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ON THE VALIDITY OF THE BOREL-HIRZEBRUCH FORMULA FOR TOPOLOGICAL ACTIONS

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Abstract

We defined an equivalence among group actions and find sufficient conditions for actions of compact connected Lie groups on Euclidean spaces for which the topological version of the Borel-Hirzebruch formula holds.

Keywords: Lie group actions, characteristic classes, cohomology, classifiying spaces, weights

1. Introduction

The usefulness of the Borel-Hirzebruch formula in the study of differentiable actions of high dimensional campact Lie groups on differentiable manifolds is undeniable, see for example [3][4]. Similar methods for topological actions fail for two reasons: lack of equivariant normal (micro)bundle and the lack of a formula for the characteristic classes of equivariant (micro)bundles to assign characteristic classes to microbundles we use the results of Kister, see [7]. In this article we will discuss the validity of a formula for the characteristic classes of some topological R^n bundles.

Since weights can be defined for actions on acyclic topological manifolds and $H^*(B_{TOP}; Q)$ has been determined by Kirby and Siebenmann, the question of topological Borel-Hirzebruch (B-H for short) formula is meaningful and may be useful in the study of topological actions on manifolds. In its simplest form, the topological B-H question can be phrased as folows:

Let G be a compact connected Lie group, H a closed connected subgroup acting or R^n, then $G \times_H R^n$ is an (equivariant) R^n bundle over G/H. $H^*(B_{TOP}; Q) \cong H^*(B_{PL}, Q) \cong H^*(B_O, Q) \cong Q[p_1, p_2, \ldots]$ has been proved by Kirby and Siebenmann [6] and Hirsch [2] so $G \times_H R^n$ has rational Pontryagin classes; the total rational Pontryagin class will be denoted by P. Let T be a maximal torus $H; \Omega' = \{\pm\omega_1, \pm\omega_2, \ldots, \pm\omega_k\}$ be the nonzero weights (defined only up to a rational multiple, possibly repeated) of the restricted T action on $R^n, \Omega^+ = \{\omega_1, \omega_2, \ldots, \omega_k\}; \pi : G/T \to /H, j : G/T \to B_T$ the canonical maps. One would like to know if (for a suitable choice of weights)

$$\pi^*(P) = j^*(\prod_{\omega \in \Omega^+} (1 + \omega^2)$$

holds; where P is the total Pontryagin class of the R^n bundle $G \times_H R^n \to G/H$. For linear actions, the geometric weights agree with the usual weights. Borel and Hirzebruch, in [1], proved this formula for the usual weights of linear actions. If such weights exist we will say the B-H formula holds for this action (this does not depend on the maximal torus chosen).

We will establish the validity of this formula for a class of actions of compact Lie groups. We will not strive for the most general case but will be content to discuss only the cases that interests us.

2. Preliminaries

Using the above formula, (when the dimension of the group is large) one can severely restrict the possibilities for the orbit types for the differentiable actions of a compact Lie group under suitable assumptions on the characteristic classes of the manifold on which the group is acting, see for example [4]. We will use concordance (as far as the author knows) not used in this context before.

Definition 2.1. *Let a compact connected Lie group H act on topological space X two ways, via φ_0 and φ_1. We say φ_0 is (topologically) concordant to φ_1, if there is an action*

Φ *of* H *on* $X \times I (I = [0, 1])$, *leaving* $X \times 0$ *and* $X \times 1$ *invariant and restricting to* φ_0 *and* φ_1 *on these subsets respectively.*

Concordance is clearly an equivalence relation. One can also define differentiable concordance of differentiable actions, with an additional condition on the behavior near the boundary (G action being a product action in a neighborhood of the boundary, i.e. *conditioned* in the language of Kirby and Siebenmann).

Let us recall briefly the definition of geometric weight system given by Hsiang in [5]. Let a torus T act on an acyclic (cohomology) manifold X, with fixed point set F (which is an acyclic cohomology manifold). X/F has the cohomology of a sphere so the generator of the top cohomology (in the Serre spectral sequence of $(X/F)_T \to B_T$) is transgressive. Its transgression in $H^*(B_T; Q)$ will split into a product of linear factors (defined only up to a rational multiple). The set of these linear factors (with multiplicities) is called by Hsiang, after including zero weights, the geometric weight system of the action and the set of nonzero weights is denoted by Ω'. They could also be defined as the duals of subtori having (locally) larger dimensional fixed point sets. For an arbitrary compact connected Lie group G acting on a contractible manifold, we fix a maximal torus T and define the geometric weight system of G (relative to T) to be the geometric weight system of the restricted T action.

3. Results

All Lie groups appearing in this article are compact and connected. We will use the following propositions in the proof of our main theorem:

Proposition 3.1. *If two actions* φ_0 *and* φ_1 *of* H *on* R^n *(or on any acyclic cohomology manifold) have concordant restrictions to some maximal torus then they have the same geometric weight system.*

Proof. First we observe that the action Φ of the maximal torus T on $R^n \times I$ be assumed to be a product action near $R^n \times 0$ and $R^n \times 1$ (i.e. *conditioned*). Since $R^n \times (0, 1)$

is contractible manifold, the action of T will have a (nonempty) connected fixed point set. Since $R^n \times 0$ and $R^n \times 1$ are also invariant and contractible, weights for the actions φ_0 and φ_1 will be same as the weights of Φ, except *a zero weight*. Thus the weights relative to T are equal. □

Observe that an action of H naturally extends to R^{n+1}, trivially on the second factor outside [0,1].

Proposition 3.2. *If two actions φ_0 and φ_1 of a Lie subgroup H of a Lie group G on R^n are concordant via a concordance Φ, then $G \times_\Phi R^{n+1} \to G/H$ (Φ acts on R^{n+1} as indicated above) is equivalent (in the sense of microbundles) to the sum of any one of $G \times_{\varphi_i} R^n \to G/H$ and a trival microbundle over G/H.*

Proof. If Φ is the product action on $U_i = R^n \times (i-\varepsilon, i+\varepsilon)(i = 0, 1, \varepsilon > 0)$ then $G \times_{\varphi_i} U_i = (G \times_{\varphi_i} R^n) \times (i-\varepsilon, i+\varepsilon) \to G/H$ are equivalent to $G \times_\Phi R^{n+1} \to G/H$ as microbundles and they are clearly the direct sum of the bundle $G \times_{\varphi_i} R^n \to G/H$ and a trivial microbundle. □

Now we can state:

Theorem 3.3. *Let G be a Lie group, H a closed subgroup, φ_0 and φ_1 two actions of H on R^n such that their restriction to some maximal torus T are concordant. Then the topological Pontryagin classes of $G \times_{\varphi_i/T} R^n \to G/T$ are equal. Hence, if $\varphi_{0/T}$ is linear, then the topological B-H Formula holds for φ_1.*

Proof. Let Φ be a concordance between $\varphi_{i|T}(i = 0, 1).\xi_i : G \times_{\varphi_i} R^n \to G/H$. Then the pull-back bundles $\pi^!(\xi)$ are the bundles $G \times_{\varphi_{i|T}} R^n \to G/T$ defined by the action restricted to T by Proposition 3.2, the fiber bundles $\pi^!(\xi_i) : G \times_{\varphi_{i|T}} R^n \to G/T$ are submicrobundles of $\xi : G \times_\Phi R^{n+1} \to G/T$ with trivial normal microbundle. Thus we have

$$P(\pi^!(\xi_1)) = P(\xi) = P(\pi^!(\xi_0)).$$

Hence

$$\pi^*(P(\xi_1)) = P(\pi^!(\xi_1)) = P(\pi^!(\xi_0)) = j^*(\prod_{\omega\in\Omega^+}(1+\omega^2)),$$

since $\varphi_{0|T}$ is assumed to be linear, where Ω' is the common geometric weight systems of φ_0 and φ_1. □

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