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LOCALLY VOLUME-MINIMIZING CODIMENSION-ONE FOLIATION OF THE SOLID TORUS*

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Abstract

The aim of this paper is to construct a specific codimension-1 foliation of $D^2 \times S^1$ with one Reeb component, and to show that this foliation locally minimizes the volume among foliations with the boundary torus as a leaf.

1. Introduction

Let M be a smooth m-dimensional manifold with tangent bundle TM. A *k-plane field* (or k-distribution) σ is the section of the Grassmann bundle $G_k(M)$ of k-planes associated to TM whose value at $x \in M$ is the k-plane $\sigma_x \subset TM_x$. If N is an injectively immersed, smooth submanifold of M such that $TN_x = \sigma_x \subset TM_x$ for all $x \in N, N$ is called an *integral submanifold* of σ. A k-plane field σ is called *completely integrable* if σ is smooth and through every point $x \in M$ there is an integral submanifold N of σ. An integrable k-plane field is also called a *foliation* and the maximal connected integral submanifolds are called *leaves* of the foliation [10].

The *volume* of a foliation $\mathcal{F}$ on a Riemannian manifold M is defined to be the Hausdorff n-dimensional measure of the image of the section $\sigma(M) \subset G_k(M)$, using the metric induced on $G_k(M)$ by the metric on M [9]. We define an *optimal foliation* on M to be one with the minimum volume among all foliations on M. An optimal foliation on a manifold may not exist. If one exists, it may not be unique.

In [2], H. Cluck and W. Ziller have asked the question that given a manifold M, what is the optimal foliation in that it minimizes the volume. In particular, they showed that on the flat torus, $S^1 \times S^1$, the optimal vector fields are precisely those of constant slope.

The question that we will address in this paper is an analogue of the original question of Gluck and Ziller for a foliation of dimension bigger than one, which is :

Among all codimension-one foliation $\mathcal{F}$ of flat solid torus, $D^2 \times S^1$, with the boundary torus a leaf of the foliation, what is the optimal foliation?

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We are able to show the existence of a codimension-one foliation $\mathcal{F}$ of the flat solid torus, $D^2 \times S^1$, consisting of one Reeb component, which is at least a *local* minimum of the volume functional. We use an extension of a slicing technique due to Gary Lawlor [6], which provides the key to showing that the foliation constructed is locally volume minimizing.

If $\mathcal{F}$ is a transversely-oriented co-dimension one foliation on a 3-manifold M, the graph $\sigma : M \to G_2(M)$ of this foliation is equivalent to a mapping $\xi : M \to T_1(M)$, sending x to a unit normal vector $\xi(x)$ to the leaf of $\mathcal{F}$ through x. (There are only two such maps, differing only in sign).

Given such a unit normal vector ξ to a codimension-one foliation $\mathcal{F}$ of a 3-manifold M, that is $\xi \in TM_x$, and $\|\xi\| = 1$, with $\xi \perp \mathcal{F}_x$, the volume functional $Vol(\mathcal{F})$ of $\mathcal{F}$ is computed by

$$Vol(\mathcal{F}) = \int_M \sqrt{1 + \sum_{i=1}^{3} \|\nabla_{e_i}\xi\|^2 + \sum_{i,j=1}^{3} \|\nabla_{e_i}\xi \wedge \nabla_{e_j}\xi\|^2 dV} \qquad (1)$$

for any orthonormal basis e_i of TM. This is of course the same formula as for the volume of the unit normal vector field to the foliation. The metric on the tangent bundle used here is defined by Sasaki, and is the natural metric on TM induced from the Riemannian metric on M [3, 9].

We recall the classical construction of a codimension-one foliation of $D^2 \times S^1$ in order to make the details of our specific foliation explicit [7].

Take any C^∞ function $f(x)$ so that $\lim_{|x|\to 1} |f^{(k)}(x)| = \infty$ for all k, and foliate the strip $\{(x, y) \in \mathbb{R}^2 : 1 \le x \le 1\}$ by the graphs of the functions $y = f(x) + c, -1 < x < 1$ and $c \in \mathbb{R}$, together with the vertical lines $x = \pm 1$. The condition on the derivatives of f are equivalent to smoothness of the construction at the boundary (cf. Figure (1-a)).

Rotate the strip $\{(x, y) \in \mathbb{R}^2 : -1 \le x \le 1\}$ about the y-axis in $\mathbb{R}^2$, generating a foliation of the cylinder $D^2 \times \mathbb{R}$. Since this foliation is invariant under vertical translations, identifying $(x, y) \sim (x, y + 1)$ we obtain a foliation of the solid torus $D^2 \times S^1$, where each non-compact leaf has the form of a snake eternally eating its tail (cf. Figure (1-b)). A foliation of a solid torus in this way is called a *Reeb component.* In general, a codimension-one foliation of a manifold M may have several Reeb components as parts (saturated sets, or components) of the total foliation.

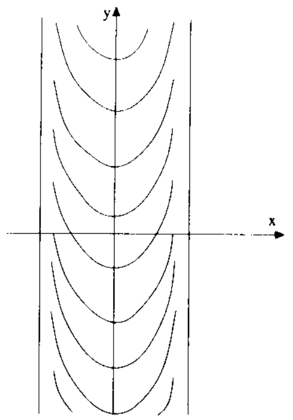


Figure 1a

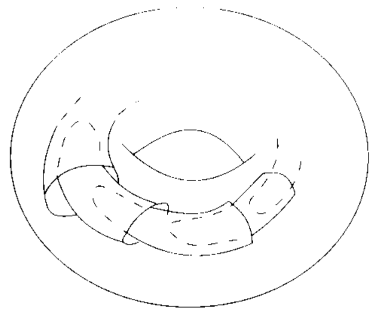


Figure 1b

2. Slicing

In [6], Gary Lawlor has constructed a more direct analogue of calibrations, which, in some cases, can directly show that a given current is mass-minimizing, by showing that it decomposes into energy-minimizing pieces. We show here a generalization of his result, which is directly applicable to this situation.

Let $\pi : E \to M$ be a fiber bundle over a Riemannian manifold M, with fiber F, which is a subbundle of a tensor bundle $\mathcal{T}$ over M. Let a functional $\mathcal{E}$ of order m on the space of sections of E, and their covariant derivatives of onder m be given by

$$\mathcal{E}(\sigma) = \int_M L(\sigma, \nabla\sigma, \ldots, \nabla^m(\sigma), x)dV, \tag{2}$$

where $\nabla^m(\sigma) := \nabla(\cdots(\nabla(\sigma))\cdots)$.

If $\mathcal{G}$ is a decomposition (slicing) of M into compact submanifolds $\{S_t\}$, i.e., $\mathcal{G} = \{S_t | t \in N\}$, indexed by a manifold N, perhaps with boundary, with an arbitrary Riemannian metric, so that $M = \cup_{t\in N} S_t$, except for a set of measure 0, then for any $f \in C^\infty(M)$ and $x \in S_t$, we can write

$$\int_M f dV = \int_N \left(\int_{S_t} f|S_t w(t,x) dV_{S_t} \right) dV_N \tag{3}$$

for some weighting function $w(t, x)$ which depends only on the decomposition and the metric chosen on N, by the general area-coarea formula of Federer [1].

Theorem 2.1. *Let $G = \{S_t | t \in N\}$ be a slicing of M and assume that for each $t \in N$,* *there exists a functional of order m*

$$\mathcal{E}_t(\sigma_t) = \int_{S_t} L_t(\sigma_t \nabla \sigma_t, \ldots, \nabla^m(\sigma_t), x) w(t,x) DV_{S_t}$$

for sections $\sigma_t : S_t \to E|_{S_t}$ *so that for each section* $\sigma : M \to E$,

$$\begin{aligned}
\mathcal{E}(\sigma) &= \int_M L(\sigma, \nabla\sigma, \ldots, \nabla^m(\sigma), x) dV \\
&= \int_N \left(\int_{S_t} (L(\sigma, \nabla\sigma, \ldots, \nabla^m(\sigma), x))|_{S_t} w(t,x) dV_{S_t} \right) dV_N \\
&\geq \int_N \left(\int_{S_t} L_t(\sigma|_{S_t}, \nabla(\sigma|_{S_t}), \ldots, \nabla^m(\sigma|_{S_t}), x) w(t,x) DV_{S_t} \right) dV_N. \qquad (4)
\end{aligned}$$

If $\sigma_0 : M \to E$ *satisfies the following conditions,*

i) $\sigma_0|_{S_t}$ *is the local minimizer of* $L_t(\sigma)$ *subject to* $\sigma|_{\partial S_t} = \psi|_{\partial S_t}$

ii) $L_t(\sigma_0|_{S_t}, \nabla(\sigma_0|_{S_t}), \ldots, \nabla^m(\sigma_0|_{S_t}), x) = L(\sigma_0, \nabla\sigma_0, \ldots, \nabla^m(\sigma_0), x)|_{S_t}$,

then, σ_0 *is a local minimizer of* $\mathcal{E}(\sigma)$ *subject to* $\sigma|_{\partial M} = \psi$ *for* $\psi : \partial M \to E|_{\partial M}$.

Proof. Let σ be a section which is a sufficiently small perturbation of σ_0, preserving the boundary conditions. Then,

$$\begin{aligned}
\mathcal{E}(\sigma_0) &= \int_N \left(\int_{S_t} L_t(\sigma_0|_{S_t}, \nabla(\sigma_0|_{S_t}, \ldots, \nabla^m(\sigma_0|_{S_t}), x) w(t,x) dV_{St} \right) dV_N \\
&\leq \int_N \left(\int_{S_t} L_t(\sigma|_{S_t}, \nabla(\sigma|_{S_t}, \ldots, \nabla^m(\sigma|_{S_t}, x) w(t,x) dV_{S_t} \right) dV_N \\
&\leq \int_N \left(\int_{S_t} (L(\sigma, \nabla\sigma, \ldots, \nabla^m(\sigma), x))|_{S_t} w(t,x) dV_{S_t} \right) dV_N \\
&= \mathcal{E}(\sigma).
\end{aligned}$$

The first equiality follows from (2) and assumption (ii), the first inequality follows from the assumption (i), second inequality from (4) and the last equality follows from (2) □

3. Main Result

Now, we describe a codimension-one foliation of the solid torus, $D^2 \times S^1$ (as the cylinder $D^2 \times [0,1]$, with identification $(x,0) \sim (x,1)$), and then we will show that it locally minimizes the volume among all foliations of the solid torus with the boundary torus as a leaf. First, recall that given a unit normal vector ξ to a codimension-one foliation $\mathcal{F}$ of a 3-manifold M, that is, $\xi \in T_*(M,x)$, and $\|\xi\| = 1$, with $\xi \perp \mathcal{F}_x$, the volume functional of $\mathcal{F}$ is defined by equation (1).

In general, a foliation can be defined locally as level sets $F(x,y,z) = c$, where each value of c determines a leaf of the foliation. The local minimizer we construct is of a special form, $F = (x,y) - z$, making it invariant under translation in the S^1-direction.

3.1. Solving the One-Dimensional Problem

Let $F(x,y,z) = f(x,y) - z$ describe a codimension-one foliation of the torus, $D^2 \times S^1$, by level sets $F(x,y,z) = c$.

Then the unit normal field ξ can be defined as

$$\xi(x,y,z) = \frac{\nabla F}{\|\nabla F\|} = \frac{(f_x, f_y - 1)}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{5}$$

Note that, in $\mathbb{R}^3$,

$$\begin{aligned}\|\nabla\xi\|^2 &= \|d\xi\|^2 \\ &= \langle \xi_x dx + \xi_y dy, \xi_x dx + \xi_y dy\rangle \\ &= \xi_x^2 + \xi_y^2,\end{aligned} \tag{6}$$

in particular, $\xi_z = 0$, and

$$\begin{aligned}\|\nabla\xi \wedge \nabla\xi\|^2 &= \|\xi_x \wedge \xi_y\|^2 \\ &= \langle \xi_x \wedge \xi_x\rangle\langle \xi_y \wedge \xi_y\rangle - (\langle \xi_x \wedge \xi_y\rangle)^2\end{aligned} \tag{7}$$

since the wedge product is just the vector product in $\mathbb{R}^3$.

We now reduce the volume problem to the one-variable case of a rotationally-invariant and translation-invariant foliation, that is, in cylindrical coordinates,

$$F(x,y,z) = f(x,y) - z = f(r,\theta) - z \tag{8}$$

with $f_\theta \equiv 0$. Rotation-invariance is equivalent to $f_\theta = 0$. It is a straightforward computation that with polar coordinates (r,θ), setting $r = \sqrt{x^2+y^2}$, that

$$\begin{aligned}\xi_x = \frac{1}{(1+f_r^2)^{3/2}}[&f_{rr}\cos^2\theta + \frac{1}{r} f_r (1+f_r^2)\sin^2\theta, \\ &f_{rr}\cos\theta\sin\theta - \frac{1}{r} f_r(1+f_r^2)\cos\theta\sin\theta, \\ &f_r f_{rr}\cos\theta]\end{aligned} \tag{9}$$

(10)

and

$$\xi_y = \frac{1}{(1+f_r^2)^{3/2}}\ \ [f_{rr}\cos\theta\sin\theta - \frac{1}{r}(1+f_r^2)\cos\theta\sin\theta,$$

$$f_{rr}\sin^2\theta + \frac{1}{r}f_r(1+f_r^2)\cos^2\theta,$$
$$f_r f_{rr}\sin\theta]. \tag{11}$$

Substitute these into the equations (6) and (7), we have

$$\begin{aligned}\|\nabla\xi\|^2 &= \xi_x^2+\xi_y^2\\ &= \frac{1}{(1+f_r^2)^2}\left(f_{rr}^2+\frac{1}{r^2}f_r^2(1+f_r^2)\right)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\|\nabla\xi\wedge\nabla\xi\|^2 &= \|\xi_x^2\wedge\xi_y^2\|^2\\ &= \frac{f_r^2 f_{rr}^2}{r^2(1+f_r2)^3}.\end{aligned} \tag{13}$$

Hence, equation (1) reduces to:

$$Vol(\mathcal{F}) = 2\pi\int_0^1\sqrt{1+\frac{1}{r^2}\frac{f_r^2}{1+f_r^2)}+\frac{1}{(1+f_r^2)^3}(1+f_r^2+\frac{1}{r^2}f_r^2)f_{rr}^2 r\,dr} \tag{14}$$

where $f_r(0)=0$, and $f_r(1)=-\infty$.

This equation can be simplified as follows:

Let $v=\tan^{-1}(f_r)$. Then, it follows that

$$Vol(\mathcal{F}) = 2\pi\int_0^1\sqrt{\left(1+\frac{1}{r^2}\sin^2 v\right)(1+v_r^2)r dr} \tag{15}$$

and that $v(0)=0, v(1)=-\pi/2$.

Hence, the volume integrand is

$$F(r,v,v_r)=\sqrt{(r^2+\sin^2 v)(1+v_r^2)}. \tag{16}$$

Using the standard method of the calculus of variations, we will show that the volume functional, with the boundary values $v(0)=0, v(1)=-\pi/2$, is locally minimized at this function v. We know that a necessary condition for a differentiable functional to have an extremum is that its first variation vanishes, that is, if we write

$$v(s,r)=v_0(r)+sh(r)$$

where $h(0)=h(1)=0$, then the necessary condition is that

$$\left.\frac{d}{ds}\right|_{s=0}\int_0^1 F(r,v,v_r)dr=0. \tag{17}$$

Applying the standard techniques of calculus of variations to the integral (15), we will have the second-order ordinary differential equation:

$$0\left(-\cos(v_0(r))\sin(v_0(r))+\frac{dv_0(r)}{dr}\right)\left(1+\left(\frac{dv_0(r)}{dr}\right)^2\right)+(r^2+\sin^2(v_0(r)))\frac{d^2v_0(r)}{dr^2} \tag{18}$$

with boundary values $v_0(0)=0$ and $v_0(1)=-\pi/2$.

By using the computer algebra program Mapple V (release 3), it is easy to solve this equation, which is given below.

Maple Program.

$$eqn := (-\cos(v(r))*\sin(v(r))+r*diff(v(r),r))*(1+(diff(v(r),r))^2)+ (r\hat{}2+(\sin(v(r)))\hat{}2)*diff(v(r),\$2);$$

$$Order := 20;$$

$$dsolve(\{eqn=0, v(0)=0, D(v)(0)=a\}, v(r), series);$$

The solution to this is :

$$v(r)=ar-\frac{1}{12}a^3r^3+\frac{1}{80}a^5r^5-\frac{1}{448}a^7r^7+\frac{1}{2304}a^9r^9-\frac{1}{11264}a^{11}r^{11}+ \frac{1}{53248}a^{13}r^{13}-\frac{1}{245760}a^{15}r^{15}+\frac{1}{1114112}a^{17}r^{17}-\frac{1}{4980736}a^{19}r^{19}+O(r^{20}).$$

Factoring the coefficient of the series $v(r)$, we notice that this series is the Taylor series of 2. arctan $(ar/2)$, which can be substituted into equation (17) to show that it is an exact solution. Note here also that since $v(1)=-\pi/2$, it follows that

$$v(1)=2\arctan(a/2)=-\pi/2,$$

implying that $a=-2$. So, $v(r)=2\arctan(-r)$, or $f_r=u=\tan v=\tan(2\arctan(-r))$. Integrating this, we have

$$\begin{aligned} f(r) &= \int \tan(2\arctan(-r))dr \\ &= \log(1-r^2). \end{aligned} \tag{19}$$

This means that the solution of the symmetrized volume, reducing the variational problem to rotationally-invariant and translation-invariant foliations, is

$$F_0(x,y,z)=\log(1-x^2-y^2)-z=c \tag{20}$$

which we will proceed to show is a local minimizer of the full variational problem.

3.2. Main Theorem

Theorem 3.2.1. *Let $\mathcal{F}_0$ be the foliation of the flat solid torus $D^2 \times S^1$ with boundary $S^1 \times S^1$ as a leaf, given by the level surfaces of $F_0(x, y, z) = \log(1 - x^2 - y^2) - z$ in $D^2 \times [0, 1]$.$\mathcal{F}_0$ is a local minimizer of the volume.*

Proof. Note that, silicing the torus, S, by discs $S_z \cong D^2$ perpendicular to the z-direction, we can write

$$\int_S Int(\mathcal{F}) dx\, dy\, dz = \int_{S^1} \left(\int_{D^2} Int_z(\mathcal{F}|_{D^2}) dxdy \right) dz.$$

Above, $Int(\mathcal{F})$ is the integrand term of the equation (1), for the foliation $\mathcal{F}$ defined by the leel sets $F(x, y, z) = c$.

Here we show that, restricted to an arbitrary scile $D^2 \times \{z\} \subset D^2 \times I$, the functional

$$\int_{D^2} \sqrt{1 + \xi_x^2 + \xi_y^2 + (\xi_x \wedge \xi_y)^2} dx\, dy$$

is minimized by $\xi = \xi_0|_{S_t}$, where ξ_0 is the unit normal vector field to $\mathcal{F}_0$. Set $Int_z(\mathcal{F}|_{D^2})$ to be the restriction of $Int(\mathcal{F})$ to the scile S_z. That is,

$$Int_z(\mathcal{F}|_{D^2}) = \sqrt{1 + \|\nabla_{e_1}\xi\|^2 + \|\nabla_{e_2}\xi\|^2 + \nabla_{e_1}\xi \wedge \nabla_{e_2}\xi\|^2}.$$

Note that

$$Int_z(\mathcal{F}|_{D^2}) \leq Int(\mathcal{F})|_{D^2}$$

for all z, so that condition (ii) of Theorem 2.1 holds. Hence, we can apply Theorem 2.1 with the weighting function $w(z, r) = 1$.

We define the first variational problem on the scile $S_z \cong D^2$ as follows: we consider variations of the sort

$$F(s, x, y, z) = \log(1 - x^2 - y^2) + sh(x, y) - z.$$

The unit normal vector ξ at each s is

$$\xi = \frac{(F_x, F_y, F_z)}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = \frac{\left(\frac{-2x}{1-x^2-y^2} + sh_x, \frac{-2y}{1-x^2-y^2} + sh_y, -1 \right)}{\sqrt{\left(\frac{-2x}{1-x^2-y^2} + sh_x \right)^2 + \left(\frac{-2y}{1-x^2-y^2} + sh_y \right)^2 + 1}}.$$

Notice that ξ is independent of z. So, obviously $\nabla_{e_3}\xi = \xi_z = \xi_{zs} = 0$. Hence the variational problem on the disk is

$$0 = \frac{d}{ds}\bigg|_{s=0} \int_{D^2} Int_z(\mathcal{F}) dxdy$$

$$= \frac{d}{ds}\bigg|_{s=0} \int_{D^2} \sqrt{1+\xi_x^2+\xi_y^2+(\xi_x\wedge\xi_y)^2}dxdy$$
$$= \int_{D^2} \frac{d}{ds}\bigg|_{s=0} \sqrt{1+\xi_x^2+\xi_y^2+(\xi_x\wedge\xi_y)^2}dxdy$$
$$= \int_{D^2} \frac{(\langle\xi_x,\xi_{xs}\rangle+\langle\xi_y,\xi_{ys}\rangle+\langle\xi_x\wedge\xi_y,\xi_{xs}\wedge\xi_y\rangle+\langle\xi_x\wedge\xi_y,\xi_x\wedge\xi_{ys}\rangle)|_{s=0}}{\sqrt{1+\xi_x^2+\xi_y^2+(\xi_x\wedge\xi_y)^2|_{s=0}}}dxdy.$$

When restricted to a slice $S_z \cong D^2, h$ and $\mathcal{F}$ can be viewed as functions of x and y. Then, we have

$$\begin{aligned} 0 &= \frac{d}{ds}\bigg|_0 \int_{D^2} Int_z(F(s,x,y))dxdy \\ &= \int_{D^2} \frac{d}{ds}\bigg|_0 Int_z(F(s,x,y))dxdy \\ &= \int_{D^2} (C_1h_x+C_2h_y+C_3h_{xx}+C_4h_{xy}+C_5h_{yy})dxdy, \end{aligned} \tag{21}$$

where

$$\begin{aligned} C_1 &= -\frac{24x(x^2-1+y^2)}{(1+x^2+y^2)^4} \\ C_2 &= -\frac{24y(x^2-1+y^2)}{(1+x^2+y^2)^4} \\ C_3 &= -\frac{(2x-2+2y^2)(x^2-1-y^2)}{(1+x^2+y^2)^3} \\ C_4 &= -\frac{8xy(x^2-1+y^2)}{(1+x^2+y^2)^3} \\ C_5 &= -\frac{(2x^2-2+2y^2)(1+x^2-y^2)}{(1+x^2+y^2)^3}. \end{aligned}$$

Using integration by parts twice, we will need to have, for the first variational problem,

$$(C_1)_x+(C_2)_y-(C_3)_{xx}-(C_4)_{xy}-(C_5)_{yy}=0,$$

which is straightforward computation to verify [4]. So,

$$\frac{d}{ds}\bigg|_{s=0} Vol(\mathcal{F})=0,$$

as required.

Now we need to show that the second variation of the volume functional is strictly positive, ie.,

$$\frac{d^2}{ds^2}\bigg|_{s=0} Vol(\mathcal{F}) > 0.$$

By a direct calculation, which was carried out with the assistance of a the computer-algebra program Maple V (release 3), the second variation of this function at this point has the following integrand form (Note that since the calculation of the integrand term is so long, we give only the output of the calculation and coefficients. For more details, we refer the reader to [4]):

$$\frac{d^2}{ds^2}\bigg|_{s=0} Vol(\mathcal{F}) = \int_{D^2} INT dxdy, \tag{22}$$

where

$$\begin{aligned} INT = {} & A_1 h_x h_y + A_2 h_x h_{xx} + A_3 h_x h_{xy} + A_4 h_x h_{yy} + A_5 h_x^2 + \\ & B_2 h_y h_{xx} + B_3 h_y h_{xy} + B_4 h_y h_{yy} + B_5 h_y^2 + C_3 h_{xx} h_{xy} + \\ & C_4 h_{xx} h_{yy} + C_5 h_{xx}^2 + D_3 h_{xy} h_{yy} + D_5 h_{yy}^2 + E_5 h_{xy}^2 \end{aligned} \tag{23}$$

with the coefficients

$$\begin{aligned} A_1 := {} & 32xy(-124x^2 - 12x^4y^2 + 22x^4 - 124y^2 + 75 - 4x^6 + 15y^8 - 4y^6 + \\ & 22y^4 + 60x^6y^2 + 15x^8 + 60x^2y^6 + 90y^4x^4 - 12y^4x^2 + \\ & 44x^2y^2)\frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^6} \end{aligned}$$

$$\begin{aligned} A_2 := {} & 16x(-8 + 2x^4y^2 - 6x^2y^2 - 2x^2y^6 - 6y^4x^4 - 2y^4x^2 - 6x^2y^2 + \\ & 7x^8y^2 + 8x^6y^4 + 2y^6x^4 + 20x^2 - 14x^4 + 3y^2 + 2x^6 - 2y^6 + 8y^4 - \\ & 2x^8 + 2x^{10} - y^{10}\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)} \end{aligned}$$

$$\begin{aligned} A_3 := {} & 16y(6x^{10} + 24x^8y^2 - 3x^8 + 36x^6y^4 - 8x^6y^2 + 10x^6 - 44x^4 - \\ & 6y^4x^4 + 24y^6x^4 + 22x^4y^2 + 6y^8x^2 - 44x^2y^2 + 32x^2 + 14y^4x^2 + 2y^6 - 1 - \\ & 2y^2 + y^8\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)} \end{aligned}$$

$$\begin{aligned} A_4 := {} & -16x(7 + 8x^4y^2 + 6x^4y^2 + 10x^2y^6 + 12y^4x^4 + 4y^4x^2 + 6x^2y^2 + \\ & 2x^8y^2 - 2x^6y^4 - 8y^6x^4 - 7y^8x^2 - 5x^2 - 8x^4 - 22y^2 + 4x^6 + 3y^8 + 14y^4 + \\ & +^8 + x^{10} - 2y^{10})\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)} \end{aligned}$$

$$A_5 := -(60 - 1192x^2 + 1812x^2 + 8y^2 - 72y^4 + 24y^4x^4 - 1136x^4y^2 - 784y^4x^2 +$$

$$240x^6y^6 + 72y^{10}x^2 + 180y^8x^4 - 200y^8x^2 + 180x^8y^4 - 1360x^6y^4 - 880y^6x^6 + 72x^{10}y^2 - 920x^8y^2 - 48x^2y^6 + 1640x^2y^2 + 8y^{10} + 12y^{12} + 36x^8 - 496x^6 - 144y^6 + 12x^{12} - 232x^{10} - 28y^8 + 80x^6y^2)\frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^6}$$

$$B_2 := 16y(-7 - 4x^4y^2 - 10x^6y^2 - 6x^2y^6 - 12y^4x^4 - 8y^4x^2 - 6x^2y^2 + 7x^8y^2 + 8x^2y^4 + 2y^6x^4 - 2y^8x^2 + 22x^2 - 14x^4 + 5y^2 - y^8 - 4y^6 + 8y^4 - 3x^8 + 2x^{10} - y^{10})\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)}$$

$$B_3 := 16x(x^2x^2 - 1 - 3y^8 + x^8 - 6y^4x^4 + 2x^6 + 32y^2 - 44x^2y^2 - 44y^4 + 6x^8y^2 + 24y^8x^2 + 6y^{10} + 24x^6y^4 + 36y^6x^4 + 10y^6 + 22y^4x^2 + 14x^4y^2 - 8x^2y^6)\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)}$$

$$B_4 := -16y(8 + 2x^4y^2 + 2x^6y^2 + 6x^2y^6 + 6y^4x^4 - 2y^2x^2 + 6x^2y^2 + 2x^2y^2 - 2x^6y^4 - 8y^6x^4 - 7y^8x^2 - 3x^2 - 8x^4 - 20y^2 + 2x^6 + 2y^8 - 2y^6 + 14y^4 + x^{10} - 2y^{10})\frac{1}{(1 + x^2 + y^2)^5(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)}$$

$$B_5 := -(60 + 8x^2 - 172x^4 - 1192y^2 + 1812y^4 + 24y^4x^4 - 784x^4y^2 - 1136y^4x^2 + 240x^6y^6 + 72y^{10}x^2 + 180y^8x^4 - 920y^8x^2 - 880x^6y^4 - 1360y^6x^4 + 72x^{10}y^2 - 200x^8y^2 + 80x^2y^6 + 1640x^2y^2 - 232y^{10} + 12y^{12} - 28x^8 - 144x^6 - 496y^6 + 12x^{12} + 8x^{10} + 36y^8 - 48x^6y^2)(\frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^6}$$

$$C_3 := \frac{8xy81 - 2x^2 - 2y^2 + x^4 + 2x^2y^2 + y^4)}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^2}$$

$$C_4 := \frac{8x^6 - 24x^4 + 24x^4y^2 - 48x^2y^2 + 24y^4x^2 + 24x^2 - 8 + 8y^6 - 24y^4 + 24y^2}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^3}$$

$$C_5 := (x^8 - 4x^6 + 4x^6y^2 - 8x^4y^2 + 6y^4x^4 + 6x^4 + 4x^2y^2 - 4y^2x^2 - 4x^2 + 4x^2y^6 + 1 + y^8 - 2y^4)\frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^2}$$

$$D_3 := -\frac{8xy(1 - 2x^2 - 2y^2 + x^4 + 2x^2y^2 + y^4)}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^2}$$

$$D_5 := (x^8 + 4x^6y^2 + 6y^4x^4 - 2x^4 - 4x^4y^2 + 4x^2y^2 - 8y^4x^2 + 4x^2y^6 + 1 + 6y^4 - 4y^2 - 4y^6 + y^8\frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^2}$$

$$E_5 := (2x^{10} + 10^8y^2 - 2x^8 + 8x^6 + 20x^6y^4 - 8x^6y^2 + 24x^4y^2 - 24x^4 + 20y^6x^4 -$$

$$12y^4x^4 - 48x^2y^2 + 24y^4x^2 + 22x^2 + 10y^8x^2 - 8x^2y^6 + 22y^2 + 2y^{10} -$$
$$24y^4 - 6 + 8y^6 - 2y^n \frac{1}{(5 + 2x^2 + x^4 + 2y^2 + 2x^2y^2 + y^4)(1 + x^2 + y^2)^3}.$$

Then, using integration by parts, we get

$$\int A_2 h_x h_{xx} dx\,dy = -\frac{1}{2}\int (A_2)_x h_x^2 dx\,dy,$$
$$\int B_2 h_y h_{xx} dx\,dy = -\int ((B_2)_x h_x h_y + B_2 h_x h_{xy}) dx\,dy,$$
$$\int B_3 h_y h_{xy} dx\,dy = -\int ((B_3)_y h_x h_y + B_3 h_x h_{yy}) dx\,dy,$$
$$\int B_4 h_y h_{yy} dx\,dy = -\frac{1}{2}\int (B_4)_y h_y^2 dx\,dy,$$
$$\int C_4 h_{xx} h_{yy} dx\,dy = \int (-(C_4)_x h_x h_{yy} + (C_4)_y h_x h_{xy} + C_4 h_{xy}^2) dx\,dy.$$

Substituting these into (22), it follows that

$$\left.\frac{d^2}{ds^2}\right|_{s=0} Vol(\mathcal{F}) = \int_{D^2} INT1 dx\,dy,$$

where

$$INT1 = \alpha h_x^2 + \alpha_1 h_y^2 + \alpha_2 h_x h_y + \alpha_3 h_x h_{xy} + \alpha_4 h_x h_{yy} + \gamma h_{xy}^2 + C_5 h_{xx}^2 + D_5 h_{yy}^2 + C_3 h_{xx} h_{xy} + D_3 h_{xy} h_{yy} \tag{24}$$

with

$$\alpha = A_5 - \frac{1}{2}(A_2)_x,$$
$$\alpha_1 = B_5 - \frac{1}{2}(B_4)_y,$$
$$\alpha_2 = A_1 - (B_2)_x - (B_3)_y,$$
$$\alpha_3 = A_3 + (C_4)_y - B_2,$$
$$\alpha_4 = A_4 - (C_4)_x - B_3,$$
$$\gamma = C_4 + E_5.$$

Applying integration by parts once more for α_4, we have

$$\int \alpha_4 h_x h_{yy} dx\,dy = \int (-(\alpha_4)_y h_x h_y + \frac{1}{2}(\alpha_4)_x h_y^2) dx\,dy.$$

Define

$$\beta = \alpha_1 + \frac{1}{2}(\alpha_4)_x,$$
$$\beta_1 = \alpha_2 - (\alpha_4)_y,$$
$$\delta = C_5,$$
$$\mu = D_5.$$

Thus, (23) becomes

$$INT1 = \alpha h_x^2 + \beta h_y^2 + \beta_1 h_x h_y + \alpha_3 h_x h_{xy} + \gamma h_{xy}^2 + \quad (25)$$
$$\delta h_{xx}^2 + \mu h_{yy}^2 + C_3 h_{xx} h_{xy} + D_3 h_{xy} h_{yy}. \quad (26)$$

The new coefficients are as follows:

$$\alpha := (20 + 2248x^2y^6 + 3112y^4x^4 + 4800y^4x^2 - 8872x^2y^2 + 2744x^8y^2 + 8704x^6y^4 + 9168y^6x^4 + 3440y^8x^2 + 432x^2 - 256x^4 + 168y^2 - 432x^6 + 496y^8 + 680y^2 + 488y^4 + 104x^8 + 1464x^6y^2 + 3688x^4y^2 - 48x^{10} + 184y^{10} + 96x^{12} + 24y^{12} + 632x^8y^4 + 528x^6y^6 + 272y^8x^4 + 104y^{10}x^2 + 392x^{10}y^2 + 312x^{12}y^2 + 832x^{10}y^4 + 1160x^8y^6 + 880x^6y^8 + 48x^{14} - 8y^{14} + 36x^{16} - 4y^{16} + 328y^{10}x^4 + 32y^{12}x^2 + 104x^{14}y^2 - 136x^{12}y^4 - 696y^{12}x^4 - 136y^{14}x^2 - 98x^{10}y^6 - 1760x^8y^8 - 1544x^6y^{10})\frac{1}{(1+x^2+y^2)^6(5+2x^2+x^4+2y^2+2x^2y^2+y^4)^2}$$

$$\beta := 4(x^2 - 1 + y^2)(-45 - 764x^2y^6 - 1434y^4x^4 - 787y^4x^2 - 346x^2y^2 - 195x^8y^2 - 430x^6y^4 - 470y^6x^4 - 255y^8x^2 - 121x^2 - 421x^4 + 267y^2 - 617x^6 - 143y^8 - 85y^6 + 75y^4 - 335x^8 - 1148x^6y^2 - 1319x^4y^2 - 35x^{10} - 55y^{10} + 33x^{12} - 15y^{12} + 255x^8y^4 + 180x^6y^6 + 15y^8x^4 - 42y^{10}x^2 + 150x^{10}y^2 + 31x^{12}y^2 + 81x^{10}y^4 + 115x^8y^6 + 95x^6y^8 + 5x^{14} + y^{14} + 45y^{10}x^4 + 11y^{12}x^2)\frac{1}{(1+x^2+y^2)^4(5+2x^2+x^4+2y^2+2x^2y^2+y^4)^3}$$

$$\beta_1 := -32xy(x^2 - 1 + y^2)(-97 + 20x^2y^6 + 30y^4x^4 - 144y^4x^2 - 266x^2y^2 + 60x^8y^2 + 120x^6y^4 + 120y^6x^4 + 60y^8x^2 - 124x^2 - 133x^4 - 124y^2 - 48x^6 + 5y^8 - 48y^6 - 133y^4 + 5x^8 + 20x^6y^2 - 144x^4y^2 + 12x^{10} + 12y^{10} + x^{12} + y^{12} + 15x^8y^4 + 20x^6y^6 + 15y^8x^4 + 6y^{10}x^2 + 6x^{10}y^2)\frac{1}{(1+x^2+y^2)^4(5+2x^2+x^4+2y^2+2x^2y^2+y^4)^3}$$

$$\alpha_3 := 16y(x^2 - 1 + y^2)(2 + 86x^2y^6 + 162y^4x^4 + 52y^4x^2 + 17x^2y^2 + 63x^8y^2 + 112x^6y^4 + 98y^6x^4 + 42y^8x^2 - 78x^2 + 4x^4 + 7y^2 + 16x^6 + 16y^8 + 18y^6 + 13y^4 +$$

$$38x^8 + 130x^6y^2 + 50x^4y^2 + 14x^{10} + 7y^{10} + 4x^{12} + y^{12} + 45x^8y^4 + 50x^6y^6 + 30y^8x^4 + 9y^{10}x^2 + 21x^{10}y^2)\frac{1}{(1+x^2+y^2)^5(5+2x^2+x^4+2y^2+2x^2y^2+y^4)^2}$$

$$\gamma := 2\frac{(x^4+2x^2y^2+y^4+1)(x^2-1+y^2)^2}{(1+x^2+y^2)^2(5+2x^2+x^4+2y^2+2x^2y^2+y^4)}$$
$$\delta := \frac{(x^2-2x+1+y^2)(x^2+2x+1+y^2)(x^2-1+y^2)^2}{(1+x^2+y^2)^2(5+2x^2+x^4+2y^2+2x^2y^2+y^4)}$$
$$\mu := \frac{(x^2+1+2y+y^2)(x^2+1-2y+y^2)(x^2-1+y^2)^2}{(1+x^2+y^2)^2(5+2x^2+x^4+2y^2+2x^2y^2+y^4)}.$$

It is not easy to see that these are all positive. However, if at some point (x, y) any of the above is negative, by the rotation-invariance of $f(x, y)$, we can rotate the coordinates so that the point has coordinates $(x, 0)$.

Notice that at $(x, 0)$,

$$\beta_1(x,0) = \alpha_3(x,0) = C_3(x,0) = D_3(x,0) = 0.$$

Hence, at $(x, 0)$, the integrand term is

$$INT1(x,0) = \alpha h_x^2 + \beta h_y^2 + \gamma h_{xy}^2 + \delta h_{xx}^2 + \mu h_{yy}^2$$

where

$$\alpha(x,0) = \frac{(36x^{10}+84x^8+216x^6+216x^4+452x^2+20)(x-1)^2(x+1)^2}{(x^2+1)^5(5+2x^2+x^4)^2}$$
$$\beta(x,0) = \frac{(20x^{12}+112x^{10}-252x^8-1088x^6-1380x^4-304x^4-180)(x^2-1)}{(5+2x^2+x^4)^3(x^2+1)^3}$$
$$\gamma(x,0) = \frac{(2x^4+2)(x-1)^2(x+1)^2}{(5+2x^2+x^4)(x^2+1)^2}$$
$$\delta(x,0) = \frac{(x-1)^4(x+1)^4}{(5+2x^2+x^4)(x^2+1)^2}$$
$$\mu(x,0) = \frac{(x-1)^4(x+1)^2}{5+2x^2+x^4}.$$

It is easy to recognize that

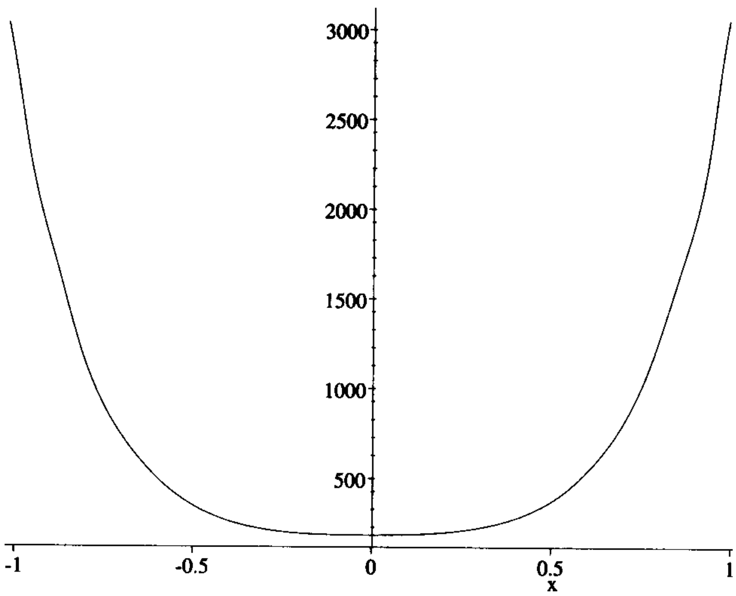
$$\alpha(x,0) > 0,$$
$$\gamma(x,0) > 0,$$
$$\delta(x,0) > 0,$$
$$\mu(x,0) > 0.$$

Note that $\beta(x,0) > 0$ follows from the fact that the numerator $\phi(x)$ of β satisfies:

$$\phi(x) = -(20x^2 + 112x^{10} - 252x^8 - 1088x^6 - 1380x^4 - 304x^2 - 180) > 0,$$

for $x \in [-1,1]$.

In particular, the following graph of $\phi(x)$ illustrates this fact; and it is easy to see that $\phi'(x) = 0$ only when $x = 0$



THE GRAPH OF ϕ .

This implies that the integrand term,

$$INT1(x,0) > 0.$$

It follows that the second variation is strictly positive, i.e.,

$$\left.\frac{d^2}{ds^2}\right|_{s=0} Vol(\mathcal{F}) > 0,$$

as desired. □

In [5], we consider the analogous question on the round 3-sphere, and find a locally volume-minimizing codimension-one foliation of S^3 consisting of two Reeb components, with common boundary the Clifford torus $S^1 \times S^1 \subset S^3$. Surprisingly, however, in that case the foliation is not C^2 at either the Clifford torus or the core circles of the two Reeb components. In contrast, it is easy to see that the local minimizer here is C^∞ at the boundary.

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