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AN IMPROPER INTEGRAL REPRESENTATION OF LINNIK'S PROBABILITY DENSITIES

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Abstract

A representation of Linnik's Probability Densities by a contour integral distinct than the one given in [2] is obtained. An Improper integral representation of the same density functions is derived. An investigation into the exceptional set is achieved as well.

1. Introduction

In 1953 Linnik introduced the probability densities $p_\alpha(x)$ defined in terms of its characteristic functions

$$\varphi_\alpha(t) = \frac{1}{1+|t|^\alpha}, \quad 0 < \alpha < 2,$$

that is,

$$p_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}dt}{1+|t|^\alpha}, \quad 0 < \alpha < 2.$$

In [1] and [2] the asymptotic behaviour of the density functions at 0 and ∞ was investigated and the expansions of $p_\alpha(x)$ into convergent series were obtained for almost all α's. Furthermore, it was proved that the exceptional set is subset of Liouville numbers, and one counter example was contructed to show that the exceptional set is not emty. In this work, inspired from the contour representation given in [2], another counter representation is given which leads to the representation of $p_\alpha(x)$ by an improper integral. Finally, by using Kronecher's theorem, a deeper characterization of the exceptional set is obtained.

Here we have to recall the representation of $p_\alpha(x)$ as contour integral given in [2, p. 515] that was the inspiration point of my work. For $\delta > 0, \alpha \in [\delta, 2-\delta]$,

$$xp_\alpha(x) = I_\delta(x,\alpha)$$
$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k\alpha}}{\Gamma(k\alpha)\cos(\frac{\pi}{2}k\alpha)} + \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{\Gamma(2k+1)\sin\frac{\pi}{\alpha}(2k+1)} \tag{1.1}$$

where

$$I_\delta(x,\alpha) = \frac{i}{4\alpha}\int_{L(\delta)}\frac{e^{z\log x}}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}$$

and $L(\delta)$ is the boundary of the region

$$G(\delta) = \{z : |z| > \frac{\delta}{2} \text{ and } |\arg z| < \frac{\pi}{4}\},$$

described in the positive direction.

2. Representation of Linnik's Probability Densities by an Improper Integral

In this section we shall represent $p_\alpha(x)$ by an improper integral. This representation will shed a light to the probability densities $p_\alpha(x)$ for irrational α's and also to the expectional set described by Theorem 9.5 in [2].

Consider $\delta > 0$ such that $\alpha \in [\delta, 2-\delta]$ and the integral

$$J_\delta(x;a) = \frac{i}{4\alpha}\int_{\wedge(\delta)}\frac{e^{z\log x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}, \quad x > 0 \tag{2.1}$$

where $\wedge(\delta)$ is the boundary of the region

$$D(\delta) = \{z : |z| > \frac{\delta}{2}; \quad |\arg z| < \frac{\pi}{2}\}$$

The transition on the boundary is the usual positive direction.

Theorem 2.1. *For any $\alpha \in (0,2)$ we have the following representation:*

$$xp_\alpha(x) = J_\delta(x;\alpha), \quad x > 0 \tag{2.2}$$

Here δ is such that $\alpha \in [\delta, 2-\delta]$.

To prove the theorem we need the following lemma.

Lemma 2.2. *For any fixed δ and M, $0 < \delta < \frac{1}{2}, 1 < M < \infty$, the integral $J_\delta(x;\alpha)$ converges absolutely and uniformly with respect to the both $\alpha \in [\delta, 2-\delta]$ and $x \in [0, M]$*

Proof. We know that

$$\left|\sin\left(\frac{\pi}{\alpha}z\right)\right| \geq \sinh\left(\frac{\pi}{\alpha}|Imz|\right), \quad \left|\cos\left(\frac{\pi}{2}z\right) \geq \sinh\left(\frac{\pi}{2}|Imz|\right)\right.$$

and on the rays $\{z : |z| > \frac{\delta}{2}$ and $\arg z = \pm\frac{\pi}{2}\}$ we have $|Imz| \geq \frac{\delta}{2}$. Therefore both functions $\sin(\frac{\pi}{\alpha}z), \cos(\frac{\pi}{2}z)$ are bounded from below on the contour $\wedge(\delta)$ by a positive constant, say C, not depending on $\alpha \in [\delta, 2-\delta]$.

Using the asymptotic expansion of $\log\Gamma(z)$ when $|\arg z| < \frac{\pi}{2}$ [3, p. 252]

$$\left|\log\Gamma(z) - \left(z - \frac{1}{z}\right)\log z + z - \frac{1}{2}\log 2\pi\right| \approx \left|\sum_{r=1}^{\infty} \frac{(-1)^{r-1}B_r}{2r(2r-1)z^{2r-1}}\right|$$

where B_r's are Bernoulli numbers, we obtain

$\log|\Gamma(z)| = (Rez)\log|z| + O(|z|)$ as $z \to \infty$ and $z \in D(\delta)$. Then the same bounds used in Lemma 13.1 of [2] can be used here, that is, there are positive constants ϵ and B such that

$$|\Gamma(z)| \geq Be^{\epsilon|z|\log|z|}, \quad z \in D(\delta).$$

If $x \in [0, M]$, then integrand in (2.1) can be estimated as follows:

$$\left|\frac{e^{z\log x}}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}\right| \leq \frac{\exp(Rez)\log x}{C^2Be^{\epsilon|z|\log|z|}} \leq \frac{\exp(Rez)\log M}{C^2Be^{\epsilon|z|\log|z|}} \tag{2.3}$$

$z \in \wedge(\delta)$. This completes the proof of Lemma. □

Proof of Theorem 2.1. First we prove the theorem for $\alpha = \frac{2\ell}{m}$. Since these rational numbers are dense in $\mathbb{R}$ and since $p_\alpha(x)$ is continuous with respect to α [1], then the Theorem will follow for all $\alpha \in (0, 2)$. To Evaluate the integral $J_\delta(x;\alpha)$, by use of the Cauchy Residue theorem, we choose our contour of integration to be the boundary of the region

$$D(\delta) \cap \{z|\ |z| < X_s\} \quad \text{denoted by} \quad \ell(\delta, X_s).$$

Here the radius of the circle $|z| = X_s$ choosen exactly like in [2 p. 514], that is, it contains none of the zeros of $\sin(\frac{\pi}{\alpha}z)$ and $\cos(\frac{\pi}{2}z)$.

Now, we consider the integral

$$J_\delta(x;a,X_s)=\frac{i}{4\alpha}\oint_{\ell(\delta,X_s)}\frac{e^{z\log^x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}$$

$$=\frac{i}{4\alpha}2\pi i\left\{\sum_{\alpha\leq k\alpha<X_s}\frac{(-1)^k x^{k\alpha}}{\Gamma(k\alpha)\cos(\frac{\pi}{2}k\alpha)}+\frac{2}{\pi}\sum_{1\leq 2k+1<X_s}\frac{(-1)^k X^{2k+1}}{\Gamma(2k+1)\sin\frac{\pi}{\alpha}(2k+1)}\right\}$$

$$=\frac{1}{2}\left\{\sum_{\alpha\leq k\alpha<X_s}\frac{(-1)^{k+1} x^{k\alpha}}{\Gamma(k\alpha)\cos(\frac{\pi}{2}k\alpha)}+\frac{1}{\alpha}\sum_{1\leq 2k+1<X_s}\frac{(-1)^k X^{2k+1}}{\Gamma(2k+1)\sin\frac{\pi}{\alpha}(2k+1)}\right\} \tag{2.4}$$

On the other hand, let $\wedge(\delta;X_s)=\gamma_1+\gamma_2+\gamma_3$ where $\gamma_1=\{z:z=ir,\ \frac{\delta}{2}\leq r\leq X_s\}$, $\gamma_2=\{z:|z|=\frac{\delta}{2},\ |\arg z|\leq\frac{\pi}{2}\}$, $\gamma_3=\{z:z=-ir,\frac{\delta}{2}\leq r\leq X_s\}$ and $C(X_s)=\{|z|=X_s;|\arg z|<\frac{\pi}{2}\}$

Then we have

$$J_\delta(x;\alpha,X_s)=\frac{i}{4\alpha}\left\{\int_{\wedge(\delta;X_s)}+\int_{C(X_s)}\right\}\frac{e^{z\log x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}. \tag{2.5}$$

Using the inequality (2.3), we have

$$\left|\frac{e^{z\log x}}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{z}z)}\right|\leq\frac{e^{X_s\log x}}{C^2Be^{\epsilon X_s\log X_s}}$$

and hence the integral along $C(X_s)$ tends to zero as $|z|\to\infty$.

Therefore

$$\lim_{|z|\to\infty}J_\delta(x>\alpha,X_s)=J_\delta(x,\alpha). \tag{2.6}$$

So, we obtain

$$J_\delta(x,\alpha)=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k\alpha}}{\Gamma(k\alpha)\cos(\frac{\pi}{2}k\alpha)}+\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{\Gamma(2k+1)\sin\frac{\pi}{\alpha}(2k+1)}, \tag{2.7}$$

and by (1.1) is follows that

$$J_\delta(x,\alpha)=xp_\alpha(x)\ ,\ x>0$$

□

Theorem 2.3. *For any $\alpha \in (0,2)$ we have the formula:*

$$xp_\alpha(x) = \frac{-i}{4\alpha}\int_{-\infty}^{\infty}\frac{e^{ir\log x}}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh(\frac{\pi}{2}r)}, \quad x>0$$

and the improper integral given above is absolutely convergent for all $\alpha \in (0,2)$.

Proof. In (2.1) we replace $\wedge(\delta)$ by its components: Let $\gamma_1' = \{z : |z| \geq \frac{\delta}{2}, \arg z = \frac{\pi}{2}\}, \gamma_2 = \{z : |z| = \frac{\delta}{2}, |\arg z| < \frac{\pi}{2}\}, \gamma_3' = \{z : |z| \geq \frac{\delta}{2}, \arg z = -\frac{\pi}{2}\}$. Then we have:

$$\begin{aligned}\int_{\wedge(\delta)}\frac{e^{z\log x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)} &= \left\{\int_{\gamma_1'}+\int_{\gamma_2}+\int_{\gamma_3'}\right\}\frac{e^{z\log x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}\\ &= \int_{\infty}^{\frac{\delta}{2}}\frac{e^{ir\log x}idr}{\Gamma(ir)\sin(\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)}+\int_{\gamma_2}\frac{e^{z\log x}dz}{\Gamma(z)\sin(\frac{\pi}{\alpha}z)\cos(\frac{\pi}{2}z)}\\ &\quad+\int_{\frac{\delta}{2}}^{\infty}\frac{e^{-ir\log x}(-idr)}{\Gamma(-ir)\sin(-\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)} \end{aligned} \tag{2.8}$$

Easily can be seen that $\lim_{\delta\to 0}\int_{\gamma_2} = 0$. Therefore we have:

$$\frac{i}{4\alpha}\left\{\int_{\infty}^{0}\frac{e^{ir\log x}dr}{\Gamma(ir)\sin(\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)}+\int_{0}^{\infty}\frac{e^{-ir\log x}(-idr)}{\Gamma(-ir)\sin(-\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)}\right\} = p_\alpha(x)$$

Replace r by $-r$ in the second integral to have

$$\frac{i}{4\alpha}\left\{-\int_{0}^{\infty}\frac{e^{ir\log x}idr}{\Gamma(ir)\sin(\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)}+\int_{0}^{-\infty}\frac{e^{ir\log x}idr}{\Gamma(ir)\sin(\frac{\pi}{\alpha}ir)\cos(\frac{\pi}{2}ir)}\right\} = xp_\alpha(x)$$

and combining these two integrals we obtain

$$\frac{-i}{4\alpha}\int_{-\infty}^{\infty}\frac{e^{ir\log x}dr}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh(\frac{\pi}{2}r)} = xp_\alpha(x)$$

which proves the first part of the theorem.

To prove that the improper integral obtained is absolutely convergent, it is enough to prove that $\int_0^{\infty}\frac{e^{ir\log x}dr}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh(\frac{\pi}{g}r)}$ is absolutely convergent

We prove it into two steps. First for $\alpha \in (0,1]$ and then for $\alpha \in (1,2)$.

On the imaginary axis, i.e., if $z = ir, r \in \mathbb{R}$, then $|\Gamma(z)| = \sqrt{\frac{\pi}{\sinh \pi r}}$, see [3].

(i) Let $\alpha \in (0,1]$, for such α's $\sqrt{\dfrac{\sinh \pi r}{\sinh \frac{\pi}{\alpha} r}} \leq 1$. Then

$$\left|\frac{e^{ir\log x}}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh(\frac{\pi}{2}r)}\right| = \frac{1}{\sqrt{\pi}}\left|\frac{\sqrt{r}\sqrt{\sinh \pi r}}{\sqrt{\sinh(\frac{\pi}{\alpha}r)}\sqrt{\sin(\frac{\pi}{\alpha}r)}\sinh(\frac{\pi}{2}r)}\right| \geq C \text{ sech } (\frac{\pi}{2}r)$$

since $\displaystyle\int_0^\infty$ sech $(\dfrac{\pi}{2}r)dr$ is convergent, means that the improper integral for $\alpha \in (0,1]$ is absolutely convergent.

(ii) Now we consider $\alpha \in (1,2)$. Here $\frac{\pi}{2} < \frac{\pi}{\alpha}$ so this time: $\frac{\sinh(\frac{\pi}{2}r)}{\sinh(\frac{\pi}{\alpha}r)} < 1$ and we have:

$$\left|\frac{e^{ir\log x}}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh\frac{\pi}{2}r}\right| = \left|\frac{\sqrt{r}\sqrt{\sin \pi r}}{\sqrt{\pi}\sinh\frac{\pi}{\alpha}r\cosh\frac{\pi}{2}r}\right|$$

$$< \frac{\sqrt{2}\sqrt{r}}{\sqrt{\sinh(\frac{\pi}{\alpha}r)}\sqrt{\sinh(\frac{\pi}{2}r}} < C\frac{1}{\sqrt{\cosh\frac{\pi}{2}r}} < C_1 e^{-\frac{\pi}{4}r}$$

which implies that the improper integral is convergent also for $\alpha \in (1,2)$, and this completer the proof of the theorem □

Corollary 2.4. *If $\varepsilon = \arg\Gamma(z)$ for $\{z : \arg z = \pm\frac{\pi}{2}\}$ and if $z = ir$ where r is a non zero real number, then for $\alpha \in (0,2)$ the following formulae hold:*

$$\frac{1}{4\alpha\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\sin[r\log x - \varepsilon]\sqrt{r\sinh\pi r}}{\sinh\frac{\pi}{\alpha}r\cosh\frac{\pi}{2}r}dr = xp_\alpha(x)$$

and

$$\int_{-\infty}^{\infty}\frac{\cos[r\log x - \varepsilon]\sqrt{r\sinh\pi r}}{\sinh\frac{\pi}{\alpha}r\cosh\frac{\pi}{2}r}dr = 0$$

Proof. For the branch of $\Gamma(z)$ we consider, on the imaginery axis we have $\Gamma(z) = \sqrt{\frac{\pi}{r\sinh \pi r}}e^{i\varepsilon}$. Replacing $\Gamma(z)$, by its value on the imaginary axis in the formula given by theorem 2.3 and then separating the real and imaginary parts the results of the corollary follow. □

Remark: To prove the Theorem 9.5. in [2], we had constructed a transcendental number α to show that the exceptional set that both series in (2.7) diverge is not empty. Now ve give another result about this exceptional set:

Theorem 2.5. *The transcendental numbers $\beta \in (0,2)$ that the series on the right hand side of (2.7) are divergent are dense in the interval (0,1).*

Proof. As a consequence of Theorem 2.3 we have for $x > 0$ and $\alpha \in (0,2)$:

$$\frac{-i}{4\alpha}\int_{-\infty}^{\infty}\frac{e^{ir\log x}dr}{\Gamma(ir)\sinh(\frac{\pi}{\alpha}r)\cosh(\frac{\pi}{2}r)} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k\alpha}}{\Gamma(k\alpha)\cos(\frac{\pi}{2}k\alpha)}$$
$$+\frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^{k}x^{2k+1}}{\Gamma(2k+1)\sin\frac{\pi}{\alpha}(2k+i)} \tag{2.9}$$

Using Kronecker's Theorem [4, p. 375], if α is an irrational number, then the numbers $\alpha n + m$ where $m,n \in \mathbb{Z}$ are dense in $\mathbb{R}$. We know that the improper integral in (2.9) is absolutely convergent for all $\alpha \in (0,2)$. Let α be the same transcendental number of theorem 9.5 in [2, p. 520], i.e., $\alpha = \sum_{j=1}^{\infty}\frac{1}{q_j}$ where $q_1 = 2$ and $q_{s+1} = (q_s!)^2 q_s$, $s = 1,2,\ldots$

We define p_s such that $\frac{p_s}{q_s} = \sum_{j=1}^{s}\frac{1}{q_j}$.

Then we have $0 < \alpha - \frac{p_s}{q_s} < \frac{2}{(q_s!)^2 q_s}$ or

$$0 < \alpha q_s - p_s < \frac{2}{(q_s!)^2} \tag{2.10}$$

Using again the Kronecker's Theorem [4, p. 376], numbers of the form $\{\alpha n\}$ where α is an irrational number is dense in (0,1). Let $\beta = \alpha n, n \in \mathbb{Z}$. Using the inequality (2.10) we have

$$\beta q_s - np_s < \frac{2n}{(q_s!)^2}$$

We can choose n to be an odd integer, p_s also is an odd integer so that

$$\left|\cos(\frac{\pi}{2}\beta q_s)\right| = \left|\sin\frac{\pi}{2}(\beta q_s - np_s)\right| \leq \frac{n\pi}{(q_s!)^2}$$

Consider the general term of the first series on the right hand side of (2.9) with index $k = q_s$ and parameter β:

$$\left|\frac{x^{\beta q_s}}{\Gamma(\beta q_s)\cos(\frac{\pi}{2}\beta q_s)}\right| > \frac{|x|^{\beta q_s}(q_s!)^2}{\Gamma(q_s+1)n\pi} = \frac{|x|^{\beta q_s}(q_s!)}{n\pi} \to \infty$$

as $s \to \infty$. Therefore the set of all α's that the first series on the right hand side of (2.8) is divergent is a dense subset of (0,1). Since the improper integral on the left hand side of (2.9) is absolutely convergent, the second series must be divergent for the same irrational numbers. This completes the proof of the theorem. □

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