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THE DUAL OF THE BOCHNER SPACE $L^p(\mu, E)$ FOR ARBİTRARY μ

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Abstract

Let μ be a finite measrue, E a Banach space, and $1 \leq p < \infty$, $1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ if, and only if, E^* has the Radon-Nikodým property with respect to μ. The aim of this article is to generalize the above result to arbitrary measures.

Let $(\Omega, \mathcal{A}, \mu)$ be a positive* measure space, and E a Banach space. If there is no possibility of ambiguity about the underlying measurable space $(\Omega, \mathcal{A})$, for any $1 \geq p \geq \infty$, $L^p(\mu, E)$ will denote the Bochner space $L^p(\Omega, \mathcal{A}, \mu, E)$. For denfinitions and properties of these spaces we refer to [4]. For two Banach spaces E and $F, E \simeq F$ will mean that they are linearly isometric. E^* will denote the topological dual of E.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, E a Banach space, and let $1 \leq p < \infty, 1 < q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $f \in L^p(\mu, E)$ and $g \in L^p(\mu, E^*)$, the function $\langle f, g \rangle$ defined on Ω by

$$\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle = g(\omega)(f(\omega)), \qquad \omega \in \Omega,$$

is integrable, and for any fixed $g \in L^q(\mu, E^*)$ the mappin ϕ_g defined on $L^p(\mu, E)$ by

$$\phi_g(f) = \int_\Omega \langle f, g \rangle d\mu, \quad f \in L^p(\mu, E),$$

* Throughout this article all scallar-valued measures are assumed to be positive.

is a bounded functional of $L^p(\mu, E)$ with norm equals $\| g \|_q$. Thus, the mapping $g \to \phi_g$ is a linear isometry from $L^p(\mu, E^*)$ into $L^p(\mu, E)^*$.

It is known that the above mentioned isometry $g \to \phi_g$ is surjective if, and only if, E^* has the Radon-Nikodým property with respect to μ, that is, each μ-continuous, E^*-valued measure of bounded variation on $\mathcal{A}$ to E^* can be represented (via integral) by an E^*-valued μ-integrable function. (This theorem is due to Bochner and Taylor [1] for the Lebesgue measure on the interval [0,1]. It was generalized to σ-finite measures by Gretsky and Uhl[5]. An eqcellent proof of it can be found in [4, pp. 98-100].)

In [3], Cengiz proves that the preceding theorem can be generalized to *arbitrary* measures, but at a price. It is proved that for an arbitrary measure μ, if E^* is separable (hence has the Radon-Nikodým property with respect to μ [4, p. 79]), then $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ still holds for $1 < p < \infty$. However, it may fail for $p = 1$ even in the scalar case (see [6, p. 349]). Instead, we have $L^1(\mu, E)^* \simeq L^\infty(\nu, E^*)$ for some *perfect* measure ν on an extremally disconnected locally compact Hausdorff space.

In this article we shall replace the separability condition on E^* by the Radon-Nikodým proprty with respect to μ. But first, we give some details about the perfect measure ν mentioned above.

We recall that a Borel measure μ on an extremally disconnected locally compact Hausdorff space is *perfect* if every nonempty open set has positive measure, every nowhere dense Borel set has measure zero, and every nonempty open set contains another nonempty open set with finite measure (see [2]).

It is proved in [3] that any arbitrary measure space $(T, \sum, \lambda)$ can be replaced by a perfect measure space $(\Omega, \mathcal{A}, \nu)$ in the sense that $L^p(\lambda, E) \simeq L^p(\nu, E)$ for every $1 \le p < \infty$ and every Banach space E. But $L^\infty(\nu, E)$ may be *enlarged*, that is, $L^\infty(\lambda, E)$ is isometric to a subspace of $L^\infty(\nu, E)$.

Some other additional nice properties of this new measure space $(\Omega, \mathcal{A}, \nu)$ are as follows:

i) Ω is the topological direct sum of a family $\{\Omega_i : i \in I\}$ of extremally disconnected compact Hausdorff spaces Ω_i, that is, $\Omega = \sum_i \oplus \Omega_i$, the spaces Ω_i are mutually disjoint and the topology on Ω is the weakest topology containing the topologies of $\Omega_i, i \in I$.

ii) The algebra $\mathcal{A}$ contains the Borel algebra. A set A belongs to $\mathcal{A}$ if, and only if, $A\cap\Omega_i$ belongs to $\mathcal{A}$ for all $i \in I$.

iii) The restriction of ν to each Ω_i is a regular Borel measure on Ω_i.

iv) Each σ-finite measurable set is contained a.e. in the union of a countable subfamily of $\{\Omega_i : i \in I\}$.

v) $\nu(A) = \sum_i \nu(A \cap \Omega_i)$ for all $A \in \mathcal{A}$. Thus every locally null set is actually null.

In view of the above discussion we may, and will assume that the given measure space $(\Omega, \mathcal{A}, \mu)$ is perfect and prove the following theorem.

Theorem *Let $(\Omega, \mathcal{A}, \mu)$ be a perfect measure space and E and Banach space. Then, for any $1 \leq p < \infty$, $1 < q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ if, and only if E^* has the Radon-Mikodým property with respect to μ; the isometry being the mapping $g \rightarrow \phi_g$, $g \in L^q(\mu, E^*)$.*

Proof. Let us assume that E^* has the Radon-Nikodým property with respect to μ, and write $\Omega = \sum_i \oplus\Omega_i$. Then, since the theorem is true for finite measures, for each $i \in I, L^p(\Omega_i, E)^* \simeq L^q(\Omega_i, E^*)$. Now let $\psi \in L^p(\mu, E)^*$. Then for each $i \in I$, there is a $g_i \in L^q(\Omega_i, E^*)$ such that

$$\psi_i(f) = \int_\Omega \langle f, g_i \rangle d\mu \quad \text{for all} \quad f \in L^p(\Omega_i, E),$$

and $\| \psi_i \| = \| g_i \|_q$, where ψ_i denotes the restriction of ψ to the *subspace* $L^p(\Omega_i, E)$ of $L^p(\mu, E)$.

for any finite subset J of I let

$$\Omega_J = \bigcup_{j\in J} \Omega_j \text{ and } g_j = \sum_{j\in J} g_i.$$

Since the functions g_i have disjoint supports, it follows that

$$\psi_J(f) = \int_\Omega \langle f, g_j \rangle d\mu \text{ for } f \in L^p(\Omega_J, E),$$

where ψ_J denotes the restriction of ψ to $L^p(\Omega_J, E)$.

If $p = 1$, then $g = \sum_i g_i$ is locally measurable, (i.e., its restriction to each measurable set of finite measure is measurable), and

$$\| g \|_\infty = \sup_i \| g_i \|_\infty \leq \| \psi \|$$

which means that $g \in L^\infty(\mu, E^*)$.

For $p > 1$, we have

$$\sum_{j \in J} \| g_j \|_j^j = \| g_J \|_q^q = \| \psi \|^q \ \leq \| \psi \|^q,$$

which shows that all but a countable number of the functions g_i are zero almost everywhere, and therefore, for the sake of simplicity, we may assume that $I = \{1, 2, 3, \ldots\}$. Consequently, $g = \sum_i g_i$ is measurable, and

$$\| g \|_q^q = \sum_i \| g_i \|_q^q \leq \| \psi \|^q,$$

which proves that $g \in L^q(\mu, E^*)$.

Next we show that

$$\psi(f) = \int_\Omega \langle f, g \rangle d\mu \text{ for all } f \in L^p(\mu, E).$$

Let $f \in L^p(\mu, E)$ and write $f = \sum_i f_i$, where, for each $i \in I, f_i = f$ on Ω_i and zero outside Ω_i. Since the support of an integrable function is σ-finite, and since each of σ-finite measurable set is contained in the union of a countable subfamily of $\{\Omega_i : i \in I\}$, we may again assume that $I = \{1, 2, 3, \ldots\}$.

For each $n = 1, 2, 3, \ldots$, let $h_n = \sum_{i=1}^n f_i$. Then, by the dominated convergence theorem, the sequence $\langle h_n \rangle$ converges to f in $L^p(\mu, E)$. It is clear that for each n,

$$\psi(h_n) = \int_\Omega \left(\sum_{i=1}^n \langle f_i, g_i \rangle \right) d\mu,$$

and, since f_i's as well as g_i's have disjoint supports, it is also clear that

$$\left|\sum_{i=1}^{n}\langle f_i(x), g_i(x)\rangle\right| \leq \| f(x) \| \| g(x) \|$$

for all $x \in \Omega$ and $n = 1, 2, 3, \ldots$ Therefore, by the dominated convergence theorem, and the fact that the ψ is continuous on $L^p(\mu, E)$, we have

$$\begin{aligned} \int_\Omega \langle f, g\rangle d\mu &= \lim_n \int_\Omega \left(\sum_{i=1}^{n}\langle f_i, g_i\rangle\right) d\mu \\ &= \lim_n \psi(h_n) = \psi(f) \end{aligned}$$

for all $f \in L^p(\mu, E)$, proving our claim.

Conversely, we now assume that $L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ for some $1 \leq p < \infty$, and show that E^* has the Radon-Nikodým property with respect to μ. To this end we let $\lambda : \mathcal{A} \to E^*$ be a μ-continuous vector measure of bounded variation. Since $L^p(\Omega_i, E)^* \simeq L^q(\Omega_i E^*)$ forall $i \in I$, and the theorem holds for finite measures, for each $i \in I$, there is an integrable function $g_i : \Omega \to E^*$ that vanishes outside Ω_i and such that

$$\lambda(A_i) = \int_{A_i} g_i d\mu \text{ for all } A_i \in \mathcal{A}_i,$$

where $\mathcal{A}_i$ is the trace of $\mathcal{A}$ on Ω_i. Let $g = \sum_i g_i$. Obviously g is locally measurable, but we want to show that it is indeed measurable (i.e., μ-essentially separably valued). Since λ is of bounded variation, $|\lambda|(\Omega)$ is finite which implies that $|\lambda|(\Omega) = 0$ for all but countably many $i \in I$, where $|\lambda|$ denotes the total variation of λ. Thus, here again we may assume that $I = \{1, 2, 3, \ldots\}$, which implies that g is measurable, and since $|\lambda|(\Omega) = \int_\Omega \| g(\cdot) \| d\mu < \infty$, we conclude that g is integrable.

Since I is countable, now by the dominated convergence theorem, it follows that

$$\lambda(A) = \int_A g d\mu$$

for all $A \in \mathcal{A}$ completing the proof.

Since reflexive Banach spaces have the Radon-Nikodým property with respect to any finite measure [4, p. 76], and the preceding proof can be used to conclude that this property with respect to any perfect measure, we have the following corollary. □

Corollary. *For any measure* μ *and reflexive Banach space* $E, L^p(\mu, E)^* \simeq L^q(\mu, E^*)$ *where* $1 \leq p < \infty$, $1 < q \leq \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$.

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