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ON THE PARABOLIC CLASS NUMBER OF SOME SUBGROUPS OF HECKE GROUPS

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Abstract

In this paper we calculate the parabolic class number of subgroups of Hecke groups $H(\sqrt{2}), H(\sqrt{3})$.

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1. Introduction

By a Funchsian group Λ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$, the group of conformal homeomorphisms of the upper-half plane. The most general presentation for Λ is

Generators;

$$\begin{array}{ll} a_1, b_1, \ldots, a_g, b_g & \text{(Hyperbolic)} \\ x_1, x_2, \ldots, x_r & \text{(Elliptic)} \\ p_1, p_2, \ldots, p_s & \text{(Parabolic)} \end{array}$$

Relations;

$$x_1^{m_1} = x_2^{m_2} = \cdots x_r^{m_r} = \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k = 1.$$

We then say Λ has signature (see [1])

$$(g, m_1, m_2, \ldots, m_r; s).$$

Hecke introduced an infinite class of discrete groups $H(\lambda_q)$ of linear fractional transformation proserving the upper-half plane. $H(\lambda_q)$ is the group generated by

$$S(z) = z + \lambda_q, \quad T(z) = \frac{-1}{z}$$

where $\lambda_q = 2\cos(\pi/q), q$ is an integer, $q \geq 3$. When $q = 3$ we have the modular group Γ. When $q = 4$ or 6 the resulting group are $H(\sqrt{2}), H(\sqrt{3})$. These two groups are of particular interest since they are the only Hecke groups, aside from the modular group, whose elements are completely known.

It is well known ([2], [3]) that $H(\sqrt{m}), m = 2, 3$, consists of the mappings of all the following two types.

i) $T(z) = \dfrac{az + b\sqrt{m}}{c\sqrt{m}z + d}, a, b, c, d \in \mathbb{Z},\ ad - bcm = 1,$

ii) $T(z) = \dfrac{a\sqrt{m}z + b}{cz + d\sqrt{m}}, a, b, c, d \in \mathbb{Z},\ adm - bc = 1.$

2. Parabolic Class Number

From now on, m will stand for 2 or 3

Lemma 1. *$H(\sqrt{m})$ act transitively on $\sqrt{m}\hat{\mathbb{Q}} = \{\frac{r}{s}\sqrt{m} : \frac{r}{s} \in \mathbb{Q}\} \cup \{\infty\}$.*

$\sqrt{m}\mathbb{Q}$ is the largest subset of $\mathbb{R}$ on which $H(\sqrt{m})$ acts transitively.

Proof. Let $\frac{x}{y}\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}} \setminus \{\infty\}$ with $(x, y) = 1$. Then $m|y$ or $m \nmid y$. Since $(x, y) = 1$, we can find $a, b \in \mathbb{Z}$ such that $xa - yb = 1$. If $m|y$, then we take

$$T(z) = \frac{xz + b\sqrt{m}}{\frac{x}{m}\sqrt{m}z + a},$$

so, we have $T(\infty) = \frac{x}{y}\sqrt{m}$.

Let $m \nmid y$. In this case $(mx, y) = 1$, and thus there exist some, $a, b \in \mathbb{Z}$ such that $mxa - yb = 1$. Similarly, if we take

$$S(z) = \frac{x\sqrt{m}z + b}{yz + \sqrt{m}a},$$

then $S(\infty) = \frac{x}{y}\sqrt{m}$.

Let $n \in \mathbb{N}$. Define

$$H_0^m(n) = \{T \in H(\sqrt{m}) : c \equiv 0(\text{mod } n)\}.$$

Then $H_0^m(n)$ is a subgroup of $H(\sqrt{m})$. □

Let Γ be the modular group, and $\Gamma_0(n)$ be the subgroup of Γ such that $c \equiv 0(\text{mod } n)$. Then,

Lemma 2. *[5]* $|\Gamma : \Gamma_0(n)| = n \prod_{p|n} \left(1 + \frac{1}{p}\right).$

Lemma 3. *If* $(m,n) = 1$, *then* $|H(\sqrt{m}) : H_0^m(n)| = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$, *if* $(m,n) = m$, *then* $|H(\sqrt{m}) : H_0^m(n)| = 2n \prod_{p|n} \left(1 + \frac{1}{p}\right)$ *where* $p \neq m$.

Proof. We will give the proof in case where $(m,n) = m$.

Let

$$H = \left\{ T \in H(\sqrt{m}) : T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d} \right\}.$$

As $m|n$, $H_0^m(n) \subset H \subset H(\sqrt{m})$. It is obvious that $|H(\sqrt{m}) : H| = 2$. Let θ be the mapping from H to $\Gamma_0(m)$ defined as follows. If

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d},$$

then

$$\theta(T)(z) = \frac{az + b}{cmz + d}.$$

It can be shown that θ is an isomorphism, and $\theta(H_0^m(n)) = \Gamma_0(mn)$.

On the other hand, $\Gamma_0(mn) \subset \Gamma_0(m) \subset \Gamma$. Therefore,

$$\begin{aligned} |H(\sqrt{m}) : H_0^m(n)| &= |H(\sqrt{m}) : H|\,|H : H_0^m(n)| = 2|\Gamma_0(m) : \Gamma(mn)| \\ &= 2\frac{|\Gamma : \Gamma_0(mn)|}{|\Gamma : \Gamma_0(m)|} = 2n \prod_{p|n} \left(1 + \frac{1}{p}\right) \text{ where } p \neq m. \end{aligned}$$

If $(m,n) = 1$, the proof can be done in a similar way. □

We now give the following lemmas without proof. Proofs are similar to those for the modular group Γ in [4].

Lemma 4. *Let* Λ *be a subgroup of finite index in* $H(\sqrt{m})$. *Then, the parabolic class number of* Λ *is the number of orbits of* Λ *on* $\sqrt{m}\hat{\mathbb{Q}}$.

Lemma 5. *Let Λ be a subgroup of finite index in $H(\sqrt{m})$. Then, the parabolic class number s of Λ satisfies $1 \leq s \leq N$, where N is the index $|H(\sqrt{m}) : \Lambda|$; in particular s is finite.*

We now give our main theorems.

Theorem 1. *If $(m,n) = 1$, then the parabolic class number of $H_0^m(n)$ is*

$$\sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

Before giving the proof we will give some lemmas.

Lemma 6. *Let $(m,n) = 1$, and $\frac{r}{s}\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}}$ with $m|s$, then we can find some $T \in H_0^m(n)$ such that $T(\frac{r}{s}\sqrt{m}) = \frac{r_1}{s_1}\sqrt{m}$ with $(m, s_1) = 1$ (we represent ∞ as $\frac{1}{0}\sqrt{m}$).*

Proof. Since $(m,n) = 1$, there exist some $a, b \in \mathbb{Z}$ such that $1 = ma - nb$. Let

$$T(z) = \frac{a\sqrt{m}z + b}{nz + \sqrt{m}}.$$

Then $T \in H_0^m(0)$, and

$$T\left(\frac{r}{s}\sqrt{m}\right) = \frac{arm + bs}{(rn+s)\sqrt{m}} = \frac{ar + bs/m}{rn + s}\sqrt{m}.$$

It can be easily shown that $(m, rn+s) = 1$. If we take $r_1 = ar + bs/m$, and $s_1 = rn + s$, then $T(\frac{r}{s}\sqrt{m}) = \frac{r_1}{s_1}\sqrt{m}$ with $(m, s_1) = 1$. □

Lemma 7. *Let $(m,n) = 1$, and $\frac{k}{s}\sqrt{m} \in \sqrt{m}\hat{\mathbb{Q}}$ with $(k,s) = 1$. If $(m,s) = 1$, then there exist some $T \in H_0^m(n)$ such that*

$$T(\frac{k}{s}\sqrt{m}) = \frac{k_1}{s_1}\sqrt{m} \text{ with } s_1|n.$$

Proof. $(km, s) = 1$ since $(k,s) = (m,s) = 1$. Let $s_1 = (s,n)$. Then $s_1 = (s,n) = (s, kmn)$. Therefore there exist some integers c_1, d_1 such that

$$\frac{kmn}{s_1}c_1 + \frac{s}{s_1}d_1 = 1.$$

Since $(d_1, \frac{kmn}{s_1}) = 1$, there exists an integer t such that $(d_1 - \frac{kmn}{s_1}t, mn) = 1$. Let $d = d_1 - \frac{kmn}{s_1}t$ and $c = c_1 + \frac{s}{s_1}t$. Then

$$\frac{kmn}{s_1}c + \frac{s}{s_1}d = 1.$$

□

On the other hand, $(d, cmn) = 1$, since $(d, mn) = (d, c) = 1$. Hence, we can find some integers x, y such that $xd - ycmn = 1$. If we take

$$T(z) = \frac{xz + y\sqrt{m}}{cn\sqrt{m}z + d},$$

then, we have $T(\frac{k}{s}\sqrt{m}) = \frac{k_1}{s_1}\sqrt{m}$ where $k_1 = xk + ys$ and $s_1 = cnmk + ds$.

It is obvious that $T \in H_0^m(n)$. On the other hand, it can be seen that $(k_1, s_1) = 1$.

Lemma 8. *Let $(m, n) = 1$. If $d_1|n$ and $(a_1 d_1) = (a_2, d_1) = 1$, then $\frac{a_1}{d_1}\sqrt{m}$ is conjugate to $\frac{a_2}{d_1}\sqrt{m}$ under $H_0^m(n)$ if and only if $a_1 \equiv a_2$ (mod t) where $t = (d_1, \frac{n}{d_1})$.*

Proof. Let $a_1 \equiv a_2$ (mod t) and $n_1 = n/d_1$. Then, $t = (d_1, n_1)$, and $(a_1 a_2, d_1)$, and $(a_1 a_2, d_1) = 1$. Furthermore, $(m, d_1) = 1$ since $(m, n) = 1$. Therefore $(a_1 a_2 m, d_1) = 1$, and thus $(n_1 a_1 a_2 m, d_1) = t$. Since $t | a_1 - a_2, mn_1 a_1 a_2 x + d_1 y = a_2 - a_1$ has a solution. That is, there exist some integers k, s such that $mn_1 a_1 a_2 k + a_1 + d_1 s = a_2$. Hence, we obtain $aa_1 + bd_1 = a_2$. On the other hand, if we take $c = n_1 d_1 k$ and $d = 1 - mn_1 a_1 k$, we obtain $mca_1 + dd_1 = d_1$.

Furthermore,

$$ad - bcm = a(1 - mn_1 a_1 k) - bmn_1 d_1 k = a - (aa_1 + bd_1)mn_1 k = 1.$$

Let

$$T(z) = \frac{az + b\sqrt{m}}{c\sqrt{m}z + d}.$$

Then it is clear that $T \in H_0^m(n)$ and $T(\frac{a_1}{d_1}\sqrt{m}) = \frac{a_2}{d_1}\sqrt{m}$.

Now let $\frac{a_1}{d_1}\sqrt{m}$ be equivalent to $\frac{a_2}{d_1}\sqrt{m}$ by some $T \in H_0^m(n)$. Then it is easily seen that

$$T(z) = \frac{az + b\sqrt{m}}{cn\sqrt{m}z + d} \text{ where } ad - bcmn = 1.$$

Hence, we obtain

$$\frac{aa_1 + bd_1}{cna_1m + dd_1}\sqrt{m} = \frac{a_2}{d_1}\sqrt{m},$$

that is,

$$\frac{aa_1 + bd_1}{cna_1m + dd_1} = \frac{a_2}{d_1}.$$

Since

$$d(aa_1 + bd_1) - b(cna_1m + dd_1) = a_1,$$

and

$$a(cna_1m + dd_1) - cnm(aa_1 + bd_1) = d_1$$

we have $(aa_1 + bd_1, cna_1m + dd_1) = 1$. Therefore, there exists some $u = \pm 1$ such that

$$aa_1 + bd_1 = ua_2,$$

and

$$cna_1m + dd_1 = ud_1.$$

It can be easily shown that $a_1 \equiv a_2 \pmod{t}$. □

Proof of Theorem 1. It is sufficient to calculate the number of orbits of $H_0^m(n)$ on $\sqrt{m}\hat{\mathbb{Q}}$. Then from Lemma 6, Lemma 7, and Lemma 8, the number of orbits of $H_0^m(n)$ on $\sqrt{m}\hat{\mathbb{Q}}$ is $\sum_{d|n} \varphi((d, \frac{n}{d}))$ where φ is Euler's function.

We can deduce the following.

Lemma 9. *If $\frac{k}{s} \in \hat{\mathbb{Q}}$ with $(k, s) = 1$, then there exists some $T \in \Gamma_0(n)$ such that $T(\frac{k}{s}) = \frac{k_1}{s_1}$ with $s_1|n$ where we represent ∞ as $\frac{1}{0}$.*

Lemma 10. *If $d_1|n$ and $(a_1, d_1) = (a_2, d_1) = 1$, then $\frac{a_1}{d_1}$ is conjugate to $\frac{a_2}{d_1}$ under $\Gamma_0(n)$ if and only if $a_1 \equiv a_2 (\mathrm{mod}\, t)$ where $t = (d_1, \frac{n}{d_1})$.*

Then it is easily seen that the number of orbits of $\Gamma_0(n)$ on $\hat{\mathbb{Q}}$ is

$$\sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right),$$

which is the parabolic class number of $\Gamma_0(n)$.

Theorem 2. *If $m|n$, then the parabolic class number of $H_0^m(n)$ is*

$$\sum_{d|n} \varphi\left(\left(d, \frac{nm}{d}\right)\right).$$

Proof. If $m|n$, then

$$H_0^m(n) = \{T \in H : c \equiv 0(\mathrm{mod} n)\}.$$

We define ϕ from the set of orbits of $\Gamma_0(mn)$ to the set of orbits of $H_0^m(n)$ as follows.

$$\phi\left(\Gamma_0(mn)\frac{r}{s}\right) = H_0^m(n)\frac{r}{s}\sqrt{m}.$$

Then ϕ is well defined one to one and onto function. The proof then follows. □

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