easily see that

$$d_jC_{(k-1)(k)}(x \otimes y) = 0.$$

If $j = k$,

$$d_kC_{(k-1)(k)}(x \otimes y) = 0 \qquad \text{since } k \in B.$$

For $j = k+2$,

$$d_{k+2}C_{(k-1)(k)}(x \otimes y) = 0 \quad \text{since } k+1 \in A.$$

and also clear that a is in NE_n. Thus

$$xy \in [(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})]$$

and hence

$$I_{<n>} \subseteq \frac{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}{d_{n+1}(NE_{n+1} \cap D_{n+1})}$$

as expected.

The opposite inclusion of this requires that each element of $NE_n \cap D_n$ be written as a product of multiplication of the form xy with

$$x \in \bigcap_{i\in A} Kerd_i \qquad \text{and} \qquad y \in \bigcap_{j\in B} Kerd_j$$

modulo $d_{n+1}(NE_{n+1}\cap D_{n+1})$. The first step in this direction is given by the above lemma since this reduces the work to checking that each element of NE_nK_n can be so written, where K_n has a semi-direct decomposition for which see [2].

We know that

$$E_n \cong \mathrm{Ker}d_n \,\mathbf{o}\, s_{n-1}(E_{n-1})$$

and that $\mathrm{Ker}d_n \subset K_n$.

If $x \in NE_n,\ y \in \mathrm{Ker}d_n$ then xy is already in the right form $(A = \{1, \ldots, n-1\}, B = \{n\})$. In $s_{n-1}(E_{n-1})$, we have the subalgebra $s_{n-1}(\mathrm{Ker}d_{n-1}^{n-1})$. Suppose $x \in NE_n$ and $y = s_{n-1}(z)$ for $z \in \mathrm{Ker}d_{n-1}^{n-1}$. Then

$$s_{n-1}(z) \in \mathrm{Ker}d_n \text{ and } s_{n-1}(z) - s_{n-2}(z) \in \mathrm{Ker}d_{n-1}^{n},$$

hence $x(s_{n-1}(z) - s_{n-2}(z))$ has the right form with $A = \{1, \ldots, n-2, n\}$, $B = \{n-1\}$, however $x(s_{n-1}(z) - s_{n-2}(z))$ is a sum of $x(-s_{n-2}(z))$ and multiplication of $(xs_{n-1}(z))$, so

$$h(a, b) = (xs_{n-1}(z)) = xy$$

which is the right form.

This process can be repeated. On the k^{th} repeat, one attacks

$$s_{n-1} \ldots s_{n-k}(E_{n-k})$$

which splits as $s_{n-1} \ldots s_{n-k}(\mathrm{Ker}d_{n-k}^{n-k})$ and $s_{n-1} \ldots s_{n-k-1}(E_{n-k-1})$. Pairing an elements of the form

$$y = s_{n-1} \ldots s_{n-k}(z) \text{ for } z \in \mathrm{Ker}d_{n-k}^{n-k} \text{ with } s_{n-1} \ldots s_{n-k+1}s_{n-k-1}(z)$$

gives an element in $\mathrm{Ker}d_{n-k}$. If $x \in NE_n, x[s_{n-1} \ldots s_{n-k+1}(s_{n-k}(z) - s_{n-k-1}(z))]$ gives something in $I_{<n>}$ modulo $\partial(NE_{n+1} \cap D_{n+1})$, but $x(-s_{n-1} \ldots s_{n-k+1}s_{n-k-1}(z))$ is in $NE_n[s_{n-1+1} \ldots s_{n-k+1}(\mathrm{Ker}d_{n-k})]$ and if this has already been shown to give a subalgebra of $I_{<n>}$ then so does

$$NE_n[s_{n-1} \ldots s_{n-k}(\mathrm{Ker}d_{n-k})].$$

The induction however stops with $k = n-1$.

This leaves us just with the term $NE_n[s_{n-1}s_{n-2} \ldots s_0d_0(NE_1)]$ to handle. First we note that if $x \in NE_n$ and $y \in NE_1$, the multiplication

$$z = s_n(x)[s_n \ldots s_2s_1(y) - s_n \ldots s_2s_0(y)] \in NE_{n+1} \cap D_{n+1}$$

and that

$$d_{n+1}z = x[s_{n-1} \ldots s_2s_1(y) - s_{n-1} \ldots s_2s_0(y)].$$

As we have shown already that $x(-s_{n-1} \ldots s_1(y))$ has the right form modulo $\partial(NE_{n+1} \cap D_{n+1})$, the usual identity for multiplication of the form $x(a+b)$ now completes the proof. □

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