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PULLBACKS OF CROSSED MODULES AND CAT^1-GROUPS

Murat Alp

Abstract

In this paper, wer define the pullback cat^1-groups and we showed that the category of bullback cat^1-group is equivalent to the category of pullback crossed modules.

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1. Introduction

Crossed modules are usefully regarded as 2-dimensional forms of groups. They were introduced by J. H. C. Whitehead in [8], and have powerful topological applications [3, 4, 5, 9]. Loday in [5] showed that the category of crossed modules is equivalent to that of cat^1-groups. We implemented crossed modules and cat^1-groups structures to the computed using the group theory language GAP [6] as a package in [7]. We also enumerated cat^1-groups of low order and group order 41-47 in [2] and [1] using this program package XMOD.

Our aim is to define pullback cat^1-groups and to show that the equivalence between cat^1-groups and crossed modules due to Loday [5] takes pullback cat^1-groups to the pullback crossed modules defined by Brown and Higgins in [3].

2. Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of two equivalent categories: The category of crossed modules and their morphisms; and the category of cat1-groups and their morphisms.

A crossed module $\chi = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary of χ, together with an action $\alpha : R \to Aut(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,*

$$\begin{aligned} \mathbf{XM1}: \quad \partial(s^r) &= r^{-1}(\partial s)r \\ \mathbf{XM2}: \quad s^{\partial s'} &= s'^{-1}ss'. \end{aligned}$$

The standard examples of crossed modules are:

1. *Any homomorphism $\partial : S \to R$ of abelian groups with R acting trivially on S may be regarded as a crossed module.*
2. *A conjugation crossed module is an inclusion of a normal subgroup $S \trianglelefteq R$, where R acts on S by conjugation.*
3. *A central extension crossed module has as boundary a surjection $\partial : \partial^{-1}r$.*
4. *An automorphism crossed module has as its range a subgroup R of the automorphism group Aut (S) of S which contains the inner automorphism group of S. The boundary maps $S \in S$ to the inner automorphism of S by s.*
5. *An R-Module crossed module has an R-module as source and ∂ as the zero map.*
6. *The direct product $\chi_1 \times \chi_2$ of two crossed modules has source $S_1 \times S_2$, range $R_1 \times R_2$ and boundary $\partial_1 \times \partial_2$, with R_1, R_2 acting trivially on S_2, S_1 respectively.*
7. *An important motivating topological example of crossed module due to Whitehead [8] is the boundary $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ from the second relative homotopy group of a based pair (X, A, x) of topological spaces, with the usual action of the fundamental group $\pi_1(A, x)$.*

A morphism between two crossed modules $\chi = (\partial : S \to R)$ *and* $\chi' = (\partial' : S' \to R')$ *is a pair* (σ, ρ), *where* $\sigma : S \to S'$ *and* $\rho : R \to R'$ *are homomorphisms satisfying*

$$\partial'\sigma = \rho\partial,\ \delta(s^{\tau}) = (\delta s)^{\rho\tau}.$$

In [5], Loday reformulated the notion of a crossed modules as a cat^1*-group, namely a group* G *with a pair of homomorphisms* $t, h : G \to G$ *having a common image* R *and satisfying certain axioms. We find it convenient to define a cat*^{1}*-group* $\mathcal{C} = (e; t, h : G \to R)$ *as a group* G *with two surjections* $t, h : G \to R$ *and an embedding* $e : R \to G$ *satisfying:*

$$\begin{array}{ll} \mathbf{CAT1}: & te = he = id_R \\ \mathbf{CAT2}: & [ker t, ker h] = \{1_G\}. \end{array}$$

The maps t, h *are often called to as the source and target, but we choose to call them tail and head of* $\mathcal{C}$, *because source is the GAP term for the domain of a function. A morphism* $\mathcal{C} \to \mathcal{C}'$ *of cat*^{1}*-groups is a pair* (γ, ρ) *where* $\gamma : G \to G'$ *and* $\rho : R \to R'$ *are homomorphisms satisfying*

$$h'\gamma = \rho h,\ t'\gamma = \rho t,\ e'\rho = \gamma e.$$

3. Pullback crossed modules

Let $\chi = (\partial : S \to R)$ *be a crossed R-module and* $\imath : Q \to R$ *be a morphism of groups. Then* $\imath^*\chi = (\partial^\bullet : \imath S \to Q)$ *is the pullback of* χ *by* $\imath$, *where* $\imath^*S = \{(q,s)| \in Q \times S | \imath q = \partial_s\}$ *and* $\partial^\bullet(q,s) = q$. *The action of* Q *on* $\imath^{**}S$ *is given by*

$$(q_1, s)^q = (q^{-1}q_1q, s^{\imath q}). \tag{0.1}$$

The verification of the crossed module axioms is given in [4] as follows

XM1

$$\begin{aligned} \partial^\bullet((q,s)^{q'}) &= \partial^\bullet(q^{q'}, s^{\imath q'}) \\ &= q^{q'} \end{aligned}$$

$$
\begin{aligned}
&= q'^{-1} q q' \\
&= q'^{-1} \partial^{\bullet}(q,s) q'
\end{aligned}
$$

XM2

$$
\begin{aligned}
(q',s')^{-1}(q,s)(q',s') &= (q'^{-1}, s'^{-1}(q,s)(q',s') \\
&= (q'^{-1} q q', s'^{-1} s s') \\
&= (q^{q'}, s^{\partial s'}) \\
&= (q,s)^{q'} \\
&= \Big((q,s)^{\partial^{\bullet} q'}, s'\Big)
\end{aligned}
$$

where $(q,s),(q',s') \in \imath^* S$.

The universal property of induced crossed modules is the following: Let $\chi = (\mu : M \to Q)$ *be a crossed module and let* $\imath^{**}\chi = (\delta : \imath_{**}M \to R)$ *be induced by the homomorphism* $\imath : Q \to R$. *In the diagram*

$$
\begin{tikzcd}
 & M \arrow[r, "\mu"] \arrow[d, "\bar{\imath}"] \arrow[ddl, "f"'] & Q \arrow[d, "\imath"] \\
 & \imath_{**}M \arrow[r, "\delta"] \arrow[dl, "g"] & R \\
C \arrow[urr, "\gamma"'] & &
\end{tikzcd}
$$

the pair $(\bar{\imath}, \imath)$ *is a morphism of crossed modules such that for any crossed R-module* $\mathcal{Y} = (\gamma : C \to R)$ *and any morphism of crossed modules* $(f, \imath) : \chi \to \mathcal{Y}$, *there is a unique morphism* $(g, 1) : \imath_{**}\chi \to \mathcal{Y}$ *of crossed R- modules such that* $g\bar{\imath} = f$.

4. Pullback Cat^1-groups

A pullback cat^1-group is defined as follows.

$$
\begin{tikzcd}
\imath^{**}G \arrow[r, "h^{**}"] \arrow[d, "t^{**}"] \arrow[dr, "\pi"] & Q \arrow[l, bend right, "e^{**}"'] \arrow[dr, "\imath"] & \\
Q \arrow[u, bend left, "e^{**}"] \arrow[dr, "\imath"] & G \arrow[r, "h"] \arrow[d, "t"] & R \arrow[l, bend right, "e"'] \\
 & R \arrow[u, bend left, "e"] &
\end{tikzcd}
$$

Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-group and let $\imath : Q \to R$ be a group homomorphism. Define $e^{**}; t^{**}, h^{**} : \imath^{**}G \to Q$ to be the pullback of G where

$$\imath^{**}G = \{(q_1, g, q_2) \in Q \times G \times | \imath q_1 = tg, \imath q_2 = hg\},$$

$t^{**}(q_1, g, q_2) = q_1, h^{**}(q_1, g, q_2) = q_2$ and $e^{**}(q) = (q, e\imath q, q)$. Multiplication in $\imath^{**}G$ is componentwise. The pair $(\pi, \imath)$ is a morphism of cat^1-groups where $\pi : \imath^{**}G \to G, (q_1, g, q_2) \mapsto g$.

We now verify the cat^1-group axioms:

$$
\begin{aligned}
t^{**}e^{**}(q) &= t^{**}(q, e\imath q, q) = q, \\
h^{**}e^{**}(q) &= h^{**}(q, e\imath q, q) = q.
\end{aligned}
$$

So $t^{**}e^{**} = h^{**}e^{**} = id_q$ and CAT1 is satisfied.

To prove CAT2, suppose $a = (q'_1, g_1, q_1) \in ker\ t^{**}, b = (q_2, g_2, q'_2) \in ker\ h^{**}$. Then $q'_1, = q'_2 = 1$ so, by the definition of $\imath^{**}$, we have $g_1 \in ker\ t, g_2 \in ker\ h$. Then $[a, b] = (1_Q, [g_1 g_2], 1_Q) = (1_Q, 1_G, 1_Q)$.

Proposition 4.1 *If $\imath^*\chi$ is the pullback of the crossed module χ over $\imath : Q \to R$ and if $\mathcal{C}, \mathcal{D}$ are the cat^1-groups obtained from $\chi, \imath^*\chi$ respectively, then $\mathcal{D} \cong \imath^{**}\mathcal{C}$.*

Proof.

$$\begin{tikzcd} \iota^* S \arrow[r] \arrow[d, "\partial^\bullet"'] & S \arrow[d, "\partial"] \\ Q \arrow[r, "\iota"'] & R \end{tikzcd}$$

Starting with the pullback crossed module $\imath^*\chi = (\partial^\bullet : \imath^* S \to Q)$, the source group of $\mathcal{D}$ is defined as the semi-direct product $Q \times \imath^* S$.

$$\begin{tikzcd} Q \ltimes \iota^* S \arrow[r] \arrow[d, shift right, "t^\bullet"'] \arrow[d, shift left, "h^\bullet"] & R \ltimes S \arrow[d, shift right, "t"'] \arrow[d, shift left, "h"] \\ Q \arrow[r, "\iota"'] & R \end{tikzcd}$$

The tail, head and embedding of $\mathcal{D}$ are respectively given by

$$\begin{aligned} t^\bullet(q', (q, s)) &= q' \\ h^\bullet(q', (q, s)) &= q' \partial^\bullet(q, s) \\ &= q' q \\ e^\bullet(q) &= (q, (1_Q, 1_S)) \end{aligned}$$

We define an isomorphism of cat^1-groups $\psi, id_Q : \mathcal{D} \to \imath^{**}\mathcal{C}$,

$$\begin{tikzcd} Q \ltimes \iota^* S \arrow[r, "\psi"] \arrow[d, shift right, "t^\bullet"'] \arrow[d, shift left, "h^\bullet"] & \iota^{**}(R \ltimes S) \arrow[d, shift right, "t^{**}"'] \arrow[d, shift left, "h^{**}"] \\ Q \arrow[r, "id"'] \arrow[u, bend left=60, "e^\bullet"] & Q \arrow[u, bend right=60, "e^{**}"'] \end{tikzcd}$$

where

$$\psi(q', (q, s)) = (q', (\imath q', s), q'q).$$

First note that $\psi(q', (q, s)) \in \imath^{**}(R \times S)$. Because

$$t(\imath q', s) = \imath q'$$

and

$$h(\imath q', s) = (\imath q')(\partial s) = (\imath q')(\imath q) = t(q'q).$$

We verify that ψ is a homomorphism as follows:

$$\begin{array}{rcl}
\psi((q_1', s_1))((q_2', (q_2, s_2 & = & \psi(q_1'q_2', (q^{q_2'}q_2, s^{\imath q_2'}s_2)) \\
& = & \psi(q_1'q_2', (\imath(q_1'q_2'), s^{\imath q_1'}s_2), q_1'q_1q_2'q_2) \\
\psi(q_1', (q_1', s_1))\psi(q_2', (q_2', s_2)) & = & (q_1', (\imath q_1', s_2), q_1'q_1)(q_2', (\imath q_2', s_2), q_2'q_1) \\
& = & (q_1'q_2', (\imath q_1', s_2)(\imath q_2', s_2), q_1'q_1q_2'q_2) \\
& = & (q_1'q_2', ((\imath q_1')(\imath q_2'), s^{\imath q_2'}s_2), q_1'q_1q_2'q_2)
\end{array}$$

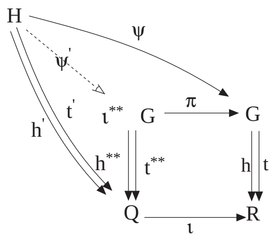
The inverse of ψ is given by $\psi^{-1}(q_1, (r, s), q_2) = (q_1, (q_1^{-1}q_2, s))$.

Then

$$\begin{array}{rcl}
t^{**}\psi(q', (q, s)) & = & t^{**}(q', (\imath q', s), q'q) \\
& = & q' \\
& = & t^{\bullet}(q', (q, s)), \\
h^{**}\phi(q', (q, s)) & = & h^{**}(q', (\imath q', s), q'q) \\
& = & q'q \\
& = & h^{\bullet}(q', (q, s)), \\
\psi e^{\bullet}(q) & = & \psi(q, (1_Q, 1_S)) \\
& = & (q, (\imath q, 1_s), q) \\
& = & e^{**}(q),
\end{array}$$

so the diagram commutes and the proof is complete. $\square$

The universal property of induced cat^1*-group is the following. Let* $\mathcal{C} = (e; t, h : G \to R)$ *be a cat*^{1}*-group and let* $\imath^{**}\mathcal{C} = (e^{**}; t^{**}, h^{**} : \imath^{**}G \to Q)$ *be induced by the homomorphism* $\imath : Q \to R$ *is given by the diagram*



the pair $(\pi, \imath)$ *is a morphism of cat*^{1}*-group such that for any cat*^{1}*-group* $\mathcal{H} = (e'; t', h' : H \to Q)$ *and any morphism of cat*^{1}*-group* $(\phi, \imath) : \mathcal{C} \to \mathcal{H}$ *there is a unique morphism* $((\psi', 1) : \imath^{**}\mathcal{C} \to \mathcal{H})$ *of cat*^{1}*-groups such that* $\pi\psi' = \psi$*.*

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Murat ALP
School of Mathematics
University of Wales, Bangor
Gwynedd, LL57 1UT, UK.
email: map006@bangor.ac.uk

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