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# ON THE DISCRETE SQUEEZING PROPERTY FOR SEMILINEAR WAVE EQUATIONS

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## 1. Introduction

We extend the results obtained in our previous paper [2] to the case where the conditions on the nonlinear term are milder,namely those that are given by the fourth condition in our basic theorem. The methods used are inspired from the results of Ladyzhenskaya (see [4], [5]) and can be considered as a direct generalization.

**Preliminaries and Notations**

In a separable Hilbert space $H$ with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ consider the following Cauchy problem:

$$Pu_{tt} + Qu_t + Au + F[u] = h, \tag{1}$$

$$u_t(0) = u_1, u(0) = u_0, \tag{2}$$

where $P, Q,$ and $A$ are linear (not necessarily bounded) , selfadjoint, positive definite operators, $F(.)$ is a nonlinear operator, and $u_0$, $u_1$, and $h$ are given elements in appropriate spaces.

Let us set $D_1 = D(P^{1/2}), D_2 = D(Q^{1/2})$ $,D_3 = D(A^{1/2})$. We will also denote the corresponding dual spaces with respect to the inner product in $H$ by $D_{-1}, D_{-2}$ and $D_{-3}$, respectively. For notational ease we will use the bracket $(u, v)$ both for the inner product in $H$ and the duality pairing between the elements of $D_i$ and $D_{-i}$ where $i = 1, 2, 3$.

From now on we will assume that the following conditions hold:

a) $D(A) = D_3 \subseteq D(Q) = D_2 \subseteq D(P) = D_1$,

b) the operator $A$ has a compact inverse,

c) the operators $P, Q$, and $A$ commute on $D(A) = D_3$,

d) there are positive constants $a, b$ and $c$ such that:

$$a \|u\| \leq \left\| Q^{\frac{1}{2}} u \right\| \leq b \left\| A^{\frac{1}{2}} u \right\|, \quad \forall u \in D_3 \tag{3}$$

$$\left\| P^{\frac{1}{2}} u \right\| \leq c \left\| Q^{\frac{1}{2}} u \right\|, \quad \forall u \in D_2 \tag{4}$$

e)the nonlinear operator $F(\cdot) : D_3 \to D_{-2}$ is continuous, bounded, and it is a gradient of some functional $G(.) : D_3 \to R^1$ .Moreover, there exists $C \geq 0$ such that

$$(F(u), u) - G(u) \geq -C, \qquad G(u) \geq -C, \quad \forall u \in D_3$$

We also set:

i) $X = D_3 \times D_1$ is a Hilbert space with the inner product:

$$((u, v), (w, z)) = (A^{\frac{1}{2}} u, A^{\frac{1}{2}} w) + (P^{\frac{1}{2}} v, P^{\frac{1}{2}} z),$$

ii)

$$\|u\|_s = \left\| A^{\frac{s}{2}} u \right\|, s \in R^1.$$

**Definition 1.** We will say that the semigroup $S(t) : X \to X, t \in R^+$, satisfies the *discrete squeezing property* on the set $\mathcal{M} \subset X$, if there exists $t_0 > 0$ such that the operator $T = S(t_0)$ is Lipschitz continuous on $\mathcal{M}$:

$$\|Tx - Ty\|_X \leq l \|x - y\|_X, \forall x, y \in \mathcal{M},$$

and for some $\delta \in (0, \frac{1}{\sqrt{2}})$ there exists $N_0(\delta)$, such that

$$\|(I - P_{N_0})(Tx - Ty)\|_X \leq \delta \|x - y\|, \forall x, y \in \mathcal{M}, \tag{5}$$

where $P_{N_0}$ is the ortogonal projection on the subspace $H_{N_0}$ ( of the Hilbert space $H$) spanned by the first $N_0$ eigenvectors of the operator $A$.

**Definition 2** A set $\mathcal{A}$ is said to be the *attractor* of a semigroup $S(t) : X \to X, t \in R^+$, if it is the minimal closed set, attracting every bounded set $B \subset X$.

**Definition 3** A set $\mathcal{M}$ is called an *exponential attractor* for the solution semigroup $\{S(t) : t > 0\}$ on the set $B$ if

(i) $\mathcal{A} \subseteq \mathcal{M} \subseteq B$,

(ii) $S(t)\mathcal{M} \subseteq \mathcal{M}$,

(iii) $\mathcal{M}$ has finite fractal dimension,

(iv) for every x in $B$, dist $(S(t)x, \mathcal{M}) \leq c_1 \exp\{-c_2 t\}$ where $c_1$ and $c_2$ are universal constants.

The following theorem is in the spirit of [5].

**Theorem 1**. Suppose that

1.the conditions a), b), c) and d) are satisfied,

2. the problem (1), (2) generates a continuous semigroup $S(t) : X \to X, t \in R^+$, that is for each pair $(u_0, u_1) \in X$ the problem (1), (2) has a unique weak solution (see [2] for a proof) , which continuosly depends on the initial data in the sense of the norm of $X$, and such that

$$u \in C(R^+; D_3), u_t \in C(R^+; D_1),$$

3.there exists a bounded set $B_0 \subset X$ such that

$$S(t)B_0 \subseteq B_0, \forall t \in R^+,$$

4. The operator $F(.) : D_3 \to D_{-2}$ is Frechet differentiable, and for some $\gamma \in (0, 1)$ and for each $(u, u_t), (v, v_t) \in B_0$ the following conditions hold

$$\| F(u) - F(v) \|_{-\gamma} \leq M_1 \| u - v \|_1, \tag{6}$$

$$\| Q^{-\frac{1}{2}}(F(u) - F(v)) \|_{\gamma} \leq M_2 \| u - v \|_1, \tag{7}$$

$$\| \frac{d}{dt}(F(u)-F(v)) \|_{-\gamma} \leq \widetilde{M}_1(\| P^{\frac{1}{2}}(u_t-v_t) \| + \| u-v \|_1), \tag{8}$$

where $M_1 = M_1(\| u \|_1, \| v \|_1)$ and $M_2 = M_2(\| u \|_1, \| v \|_1)$ are some continuous, positive functions on $R^+ \times R^+$ and $\widetilde{M}_1 = \widetilde{M}_1(\| u \|_1, \| v \|_1,$ $\| P^{\frac{1}{2}} u_t \|, \| P^{\frac{1}{2}} v_t \|)$ is some continuous,positive function on $R^+ \times R^+ \times R^+ \times R^+$.
Then the semigroup $S(t) : X \to X, t \in R^+$, generated by the problem (1),(2) has the discrete squeezing property in $B_0$.

**Proof** Let $w_1, w_2, ..., w_n, ...$ be the orthonormal system of eigenvectors of the operator $A$, which forms a basis of $H$, and let $\lambda_1, \lambda_2, ..., \lambda_n, ...$ be the corresponding system of eigenvalues of $A$. Suppose that $(u_0, u_1)$ and $(v_0, v_1)$ are two arbitrary elements of $B_0$. Then for the corresponding solutions $u(t)$ and $v(t)$ of the problem (1),(2) due to the condition 3. of the Theorem 1 we have:

$$(u(t), u_t(t)), (v(t), v_t(t)) \in B_0, \forall t \in R^+.$$

It is clear that $z(t) = u(t) - v(t)$ satifies the equation

$$P z_{tt} + Q z_t + Az + \delta F = 0, \tag{9}$$

where $\delta F = F(u) - F(v)$. Let $z^* = (I - P_N)z$ and $\mu$ be some positive parameter,to be specified later. Multiplying (9) by $z_t^* + \mu z^*$, we get:

$$\frac{d}{dt}[\left\| \frac{1}{2} P^{\frac{1}{2}} z_t^* \right\|^2 + \frac{1}{2} \left\| A^{\frac{1}{2}} z^* \right\|^2 +$$

$$\frac{\mu}{2} \left\| Q^{\frac{1}{2}} z^* \right\|^2 + \mu(P^{\frac{1}{2}} z_t^*, P^{\frac{1}{2}} z^*) + (\delta F, z^*)] +$$

$$\mu \left\| A^{\frac{1}{2}} z^* \right\|^2 + \left\| Q^{\frac{1}{2}} z^* \right\|^2 - (\frac{d}{dt}\delta F, z^*) + \mu(\delta F, z^*) = 0. \tag{10}$$

Consider the functional

$$E_1(z^*, z_t^*) \equiv \frac{1}{2} \left\| P^{\frac{1}{2}} z_t^* \right\|^2 + \frac{1}{2} \left\| A^{\frac{1}{2}} z^* \right\|^2 + \frac{\mu}{2} \left\| Q^{\frac{1}{2}} z^* \right\|^2 + \mu(P^{\frac{1}{2}} z_t^*, P^{\frac{1}{2}} z^*) + (\delta F, z^*).$$

It follows from (10) that

$$\frac{d}{dt} E_1(z^*, z_t^*) + \mu \left\| A^{\frac{1}{2}} z^* \right\|^2 + \left\| Q^{\frac{1}{2}} z^* \right\|^2 \leq \left| (\frac{d}{dt}\delta F, z^*) \right| + \mu \left| (\delta F, z^*) \right|. \tag{11}$$

It is not difficult to see that for each $u \in D_3,$ the following inequality holds:

$$\|u^*\| \le \lambda_0 \left\|A^{\frac{1}{2}}u^*, \right\| \tag{12}$$

where $\lambda_0 = \lambda_{N+1}^{-\frac{1}{2}(1-\gamma)}$ and $u^* = (I - P_N)u$. Taking into account conditions (6)-(8) and the last inequality we get

$$\frac{d}{dt}E_1(z^*, z_t^*) + \mu \left\|A^{\frac{1}{2}}z^*\right\|^2 + \left\|Q^{\frac{1}{2}}z^*\right\|^2$$

$$\le \mu M_1 \|z\|_1 \|z^*\|_\gamma + \widetilde{M}_1 \left[\| P^{\frac{1}{2}} z_t \| + \|z\|_1\right] \|z^*\|_\gamma \le$$

$$\le \left[\mu M_1 + \widetilde{M}_1\right] \left[\| P^{\frac{1}{2}} z_t \| + \|z\|_1\right] \lambda_0 \left\|A^{\frac{1}{2}}z^*\right\| \le$$

$$\le \lambda_0 \left\|A^{\frac{1}{2}}z^*\right\|^2 + \widetilde{M}_2 \lambda_0 \left[\| P^{\frac{1}{2}} z_t \|^2 + \left\|A^{\frac{1}{2}}z\right\|^2\right].$$

So we have:

$$\frac{d}{dt}E_1(z^*, z_t^*) + \frac{\mu}{2} \left\|A^{\frac{1}{2}}z^*\right\|^2 + \left\|Q^{\frac{1}{2}}z^*\right\|^2 \le$$

$$(\lambda_0 - \frac{\mu}{2}) \left\|A^{\frac{1}{2}}z^*\right\|^2 + \widetilde{M}_2 \lambda_0 \left[\| P^{\frac{1}{2}} z_t \|^2 + \left\|A^{\frac{1}{2}}z\right\|^2\right] \tag{13}$$

It is not difficult to see that

$$2E_1(z^*, z_t^*) \ge \left\|P^{\frac{1}{2}}z_t^*\right\|^2 + \left\|A^{\frac{1}{2}}z^*\right\|^2 + \mu \left\|Q^{\frac{1}{2}}z^*\right\|^2 -$$

$$2\mu \left\|P^{\frac{1}{2}}z_t^*\right\| \left\|P^{\frac{1}{2}}z^*\right\| - 2\|\delta F\|_{-\gamma} \|z^*\|_\gamma \tag{14}$$

Using the conditions (3),(4), (6) and the inequality (12) we obtain from (14):

$$2E_1(z^*, z_t^*) \ge (1-\mu) \left\|P^{\frac{1}{2}}z_t^*\right\|^2 + (1-\mu c^2 b^2) \left\|A^{\frac{1}{2}}z^*\right\|^2 + \mu \left\|Q^{\frac{1}{2}}z^*\right\|^2 -$$

$$-M_1 \lambda_0 \|z\|_1^2 - M_1 \lambda_0 \|z^*\|_1^2$$

Choosing in the last inequality $\mu < \min\left\{1, (c^2b^2)^{-1}\right\}$, and $N_1$ sufficiently large, we obtain:

$$E_1(z^*, z_t^*) \ge C_2 \|\{z^*, z_t^*\}\|_X^2 - \frac{1}{2} M_1 \lambda_0 \|\{z, z_t\}\|_X^2, \forall N \ge N_1. \tag{15}$$

Let us choose $N_2$ so that $(\frac{\mu}{2} - \lambda_0) > 0$ is satisfied $\forall N > N_2.$ Then it follows from (13) that

$$\frac{d}{dt}E_1(z^*, z_t^*) + \frac{\mu}{2}\left\|A^{\frac{1}{2}}z^*\right\|^2 + \left\|Q^{\frac{1}{2}}w\right\|^2 \leq$$
$$\leq \widetilde{M}_2\lambda_0 \left\|\{z, z_t\}\right\|_X^2, \forall N > N_2.$$

From the inequality (15) and from the last inequality for sufficiently small $\delta_1 > 0$ it follows:

$$\frac{d}{dt}E_1(z^*, z_t^*) + \delta_1 E_1(z^*, z_t^*) \leq \widetilde{M}_3\lambda_0 \left\|\{z, z_t\}\right\|_X^2, \forall N > N_1 + N_2.$$

This inequality in turn implies that:

$$E_1(z^*(t), z_t^*(t)) \leq e^{-\delta_1 t}E_1(z^*(0), z_t^*(0)) + \widetilde{M}_3\lambda_0 \int_0^t \left\|\{z(s), z_s(s)\}\right\|_X^2 ds \qquad (16)$$

Taking the inner product of (10) by $z_t$, and using the condition (7) we get:

$$\frac{d}{dt}\left[\frac{1}{2}\left\|P^{\frac{1}{2}}z_t\right\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}z\right\|^2\right] + \left\|Q^{\frac{1}{2}}z\right\|^2 = (\delta F, z_t) =$$
$$= (Q^{-\frac{1}{2}}\delta F, Q^{\frac{1}{2}}z_t) \leq M_2 \left\|z\right\|_1 \left\|Q^{\frac{1}{2}}z_t\right\| \leq M_2^2 \left[\frac{1}{2}\left\|P^{\frac{1}{2}}z_t\right\|^2 + \frac{1}{2}\left\|A^{\frac{1}{2}}z\right\|^2\right].$$

Therefore we have

$$\left\|\{z(t), z_t(t)\}\right\|_X^2 \leq \exp(M_2^2 \cdot t) \left\|\{z(0), z_t(0)\}\right\|_X^2 \qquad (17)$$

Using (7) it is not difficult to prove that

$$E_1(z^*(0), z_t^*(0)) \leq M_3 \left\|\{z(0), z_t(0)\}\right\|_X^2 \qquad (18)$$

So thanks to (16),(17) and (18) we obtain from (15) the following inequality:

$$C_2 \left\|\{z^*(t), z_t^*(t)\}\right\|_X^2 \leq \left\{e^{-\delta_1 t} + \lambda_0\left[\widetilde{M}_3 M_2^{-2} + \frac{M_1}{2}\exp(M_2^2(.)t)\right]\right\} \left\|\{z(0), z_t(0)\}\right\|_X^2$$

It follows from the last inequality that, the numbers $t_0$ and $N_3 \geq N_2 + N_1$ can be choosen so that :

$$\left\|\{z^*(t_0), z_t^*(t_0)\}\right\|_X^2 \leq \frac{1}{16}\left\|\{z(0), z_t(0)\}\right\|_X^2, \forall N \geq N_3. \qquad (19)$$

As a closing remark let us mention that the improvement over the previous work stems basically from the weaker assumtion on the nonlinear term, namely, those given in condition 4 of the theorem, instead of the assuptions (V) and (W) as in [2].

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## References

[1] Eden A.,Foias C,Nicolaenko B. and R.Temam, Exponential attractors for dissipative evolution equations,Masson Publications,1994.

[2] Eden A., Kalantarov V.K. Finite dimensional attractors for a class of semilinear wave equations, Turkish Math.Journal, v.20, 425-450, (1996).

[3] Kapitanski L., Minimal compact global attractors for a damped semilinear wave equation, Commun. in P.D.E. v.20,1303-1323 (1995).

[4] Ladyzhenskaya O.A.On the finiteness of the dimension of bounded invariant sets for the Navier-Stokes equations and other related dissipative systems,Zapiski nauchnich seminarov LOMI, Leningrad, 115, 137-155, (1982), see also J.Soviet Math.,28,no.5, 714-725,(1985).

[5] Ladyzhenskaya O.A., Finding minimal global attractors for the Navier-Stokes equations and other Partial Differential Equations, Uspekhi Mat. Nauk,v.42,6, 25-60 (1987) MR 89m:58031.

[6] Ladyzhenskaya O.A.,Attractors for semigroups and evolution equations, Accademia Nazionae dei Lincei series, Cambridge University press,Cambridge,1991.

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