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A NEW INTEGRABLE REDUCTION OF THE COUPLED NLS EQUATION

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Abstract

The method of multiple scales is used to derive a new integrable coupled nonlinear Schrödinger equation (CNLS) as an amplitude equation from the coupled nonlinear Klein-Gordon Equation (CNKG). We also give the corresponding spectral problem and further reduce the derived equation into a finite dimensional integrable Hamiltonian system. Finally the integrability of the reduced system is deduced by using a perturbation analysis.
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1. Introduction

In references [2, 3] Fordy and Gibbons introduced a two component nonlinear Klein-Gordon equation as the system of equations

$$\begin{aligned} u_{tt} - u_{xx} &= -e^{2u} + e^{-u}\cosh 3v, \\ v_{tt} - v_{xx} &= -e^{-u}\sinh 3v, \end{aligned} \tag{1.1}$$

where subscribes indicate partial differentiation with respect to some variables. They considered the 3×3 scattering problem:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_\xi = \begin{pmatrix} u_\xi + v_\xi & \lambda & 0 \\ 0 & -2v_\xi & \lambda \\ \lambda & 0 & -u_\xi + v_\xi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{1.2}$$

and the time evolution of $\mathbf{\Phi} = (\phi_1,\, \phi_2,\, \phi_3)^T$:

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}_\tau = \frac{1}{\lambda}\begin{pmatrix}0 & 0 & e^{2u}\\ e^{-u-3v} & 0 & 0\\ 0 & e^{3v-u} & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \tag{1.3}$$

then obtained, as integrability conditions

$$\begin{aligned} u_{\xi\tau} =& \quad e^{2u} - e^{-u}\cosh 3v,\\ v_{\xi\tau} =& \quad e^{-u}\sinh 3v, \end{aligned} \tag{1.4}$$

where we used the laboratory coordinates $x = \xi + \tau$, and $t = \xi - \tau$ for the above system in order to get the system (1.1).

A particular case of the system (1.1) is

$$u_{tt} - u_{xx} = -e^{2u} + e^{-u} \tag{1.5}$$

by retaining u. This is an example of a single component nonlinear Klein-Gordon equation, known as Dodd-Bullough (DB) equation.

It is well known that a multiple scales analysis of the Sine-Gordon equation (and, indeed many other equation) leads to the NLS equation for the modulated amplitude. In this paper, we use the method of multiple scales [7] to derive a coupled NLS equation from the system (1.1) and give the corresponding spectral problem to show its integrability. We also reduce the derived system into a four-degrees of freedom Hamiltonian system. Then we prove its complete integrability by a perturbation method [5, 8].

In section 2 we present the multiple scales method to derive a new integrable coupled NLS equation. We then give the corresponding spectral problem and Hamiltonian formulation. Sections 3 and 4 are respectively concerned with the related integrable Hamiltonian system and the background material on perturbation method used to obtain the integrals of motion.

Throughout the paper we make extensive use of Maple V [1] and Reduce [6] to calculate and simplify our results.

2. The Multiple Scales Method

In this section we consider the multiple scales method of the system (1.1) to derive an integrable coupled NLS equation and give the corresponding spectral problem and Hamiltonian formulation. To illustrate the method we seek a solution to the system (1.1) in the following series form:

$$u(x,t) = \sum_{n=1}^{\infty} \epsilon^n \, u_n(t,\xi,\tau), \quad v(x,t) = \sum_{n=1}^{\infty} \epsilon^n \, v_n(t,\xi,\tau), \tag{2.1}$$

where the scaling parameter ϵ dependence is defined as

$$\xi = \epsilon x, \qquad \tau = \epsilon^2 t. \tag{2.2}$$

Then $D_t = \partial/\partial t + \epsilon^2 \partial/\partial \tau,\ D_x = \epsilon \partial/\partial \xi$. In general, the scaling parameter ϵ dependence could be considered as follows:

$$\xi = (a_1 \epsilon + a_2 \epsilon^2 + \cdots)x, \qquad \tau = (b_1 \epsilon + b_2 \epsilon^2 + \cdots)t.$$

where a_j, b_j are unknown constants to be determined. The reason of defining ξ and τ as in (2.2) is to get the coupled NLS equation (2.10) below.

Inserting (2.1) with (2.2) into (1.1) and using the Taylor series expansions of the right hand side functions at zero, and then equating to zero coefficients of like powers of ϵ, we respectively find the following:

$$\begin{aligned} u_{1tt} + 3u_1 &= 0, \\ v_{1tt} + 3v_1 &= 0. \end{aligned} \tag{2.3}$$

$$\begin{aligned} u_{2tt} + 3u_2 &= -\tfrac{3}{2}(u_1^2 - 3v_1^2), \\ v_{2tt} + 3v_2 &= 3u_1 v_1. \end{aligned} \tag{2.4}$$

$$\begin{aligned} u_{3tt} + 3u_3 &= -\tfrac{3}{2}(u_1^2 + 3v_1^2)u_1 - 3u_1u_2 + 9v_1v_2 + u_{1\xi\xi} - 2u_{1t\tau}, \\ v_{3tt} + 3v_3 &= -\tfrac{3}{2}(3v_1^2 + u_1^2)v_1 - 3v_1u_2 + 3u_1v_2 + v_{1\xi\xi} - 2v_{1t\tau} \end{aligned} \tag{2.5}$$

and so on. These system of equation can be solved iteratively to give:

$$\begin{aligned} u_1(t,\xi,\tau) &= A\, e^{\sqrt{3}it} + c.c. \\ v_1(t,\xi,\tau) &= B\, e^{\sqrt{3}it} + c.c., \end{aligned} \tag{2.6}$$

$$\begin{aligned} u_2(t,\xi,\tau) &= \tfrac{1}{6}((A^2-3B^2)e^{2\sqrt{3}it}+c.c.)-AA^*+3BB^*,\\ v_2(t,\xi,\tau) &= -\tfrac{1}{3}ABe^{2\sqrt{3}it}+AB^*+c.c. \end{aligned} \tag{2.7}$$

$$\begin{aligned} u_3(t,\xi,\tau) &= \tfrac{1}{12}(3B^2+A^2)\,A\,e^{3\sqrt{3}it}+c.c.\\ v_3(t,\xi,\tau) &= \tfrac{1}{12}(3B^2+A^2)\,B\,e^{3\sqrt{3}it}+c.c. \end{aligned} \tag{2.8}$$

where A^*, and B^* are complex conjugates of A, and B respectively and *c.c.* denotes complex conjugates. The non-oscillating terms in u_2 and v_2 were choosen to avoid a secularity at the level ϵ^3. Further secular terms are avoided by choosing $A = A(\xi,\tau)$, $B = B(\xi,\tau)$, A^*, and B^* to satisfy a system of amplitude equations, which, in terms of

$$q = A, \qquad r = \sqrt{3}B \tag{2.9}$$

is the coupled nonlinear Schöredinger equation:

$$\begin{aligned} iq_\tau &= q_{\xi\xi} - 2q(\mid q\mid^2 + 2\mid r\mid^2) + 2r^2q^*,\\ ir_\tau &= r_{\xi\xi} - 2r(\mid r\mid^2 + 2\mid q\mid^2) + 2q^2r^*, \end{aligned} \tag{2.10}$$

together with their complex conjugates. Here q^* and r^* are complex conjugate of q and r.

We thus have derived a system of nonlinear Schrödinger equations as a system of amplitude equations, which, in terms of

$$\mathbf{Q} = \begin{pmatrix} q & r\\ -r & q \end{pmatrix}, \qquad \mathbf{R} = \mathbf{Q}^T = \begin{pmatrix} q & -r\\ r & q \end{pmatrix} \tag{2.11}$$

can be written in the matrix form as follows:

$$i\mathbf{Q}_\tau = \mathbf{Q}_{\xi\xi} - 2\mathbf{Q}\mathbf{R}^*\mathbf{Q}, \tag{2.12}$$

from integrable nonlinear Klein-Gordon equation by applying the method of multiple scales. Here the series in (2.1) with slow space and time coordinates give us a perturbative series which is free of secularities (uniform expansion). We also let $\epsilon \to 0$ after the third order (ϵ^3) term which might be considered as the order of the error.

The above form of the multiple scales method for elliptic equations, including the first order terms (u and v) in the right hand side functions, could also be applied. The result might be the same as above with the sign differences. However, for parabolic equations with an appropriate right hand functions, the usual multiple scales method with the dispersion relations [4, 8, 9] might be applied to get NLS type systems. In general, it is not easy to say that the multiple scales method for elliptic or parabolic equations give rise to some integrable equations other than NLS systems without knowing the exact form of the equations.

Remark 1[DB Equation] If we take v's equal to zero then we end up with the NLS equation

$$iq_\tau = q_{\xi\xi} - 2q \mid q \mid^2 \tag{2.13}$$

for the equation (1.5).

2.1. The Spectral Problem

Following [4, 9], we can write the system of NLS equations (2.10) as integrability condition of the following spectral problem:

$$\begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\xi = i \begin{pmatrix} -\mu & \mathbf{Q} \\ -\mathbf{Q}^* & \mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}. \tag{2.14}$$

$$\begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\tau = \begin{pmatrix} i(2\mu^2 + \mathbf{Q}\mathbf{Q}^*) & (\frac{\partial}{\partial\xi} - 2i\mu)\mathbf{Q} \\ (\frac{\partial}{\partial\xi} + 2i\mu)\mathbf{Q}^* & -i(2\mu^2 + \mathbf{Q}\mathbf{Q}^*) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}, \tag{2.15}$$

where

$$\mathbf{Q} = q + ir, \quad \mathbf{Q}^* = q^* - ir^* \tag{2.16}$$

or alternatively

$$\begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\xi = i \begin{pmatrix} -\mu I & \mathbf{Q} \\ -\mathbf{R}^* & \mu \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}. \tag{2.17}$$

$$\begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}_\tau = \begin{pmatrix} i(2\mu^2 + \mathbf{Q}\mathbf{R}^*) & (\frac{\partial}{\partial\xi} - 2i\mu)\mathbf{Q} \\ (\frac{\partial}{\partial\xi} + 2i\mu)\mathbf{R}^* & -i(2\mu^2 + \mathbf{Q}\mathbf{R}^*) \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_1^* \end{pmatrix}, \tag{2.18}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are given by (2.11). The integrability conditions of these give rise to the coupled NLS equation (2.12). We use the latter one for the Hamiltonian formulation in the following section.

2.2. Hamiltonian Formulation

Let $\mathcal{H}(q,q^*)$ and $\mathcal{F}(q,q^*)$ be respectively a conserved density and flux of the usual scalar NLS hierarchy. Since the matrices $\mathbf{Q}, \mathbf{Q}_\xi, \ldots$ all commute $\mathcal{H}(\mathbf{Q},\mathbf{R}^*)$ is a matrix of conserved densities. We thus have the matrix conservation law:

$$\partial_\tau \mathcal{H}(\mathbf{Q},\mathbf{R}^*) = \partial_\xi \mathcal{F}(\mathbf{Q},\mathbf{R}^*).$$

The first few are:

$$\begin{aligned}
\mathcal{H}_0 &= \mathbf{Q}\,\mathbf{R}^* = \begin{pmatrix} \mathcal{H}_0^0 & \mathcal{H}_0^1 \\ -\mathcal{H}_0^1 & \mathcal{H}_0^0 \end{pmatrix} \\
\mathcal{H}_1 &= \tfrac{1}{2} i(\mathbf{Q}\,\mathbf{R}_\xi^* - \mathbf{R}^*\,\mathbf{Q}_\xi) = \begin{pmatrix} \mathcal{H}_1^0 & \mathcal{H}_1^1 \\ -\mathcal{H}_1^1 & \mathcal{H}_1^0 \end{pmatrix} \\
\mathcal{H}_2 &= \mathbf{Q}_\xi\,\mathbf{R}_\xi^* + (\mathbf{R}^*\,\mathbf{Q})^2 = \begin{pmatrix} \mathcal{H}_2^0 & \mathcal{H}_2^1 \\ -\mathcal{H}_2^1 & \mathcal{H}_2^0 \end{pmatrix} \\
\mathcal{H}_3 &= i(\mathbf{Q}_{\xi\xi}\,\mathbf{R}_\xi^* - \mathbf{Q}_\xi\,\mathbf{R}_{\xi\xi}^* + 3\mathbf{R}^*\,\mathbf{Q}(\mathbf{R}^*\,\mathbf{Q}_\xi - \mathbf{R}_\xi^*\,\mathbf{Q})) = \begin{pmatrix} \mathcal{H}_3^0 & \mathcal{H}_3^1 \\ -\mathcal{H}_3^1 & \mathcal{H}_3^0 \end{pmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{H}_0^0 &= |q|^2 + |r|^2, \qquad \mathcal{H}_0^1 = rq^* - qr^*, \\
\mathcal{H}_1^0 &= qq_\xi^* + rr_\xi^* - q^*q_\xi - r^*r_\xi, \quad \mathcal{H}_1^1 = rq_\xi^* + q_\xi r^* - q^*r_\xi - qr_\xi^*, \\
\mathcal{H}_2^0 &= |q_\xi|^2 + |r_\xi|^2 + (|q|^2+|r|^2)^2 - (rq^* - qr^*)^2, \\
\mathcal{H}_2^1 &= q_\xi^* r_\xi - q_\xi - r_\xi^* + 2(|q|^2+|r|^2)(rq^* - qr^*) \\
\mathcal{H}_3^0 &= \tfrac{1}{2} i\Big(q_{\xi\xi}q_\xi^* - q_\xi q_{\xi\xi}^* - r_\xi r_{\xi\xi}^* + 6|q|^2(r_\xi r^* - rr_\xi^*) + 6|r|^2(q_\xi q^* - qq_\xi^*) \\
&\quad + r_{\xi\xi}r_\xi^* + 3(q^2 - r^2)(r^*r_\xi^* - q^*q_\xi^*) + 3(r^{*2} - q^{*2})(rr_\xi - qq_\xi)\Big).
\end{aligned}$$

Thus the Hamiltonian structure:

$$i\dot{\mathbf{Q}}_{ij} = -\frac{\delta \mathcal{H}}{\delta \mathbf{R}^*_{ji}}$$

reduces to :

$$i\dot{q}_k = -(\sigma_1)_{kj}\frac{\delta \mathcal{H}}{\delta q^*_j},$$

giving a Poisson bracket

$$\{\mathcal{H}, \mathcal{G}\} = i\int_{-\infty}^{\infty}\left[\left(\frac{\delta \mathcal{G}}{\delta \mathbf{q}}\right)^T \sigma_1 \frac{\delta \mathcal{H}}{\delta \mathbf{q}^*} - \left(\frac{\delta \mathcal{G}}{\delta \mathbf{q}^*}\right)^T \sigma_1 \frac{\delta \mathcal{H}}{\delta \mathbf{q}}\right] d\xi,$$

where $\mathbf{Q}$ and $\mathbf{R}$ are given by (2.11), $\mathbf{q} = (q, r)^T$, and $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The coupled NLS equation (2.10) is generated by $\mathcal{H}^0_2$. The Hamiltonian $\mathcal{H}^0_3$ generates a coupled complex modified KdV flow:

$$\begin{aligned} iq_{\tau_3} &= q_{\xi\xi\xi} - 6q_\xi(\mid q\mid^2 + 2\mid r\mid^2) + 6(rq^* - qr^*)r_\xi, \\ ir_{\tau_3} &= r_{\xi\xi\xi} - 6r_\xi(\mid r\mid^2 + 2\mid q\mid^2) - 6(rq^* - qr^*)q_\xi, \end{aligned} \tag{2.19}$$

together with the complex conjugate equations.

3. Related Finite Dimensional Hamiltonian System

We relate the system of NLS equations (2.10) derived in previous section to the finite dimensional Hamiltonian system. We assume solutions of the NLS equation (2.10) of the form:

$$q(\xi, \tau) = e^{iw_1^2\tau}\, U(\xi), \quad r(\xi, \tau) = e^{iw_2^2\tau}\, V(\xi). \tag{3.1}$$

We insert this into the system (2.10) then consider the real equations satisfied by the real and the imaginary parts of U, V.

We now illustrate this process with some examples.

3.1. Examples

Example 3.1 (NLS Equation) As our first example we consider NLS equation (2.13). Taking $w_1 = w$ and inserting (3.1) with $r = 0$ into this equation, we obtain

$$U_{\xi\xi} + w^2 U - 2 \mid U \mid^2 U = 0, \tag{3.2}$$

with $U(\xi) = q_1(\xi) + iq_2(\xi)$, the real and imaginary part of the equation (3.2) are:

$$\begin{aligned} q_{1\xi\xi} + w^2 q_1 - 2q_1(q_1^2 + q_2^2) = 0, \\ q_{2\xi\xi} + w^2 q_2 - 2q_2(q_1^2 + q_2^2) = 0. \end{aligned} \tag{3.3}$$

It is a Hamiltonian system with the Hamiltonian $H = H_0 + H_2$ defined by

$$\begin{aligned} H_0 &= \tfrac{1}{2}(p_1^2 + p_2^2 + w^2(q_1^2 + q_2^2)), \\ H_2 &= -\tfrac{1}{2}(q_1^4 + q_2^4 + 2q_1^2 q_2^2) \end{aligned} \tag{3.4}$$

where $p_j = q_{j\xi}$ for j=1,2.

This system (3.3) has rotational symmetry with Noether constant

$$k = (p_1\, q_2 - p_2\, q_1), \tag{3.5}$$

which is the second integral.

Example 3.2 (2D NLS Equation) We now consider the coupled NLS equation (2.10) and assume solutions in the form of (3.1). Inserting (3.1) into the system (2.10) we find:

$$\begin{aligned} \big(U_{\xi\xi} + w_1^2 U - 2U(\mid U \mid^2 + 2 \mid V \mid^2)\big)\, e^{iw_1\tau} + 2U^* V^2 e^{i(2w_2 - w_1)\tau} = 0, \\ \big(V_{\xi\xi} + w_2^2 V - 2V(\mid V \mid^2 + 2 \mid U \mid^2)\big)\, e^{iw_2\tau} + 2V^* U^2\, e^{i(2w_1 - w_2)\tau} = 0. \end{aligned} \tag{3.6}$$

For nontrivial U and V we find that $w_1 = w_2 = w$. Then we define

$$U(\xi) = q_1(\xi) + iq_2(\xi), \quad V(\xi) = q_3(\xi) + iq_4(\xi), \tag{3.7}$$

in order to find a finite dimensional Hamiltonian system of the form:

$$\begin{aligned} q_{1\xi\xi} + w^2 q_1 - 2q_1(q_1^2 + q_2^2 + q_3^2 + 3q_4^2) + 4q_2q_3q_4 = 0, \\ q_{2\xi\xi} + w^2 q_2 - 2q_2(q_1^2 + q_2^2 + 3q_3^2 + q_4^2) + 4q_1q_3q_4 = 0, \\ q_{3\xi\xi} + w^2 q_3 - 2q_3(q_3^2 + q_4^2 + q_1^2 + 3q_2^2) + 4q_1q_2q_4 = 0, \\ q_{4\xi\xi} + w^2 q_4 - 2q_4(q_3^2 + q_4^2 + 3q_1^2 + q_2^2) + 4q_1q_2q_3 = 0. \end{aligned} \tag{3.8}$$

with the Hamiltonian $H = H_0 + H_2$ for which

$$\begin{aligned} H_0 = &\ \tfrac{1}{2}({p_1}^2 + {p_2}^2 + {p_3}^2 + {p_4}^2 + w^2({q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2)), \\ H_2 = &\ \tfrac{1}{2}\Big(-{q_1}^4 - 2\,{q_1}^2\,{q_2}^2 - {q_2}^4 - 2\,{q_1}^2 {q_3}^2 - 6\,{q_2}^2\,{q_3}^2 - {q_3}^4 \\ &+8\,q_1\,q_2\,q_3\,q_4 - 6\,{q_1}^2\,{q_4}^2 - 2\,{q_2}^2\,{q_4}^2 - 2\,{q_3}^2\,{q_4}^2 - {q_4}^4\Big), \end{aligned} \tag{3.9}$$

where $p_j = \dot{q_{j\xi}}$ for $j = 1, ..., 4$.

4. Perturbation Theory

In this section we start with giving the background material on a perturbation method [5, 8] to construct the integrals of motion for the Hamiltonian (3.9).

We consider some Hamiltonians whose quadratic part is H_0:

$$H = H_0 + H_1 + H_2 + \dots, \tag{4.1}$$

where H_j is a homogeneous polynomial of degree j+2. We write H_0 for representing our approach as

$$H_0 = \frac{1}{2}\sum_{j=1}^{n} w_j (p_j^2 + q_j^2) \tag{4.2}$$

after following the canonical transformation

$$(q_j, p_j) \longrightarrow (w_j^{-1/2} q_j, w_j^{1/2} p_j). \tag{4.3}$$

It is convenient to transform into characteristic coordinates, which diagonalize the operator $\{H_0, .\}$:

$$z_j = q_j + ip_j, \tag{4.4}$$

which satisfy :

$$\{z_j, z_k\} = \{z_j^*, z_k^*\} = 0,\ \{z_j, z_k^*\} = -2i\delta_{jk}. \tag{4.5}$$

The Hamiltonian (4.2) now takes the form:

$$H_0 = \frac{1}{2}\sum_{j=1}^{n} w_j z_j z_j^*, \tag{4.6}$$

so that

$$\{F, H_0\} = X_{H_0} F = -i \sum_{j=1}^{n} w_j \left(z_j \frac{\partial}{\partial z_j} - z_j^* \frac{\partial}{\partial z_j^*} \right) F. \tag{4.7}$$

4.1. Resonances and Non-Resonances

The eigenfunction of X_{H_0} are the monomials:

$$z_1^{m_1} \ldots z_n^{m_n} z_1^{*n_1} \ldots z_n^{*n_n}, \tag{4.8}$$

with corresponding eigenvalues:

$$i \sum_{j=1}^{n} w_j (n_j - m_j),$$

and weight

$$l = \sum_{j=1}^{n} (n_j + m_j).$$

Functions which are in involution with H_0 have zero eigenvalues , so satisfy:

$$\sum_{j=1}^{n} w_j (n_j - m_j) = 0. \tag{4.9}$$

For a given $\mathbf{w} = (w_1, \ldots, w_n) \in \Re^n$ and H_0, defined as above with (4.6), $\mathbf{w}$ is called <u>*resonant*</u> if and only if there exist $\mathbf{r} = (r_1, \ldots, r_j) \neq 0 \in Z^n$ such that $\mathbf{w}.\mathbf{r} = 0$.

We now consider Hamiltonian defined by

$$H_{\mathbf{mn}} = z^{\mathbf{m}} z^{*\mathbf{n}} = \prod_{k=1}^{n} z_k^{m_k} z_k^{*n_k} \tag{4.10}$$

It is said to be <u>*resonant*</u> <u>*Hamiltonian*</u> if and only if (4.9) is satisfied.

In the non-resonant case for which $w_1, \ldots, w_n$ are rationally independent, the only solutions are $n_j = m_j$. The corresponding functions of weight 2 are: $\mid z_j \mid^2, \quad j = 1, \ldots, n.$ All other commuting functions are of even weight and are functions of the above weight 2 building blocks.

In the resonant case for which some or all of the w_j are rationally related , the above equations have other solutions, which give rise to further commuting integrals.

To illustrate this, we consider 2-degrees of freedom system. We list the solution as (m_1, m_2, n_1, n_2).

<u>$Case\, w_1 = w_2 = 1:$</u> In this case $m_1 + m_2 = n_1 + n_2$ and the first few invariant functions are:

<u>$l = 2$</u>

(1, 0, 1, 0)	$\mid z_1 \mid^2$
(0, 1, 0, 1)	$\mid z_2 \mid^2$
(0, 1, 1, 0)	$z_1^* z_2$

<u>$l = 3$</u>

no solution.

<u>$l = 4$</u>

(2, 0, 2, 0)	$\mid z_1 \mid^4$
(0, 2, 0, 2)	$\mid z_2 \mid^4$
(1, 1, 1, 1)	$\mid z_1 \mid^2 \mid z_2 \mid^2$
(2, 0, 0, 2)	$z_1^2 z_2^{*2}$
(2, 0, 1, 1)	$z_1^2 z_1^* z_2^*$
(0, 2, 1, 1)	$z_2^2 z_1^* z_2^*$

<u>$l = 5$</u>

no solution.

together with their complex conjugates.

We now assume an integral of H in a polynomial form as:

$$k = I_0 + I_1 + I_2 + \dots . \tag{4.11}$$

The condition that Poisson bracket of H and k vanishes gives the following equations:

$$\begin{aligned} \frac{dk}{dt} &= \{H, k\} \\ &= \{H_0, I_0\} + \{H_1, I_0\} + \{H_0, I_1\} + \{I_0, H_2\} + \{H_0, I_2\} + \dots \\ &= 0. \end{aligned} \tag{4.12}$$

for which we have

$$\begin{aligned} &\{H_0, I_0\} = 0, \\ &\{H_0, I_1\} = \{H_1, I_0\}, \\ &\{H_0, I_2\} = \{I_0, H_2\} + \{I_1, H_1\}, \end{aligned} \tag{4.13}$$

and so on. We use one of the functions from the set of resonant Hamiltonians which commute with H_0 as a <u>seed</u> for an integral of H. Then we check that if

$$\{H_1, I_0\} \tag{4.14}$$

contains terms of $Ker\{H_0,.\}$ or not in order to decide which seed to start.

(a) If (4.14) contains no terms of $Ker\{H_0,.\}$ we solve I_1 in general form as a summation of paticular and homegeneous solutions:

$$I_1 = I_1^{(p)} + I_1^{(h)} \tag{4.15}$$

where $I_1^{(h)}$ is the linear combination of resonant Hamiltonians.

Then we check whether

$$\{I_0, H_2\} + \{I_1, H_1\} \tag{4.16}$$

contains terms of $Ker\{H_0,.\}$ or not. If it does not we solve I_2, and so on.

(b) If (4.14) contains terms of $Ker\{H_0,.\}$ we choose another seed for I_0 then repeat the above procedure **(a)**.

We now illustrate this procedure by considering the above derived system (0.32) in order to find integrals of motion.

4.2. Integrals of Motion

We now consider sytem (3.8) with the Hamiltonian (3.9) and find integrals of motion by the perturbation analysis [5, 8]. This case we have the resonance case of $w_1 = w_2 = w_3 = w_4 = w = 1$. In complex coordinates:

$$z_j = w_j^{-1/2} q_j + i w_j^{1/2} p_j, \;\; for \; j = 1,..,4, \tag{4.17}$$

we have

$$\begin{aligned} H_0 = \; & \tfrac{1}{2}(z_1\, z_1^* + z_2\, z_2^* + z_3\, z_3^* + z_4\, z_4^*), \\ H_2 = \; & \tfrac{-1}{32}\Big((z_1 + z_1^*)^4 + (z_2 + z_2^*)^4 + 6\,(z_2 + z_2^*)^2 (z_3 + z_3^*)^2 \\ & +(z_3 + z_3^*)^4 - 8\,(z_1 + z_1^*)\,(z_2 + z_2^*)\,(z_3 + z_3^*)\,(z_4 + z_4^*) \\ & +6\,(z_1 + z_1^*)^2 (z_4 + z_4^*)^2 + (z_4 + z_4^*)^4 + 2\big((z_2 + z_2^*)^2 + \\ & (z_3 + z_3^*)^2\big)\,\big((z_1 + z_1^*)^2 + (z_4 + z_4^*)^2\big)\Big). \end{aligned} \tag{4.18}$$

We consider the seed:

$$I_0 = \frac{1}{2}\left(z_3\, z_2^* + z_2\, z_3^* - z_4\, z_1^* - z_1\, z_4^*\right), \tag{4.19}$$

and we find I_1 as at he general solution of

$$\{H_2, I_0\} + \{H_0, I_1\} = 0,$$

$$\begin{aligned} I_1 = \;& \tfrac{1}{8}\Big({z_1}^3\,(z_4 + z_4^*) - {z_2}^3\,(z_3 + z_3^*) - {z_3}^3\,(z_2 + z_2^*) + {z_4}^3\,(z_1 + z_1^*) \\ & +({z_2}^2 + {z_3}^2)\,(z_1\, z_4 + z_4\, z_1^* + z_1\, z_4^* + 3\, z_1^* z_4^* - 3\, z_2^*\, z_3^*) - \\ & ({z_1}^2 + {z_4}^2)\,(z_2\, z_3 + z_3\, z_2^* + z_2\, z_3^* + 3\, z_2^* z_3^* - 3\, z_1^*\, z_4^*) - \\ & 3\,({z_2}^2\, z_3\, z_2^* + {z_3}^2\, z_2\, z_3^* - {z_1}^2\, z_4\, z_1^* + {z_4}^2\, z_1\, z_4^*) + \\ & 2\,(z_2^*\, z_1\, z_4\, z_2 + z_3^*\, z_1\, z_3\, z_4 - z_1\, z_1^*\, z_2\, z_3 - z_4^*\, z_2\, z_3\, z_4) + c.c.\Big) \\ & + c_3\Big(z_1\, z_2^* + z_3\, z_4^* - c.c.\Big)^2 + c_4\Big(z_3\, z_1^* + z_4\, z_2^* - c.c.\Big)^2, \end{aligned}$$

so that $I_0 + I_1$ commutes with H. Here c_3 and c_4 are arbitrary constants, and their coefficients are constant of motion.

As a result we can write the integrals of motion in usual qp-coordinates in the following forms:

$$\begin{aligned} H_0 = \;& \tfrac{1}{2}({p_1}^2 + {p_2}^2 + {p_3}^2 + {p_4}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2), \\ H_2 = \;& -\tfrac{1}{2}\Big(({q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2)^2 + 4(q_1\, q_4 - q_2\, q_3)^2 \Big), \end{aligned} \tag{4.20}$$

$$\begin{aligned} k_1 = \;& (p_1\, q_2 - p_2\, q_1 - p_4\, q_3 + p_3\, q_4)\,, \\ k_2 = \;& (p_1\, q_3 - p_3\, q_1 - p_4\, q_2 + p_2\, q_4)\,, \\ k_3 = \;& (p_2\, p_3 + q_2\, q_3 - p_1\, p_4 - q_1\, q_4 - (p_1\, p_3 - p_2\, p_4)\,(q_1\, q_2 - q_3\, q_4) - \\ & (p_1\, p_2 - p_3\, p_4)\,(q_1\, q_3 - q_2\, q_4) - 2(q_2\, q_3 - q_1\, q_4)({q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2)) - \\ & q_1\, q_4({p_2}^2 + {p_3}^2) + q_2\, q_3({p_1}^2 + {p_4}^2) + p_2\, p_3({q_1}^2 + {q_4}^2) - p_1\, p_4({q_2}^2 + {q_3}^2). \end{aligned} \tag{4.21}$$

Here these integrals of motion are in involution with respect to the canonical Poisson bracket.

5. Conclusion

We have used the multiple scales method to derive a new integrable system of NLS equations and give the corresponding spectral problem. The starting point was the two-component nonlinear Klein-Gordon equation. We also reduce the coupled NLS equation to a four-degrees of freedom Hamiltonian system and find the integrals of motion with a perturbation method [5, 8].

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