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FROM SIMPLICIAL GROUPS TO CROSSED COMPLEXES

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Abstract

In this paper, we will give a short proof of the construction of the crossed complex of groups by using the higher order Peiffer elements.

Introduction

Crossed complexes in combinatorial and cohomological algebra theory has the advantage of being less cumbersome than the full simplicial theory, but certain structural invariants are lost when they are used, as such crossed resolutions do not represent all the possible homotopy types available. It is therefore important to be able to go from the simplicial context to the crossed one and to study what is lost in the process.

Carrasco and Cegarra [5], calculated the relative homotopy group π_n (sk_n $\mathbf{G}$, sk_{n-1} $\mathbf{G}$) for a simplicial group $\mathbf{G}$ and proved it equal to

$$C_n(\mathbf{G}) = \frac{NG_n}{(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})}$$

for each n. This construct a crossed complex of groups from the Moore complex $\mathbf{NG}$ of $\mathbf{G}$. Their proof requires and understanding of hypercrossed complexes. Ehlers and Porter [7] developed a direct proof for simplicial groups/groupoids independently of [5]. Here we will do a short proof by using the higher order Peiffer elements. The following results are analogues of the commutative algebra version given by [1]. For more details about higher dimensional Peiffer elements in simplicial commutative algebras, see [3].

We begin by recalling the following results from [2]:

In [2], we showed the following. Let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$ and for $n > 1$ and let D_n be the normal subgroup generated by the degenerate elements in dimension n. If $G_n = D_n$, then

$$\partial_n(NG_n) = \partial_n(N_n) \text{ for all } n > 1.$$

where N_n is a normal subgroup in G_n generated by a fairly small explicitly given set of elements (see below).

If $n = 2, 3$ or 4, then the image of the Moore complex of the simplicial group $\mathbf{G}$ can be given in the form

$$\partial_n(NG_n) = \prod_{I,J}[K_I, K_J],$$

where the square brackets denote the commutator subgroup and $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$, and where

$$K_I \bigcap_{i \in I} \mathrm{Ker} d_i \text{ and } K_J = \bigcap_{j \in J} \mathrm{Ker} d_j.$$

In general, for $n > 4$, there is an inclusion

$$\prod_{I,J}[K_I, K_J] \subseteq \partial_n(NG_n).$$

1. From Simplicial Groups to Crossed Complexes

1.1. Higher Order Peiffer Elements

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from the ordered set $[n] = \{0, 1, \ldots, n\}$ to the ordered set $[n-r] = \{0, 1, \ldots, n-r\}$. This can be generated from the various σ_i^n by composition. The composition of these generating maps is subject to the following rule: $\sigma_j \sigma_i = \sigma_{i-1} \sigma_j, j < i$. This implies that every element $\sigma \in S(n, n-r)$ has a unique expression as $\sigma = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq n-1$, where the indices i_k are the elements of $[n]$ such that $\{i_1, \ldots, i_r\} = \{i : \sigma(i) = \sigma(i+1)\}$. We thus can identify $S(n, n-r)$ with the set $\{(i_r, \ldots, i_1) : 0 \leq i_1 < i_2 < \cdots < i_r \leq n-1\}$. In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$ corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly, the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq r \leq n} S(n, n-r).$$

Let $P(n)$ be a set consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$, where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n)$. We write $\#\alpha = r$, i.e. the length of the string α. The linear morphisms that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \to NG_n : (\alpha, \beta) \in P(n), n \geq 0\},$$

are given a composites $F_{\alpha,\beta} = p\mu(s_\alpha \times s_\beta)$, where

$$s_\alpha = s_{i_r} \ldots s_{i_1} : NG_{n-\#\alpha} \to G_n, s_\beta = s_{j_s} \ldots s_{j_1} : NG_{n-\#\beta} \to G_n,$$

$p : G_n \to NG_n$ is defined by composite projections $p = p_{n-1} \ldots p_0$, where $p_j(z) =$

$zs_jd_j(z)^{-1}$ with $j = 0, 1, \ldots, n-1$ and $\mu : G_n \times G_n \to G_n$ is given by the commutator. Thus

$$F_{\alpha,\beta}(x_\alpha, y_\beta) = (1s_{n-1}d_{n-1}^{-1}) \cdots (1s_0d_0^{-1})[s_\alpha(x_\alpha), s_\beta(y_\beta)].$$

Define the normal subgroup N_n to be that generated by elements of the form $F_{\alpha,\beta}(x_\alpha, y_\beta)$, where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$.

The idea for the construction of N_n and the use of the structure maps came from examining the thesis of Carrasco [4] (see also Carrasco and Cegarra, [5]).

The final elements that we need are the definition of a crossed complex of groups, and a construction of a crossed complex from a simplicial group. The proof that this works by using $F_{\alpha,\beta}$ maps in an innovative way.

A crossed complex of groups is a sequence of groups

$$\mathcal{C} : \qquad \cdots \to C_n \overset{\partial_n}{\to} C_{n-1} \to \cdots \to C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$$

such that

i) (C_1, C_0, ∂_1) is a crossed module, i.e., if $x, y, \in C_1$, then ${}^{\partial_1 x}y = xyx^{-1}$;

ii) for $i > 1, C_i$ is an C_0-module (Abelian) on which $\partial_1 C_1$ operates trivially and each ∂_i is an operator morphism; and

iii) for $i \geq 1, \partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way.

1.2. The Ultimate Dold-Kan Theorem: Hypercrossed Complexes

An important observation in Conduché's work, (cf. [6]), is the existence of a semidirect product decomposition of the group, G_n, of n-simplices in a simplicial group $\mathbf{G}$. These semidirect product decompositions are the analogue in the non-Abelian case of the direct sum decompositions used in the Dold-Kan theorem and have been studied in depth by Carrasco, [4] (see also [5]). By encoding the multiplication of the simplicial group in terms of this decomposition, she is able to make precise the extra structure carried by the Moore complex of a simplicial group which makes it possible to reconstruct the simplicial group up to isomorphism. This gives the most general non-Abelian form of A Dold-Kan type theorem.

We will give briefly this construction and importance of that in the following :

If $\mathbf{G}$ is a $(n-1)$-truncated simplicial group the nth simplicial kernel $\Delta^n(\mathbf{G})$ is a subgroup of $(G_{n-1})^{n+1}$ whose elements are those $(x_0, \ldots, x_n)$ such that $d_ix_j = d_{j-1}x_i$ for $i < j$. Thus $\Delta^n\mathbf{G}$ consists of all formal boundaries of the n-simplices that can be attached to G_{n-1}.

If $\mathbf{G}$ is a simplicial group, there is a canonical group extension

$$\text{Ker}\partial_n \to G_n \to \Delta^n(\text{tr}_{n-1}\mathbf{G}).$$

Similarly, if Λ_0^n $(\text{tr}_{n-1}\mathbf{G})$ denotes the group of $(n, 0)$-horns of the $(n-1)$-truncation, i.e.,

$$\Lambda_0^n(\mathrm{tr}_{n-1}\mathbf{G}) \subseteq (G_{n-1})$$

is defined by the rule:

$$\mathbf{x} \in \Lambda_0^n(\mathrm{tr}_{n-1}\mathbf{G}) \text{ if and only if } \mathbf{x} = \{x_1, \ldots, x_n),$$

where $d_i x_j = d_{j-1} x_i$ for all $i < j$, then there is a group extension

$$NG_n \to G_n \to \Lambda_0^n(\mathbf{tr}_{n-1}\mathbf{G}).$$

These two extensions are related as the first is contained in the second.

As with any group extensions, these can be specified by 2-cocycles: Carrasco and Cegarra [5], using the alternative form of the Moore complex, analyse this in the abstract discrete case.

A multi index 2-cocycle consists of the following: for each multi index $\alpha = (i_r, \ldots, i_1)$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq n-1$,

$$\text{an action } \Phi_\alpha^n : NG_{n-\#\alpha} \times NG_n \to NG_n$$

and for each pair of multi indices α, β with $\emptyset < \alpha < \beta$, $\alpha \cap \beta = \emptyset$ a pairing

$$F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \to NG_n.$$

Here if, α and $\beta = (i_r, \ldots, i_1)$ are multi indices,

$$\begin{array}{lll} \alpha < \beta \text{ if} & i_1 = j_1, \ldots, i_k = j_k & \text{but } i_{k+1} > j_{k+1} (k \geq 0) \text{ or} \\ \quad \text{if} & i_1 = j_1, \ldots, i_r = j_r & \text{and } r < s, \end{array}$$

so for instance $n = 3$,

$$\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0).$$

Remark. Group extensions

$$N \to E \xrightarrow{p} Q$$

of groups depend for their 2-cocycle definition on the result giving the existence of a section for p. For the application here, this section is given in the structure of the simplicial group so that

$$\Phi_\alpha^n : NG_{n-\#\alpha} \times NG_n \to NG_n$$

is 'conjugation via s_α':

$$\Phi_\alpha^n(x_\alpha, y_n) = s_\alpha(x_\alpha) x_n s_\alpha(x_\alpha)^{-1}.$$

As $F^n_{\alpha,\beta}(x_\alpha, y_\beta) = p[s_\alpha(x_\alpha), s_\beta(y_\beta)]$, it is clear that there must be a set of relations between the Φ^n_α and $F^n_{\alpha,\beta}$. These can be summarised as follows:

Let $S^n(\Delta^n(\mathbf{G})) = (k_\alpha, \kappa^n_{\alpha,\beta})$ denote the 2-cocycle corresponding to the first extension

$$\mathrm{Ker}\ \partial_{n-1} \to \Delta^n(\mathbf{G}) \to \Lambda^n_0(\mathbf{G})$$

and $S^n(G) = (\Phi^n_\alpha, F^n_{\alpha,\beta})$ to the extension

$$NG_n \to G_n \to \Lambda^n_0(\mathbf{G}).$$

Then

$$\partial F^n_{\alpha,\beta}(x,y) = \kappa^n_{\alpha,\beta}(x,y) \in \ \mathrm{Ker}\partial_{n-1} \subset NG_{n-1}$$

and

$$\partial(\Phi^n_\alpha(x, x_n)) = k_\alpha(x, x_n).$$

It is clear that these two conditions generalise the crossed module conditions to arbitrary dimensions. It is possible to expand the two conditions to arbitrary dimensions but the number of multi indices involved makes their immediate use difficult in dimensions greater than about 5. The author and T. Porter [2], [3] have calculated the corresponding conditions and $F^n_{\alpha,\beta}(x,y)$ pairing for simplicial groups and algebras. Carrasco gives an expanded version of certain types of conditions in all dimensions and also makes the following observations on their explicit interpretation in dimensions one and two. She calls such a system

$$(\mathbf{NG}; \Phi^n_\alpha, F_{\alpha,\beta})_{a,\beta\in S(n), n\geq 1, \alpha<\beta, \alpha\cap\beta=\emptyset}$$

a hypercrossed complex, in general.

- A 1-truncated hypercrossed complex is simply a crossed module.
- A 2-truncated hypercrossed complex is a system

$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

with actions

$$\Phi_i : C_1 \times C_2 \to C_2, \qquad i = 0, 1$$
$$\Phi_{1,0} : C_0 \times C_2 \to C_2$$

and

$$\Phi : C_0 \times C_1 \to C_1$$

and a pairing

$$F_{0,1} : C_1 \times C_1 \to C_2$$

satisfying conditions (that we will not give explicitly, as some are redundant but, more importantly) where the remainder reduce to 2-crossed module (cf. [6]) axioms of Conduché, (see p. 228-229 of Carrasco and Cegarra [5]). In both these cases the 'complex' involved was truncated.

- If the hypercrossed complex satisfies $F^n_{\alpha,\beta} = 1$ for all $n \geq 1$ then it is a 'crossed complex'.

We refer the reader to paper by Carrasco and Cegarra, [5]. It will be sufficient for our purposes to think of the hypercrossed complexes as being Moore complexes with the actions and pairings precisely laid out.

1.3. Construction of Crossed Comxlex

We should first show that the quotient group does exist.

Lemma 1. *The subgroup $(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})$ is a normal subgroup is G_n.*

Proof. It is straightforward from a direct calculation. □

Theorem 1. *Let **G** be a simplicial group. Then defining*

$$C_n(G) = \frac{NG_n}{(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})}$$

with

$$\partial_n(\overline{z}) = \overline{d_n z}$$

gives a crossed complex $C(\mathbf{G})$ of groups.

Proof. (i) follows since

$$C_1(\mathbf{G}) = \frac{NG_1}{\partial_2(NG_2 \cap D_2)} = \frac{NG_1}{[\mathrm{Ker} d_0,\ \mathrm{Ker} d_1]},$$

and in [2] we showed that the normal subgroup $[\mathrm{Ker} d_0,\ \mathrm{Ker} d_1]$ contains the Peiffer elements so $(C_1(\mathbf{G}), C_0(\mathbf{G}), \partial)$ is a crossed module, i.e.

$$d_1 : \frac{NG_1}{[\mathrm{Ker} d_0,\ \mathrm{Ker} d_1]} \to NG_0.$$

(ii) Take the generators $F_{\alpha,\beta}(x_\alpha, y_\beta)$ of the normal subgroup $NG_{n+1} \cap D_{n+1}$. For $x \in NG_r$ and $y \in NG_{r-1}$, by taking $\alpha = (n, n-1, \ldots, r), \beta = (r-1)$, it is easy to see that

$$F_{(n,n-1,\ldots,r),(r-1)}(x,y) = [s_n \ldots s_r(x), s_{r-1}(y)] \prod_{k=0}^{n-r} [s_{r+k}(y), s_n \ldots s_r(x)]^{(-1)^k},$$

and then

$$\begin{aligned} d_{n+1}F_{\alpha,\beta}(x,y) &= [s_{n-1} \ldots s_r(x), s_{r-1}d_n(y)] \prod_{k=0}^{n-r-1} [s_{r+k-1}d_n(y), s_{n-1} \ldots s_r(x)]^{(-1)^k} \\ &\quad [y, s_{n-1} \ldots s_r(x)]^{(-1)^{n-r}} \\ &= [s_r^{(n-r)}(x), s_{r-1}d_n(y)] \prod_{k=0}^{n-r-1} [s_{r+k-1}d_n(y), s_r^{(n-r)}(x)]^{(-1)^k} \\ &\quad [y, s_r^{n-r}(x)]^{(-1)^{n-r}}. \end{aligned}$$

This implies

$$[s_r^{(n-r)}(x), y] \in (NG_n \cap D_n d_{n+1}(NG_{n+1} \cap D_{n+1})),$$

which shows that the actions of NG_r on NG_n, defined by commutator

$$x \cdot y = [s_r^{(n-1)}(x), y]$$

via degeneracies, are trivial if $r \geq 1$. For $r = 1$, this gives $\alpha = (n, n-1, \ldots, 1), \beta = (0)$ and

$$\begin{aligned} F_{(n,n-r,\ldots,1)(0)}(x,y) = [s_n s_{n-1} \ldots s_1(x), s_0(y)][s_n s_{n-1} \ldots s_1(x), s_1(y)]^{-1} \ldots \\ [s_n s_{n-1} \ldots s_1(x), s_n(y)]^{(-1)^n} \end{aligned}$$

where $x \in NG_1, y \in NG_n$ and it is easily checked that

$$d_{n+1}F_{\alpha,\beta}(x,y) = [s_{n-1} \ldots s_1(x), s_0 d_n y] \ldots [s_{n-1} \ldots s_1(x), s_{n-1} d_n y][s_{n-1} \ldots s_1(x), y]^{(-1)^n}.$$

Then

$$[s_{n-1} \ldots s_1(x), y] \equiv 0 \bmod (NG_n \cap D_n d_{n+1}(NG_{n+1} \cap D_{n+1})).$$

This gives the following ; if $\bar{x} \in C_1$ then $\bar{x}$ and $\partial_1 \bar{x}$ act on C_n in the same way, and so $\partial_1 C_1$ acts trivially on C_n.

(iii) By defining

$$\partial_n(\bar{z}) = \overline{d_n^n(z)} \text{ with } z \in NG_n,$$

on obtains a well defined map $\partial : C_n(\mathbf{G}) \to C_{n-1}(\mathbf{G})$ verifying $\partial\partial = 0$. □

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