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ON THE DIFFERENTIAL PRIME RADICAL OF A DIFFERENTIAL RING

*D. Khadjiev**& F. Çallıalp*

Abstract

In this paper we have obtained the following results for a differential ring (associative or nonassociative):

(1) For a differential ring ($\mathcal{D}$-ring) we introduce definitions of a $\mathcal{D}$-prime $\mathcal{D}$-ideal, $\mathcal{D}$-semiprime $\mathcal{D}$-ideal and a strongly $\mathcal{D}$-nilpotent element. We define the $\mathcal{D}$-prime radical as the intersection of all $\mathcal{D}$-prime $\mathcal{D}$-ideals. For any $\mathcal{D}$-ring the $\mathcal{D}$-prime radical, the intersection of all $\mathcal{D}$-semiprime $\mathcal{D}$-semiprime $\mathcal{D}$-ideals and the set of all strongly $\mathcal{D}$-nilpotent elements are equal.

(2) For a $\mathcal{D}$-ring we introduce a definition of an s-nilpotent $\mathcal{D}$-ideal. If a $\mathcal{D}$-ring satisfies the ascending chain condition for $\mathcal{D}$-ideals then its $\mathcal{D}$-prime radical is s-nilpotent.

(3) Let $\mathcal{Q}$ be a field of rational numbers. If δ is a differentiation of a $\mathcal{Q}$-algebra R with 1 then $\delta(Pr.rad(R)) \subseteq Pr.rad(R)$.

(4) Let K be a differential ring. Then every radical $\mathcal{D}$-ideal of K is an intersection of $\mathcal{D}$-prime $\mathcal{D}$-ideals.

1. The differential prime radical

This paper is a continuation of our papers [1-3]. Further, we use notions and notations of books [4-6].

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Definition *([7], p.556). A differential ring ($\mathcal{D}$-ring) is a system (K,$\mathcal{D}$), where K is a ring (associative or nonassociative) and $\mathcal{D}$ is a set of differentiations of K. A $\mathcal{D}$-subgroup H is an additive subgroup of the ring K such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.*

Let K be a $\mathcal{D}$-ring. Denote by $Ad(K)$ the set of all $\mathcal{D}$-subgroups of K. $Ad(K)$ is a complete lattice with respect to the inclusion relation. Introduce a multiplication operation on it by the following manner ([5], p.12): For $A, B \in Ad(K) A \cdot B$ consists of all finite sums $\sum_{i=1}^{n} a_i b_i$, where $a_i \in A, b_i \in B$.

Definition *([7], p.556). A differential ideal ($\mathcal{D}$-ideal) of K is an ideal H of K such that $dh \in H$ for all $h \in H, d \in \mathcal{D}$.*

Denote by $Id(K)$ the set of all $\mathcal{D}$-ideals of K. $Id(K)$ is a complete lattice with respect to the inclusion relation. If K is associative then $A \cdot B \in Id(K)$ for all $A, B \in Id(K)$.

But there is nonassociative $\mathcal{D}$-ring K and $A, B \in Id(K)$ such that $A \cdot B \notin Id(K)$.

Therefore for any $\mathcal{D}$-ring we define a multiplication operation of $\mathcal{D}$-ideals in the following manner. For $A, B \in Id(K)$ denote by $A \cdot B$ the intersection of all $\mathcal{D}$-ideals of K containing the set $\{x \in K : x = a \cdot b, a \in A, b \in B\}$.

Proposition 1.1. *For any $\mathcal{D}$-ring K the lattice $Id(K)$ with above multiplication operation is a complete l-groupoid.*

Proof. Let $A, B_t \in Id(K)$, $t \in T$. The inequality

$$A \cdot (\vee_{t \in T} B_t) \geq V_{t \in T}(A \cdot B_t)$$

is obvious. We now prove the inverse inequality. The $\mathcal{D}$-ideal $A \cdot (\vee_{t \in T} B_t)$ is the smallest $\mathcal{D}$-ideal containing all elements $a \cdot (b_1 + b_2 + \ldots + b_k)$, where $a \in A$, $b_i \in B_{t_i}$ From the equality $a \cdot (b_1 + b_2 + \ldots + b_k) = ab_1 + \ldots + ab_k$ we obtain that $a \cdot (b_1 + b_2 + \ldots + b_k) \in \vee_{t \in T}(A \cdot B_t)$. Therefore

$$A \cdot (\vee_{t \in T} B_t) \leq \vee_{t \in T}(A \cdot B_t).$$

A proof of the equality

$$(\vee_{t\in T}B_t)\cdot A=\vee_{t\in T}(B_t\cdot A)$$

is similar. $\square$

Definition *A $\mathcal{D}$-ideal P of K is $\mathcal{D}$-prime if $P\neq K$ and $A\cdot B\subseteq P$, $A,B\in Id(K)$, implies that $A\subseteq P$ or $B\subseteq P$.*

For $A\in Id(K)$, $A\neq K$, denote by $R(A)$ the intersetion of all $\mathcal{D}_{s^-}$ semiprime $\mathcal{D}$-ideals of K containing A. Put $r^s(A)=K$ if there are none.

For $A\in Id(K)$, denote by $<A>$ the groupoid generated by A. An element of the groupoid $<A>$ will be denoted by $f(A)$.

Definition *A $\mathcal{D}$-ideal H of K is $\mathcal{D}_w$-semiprime if $H\neq K$ and $f(A)\subseteq H$, $A\in Id(K)$, $f(A)\in<A>$, implies that $A\subseteq H$.*

For $A\in Id(K)$, $A\neq K$, denote by $r^w(A)$ the intersection of all $\mathcal{D}_{w^-}$ semiprime $\mathcal{D}$-ideals of K containing A. Put $r^w(A)=K$ if there are none. It is clear $r^s(A)\subseteq r^w(A)\subseteq R(A)$ for all $A\in Id(K)$.

For $A\in Id(K)$ the $\mathcal{D}$-ideal $R(A)$ will be called $\mathcal{D}$-radical of A.

Definition *A $\mathcal{D}$-ideal M of K is $\mathcal{D}$-maximal if $M\neq K$ and $M\subseteq B\subseteq K$, $B\in Id(K)$, implies that $M=B$ or $B=K$.*

Proposition 1.2 *Let K be a $\mathcal{D}$-ring satisfying the ascending chaim condition for $\mathcal{D}$-ideal. Then every $\mathcal{D}$-ideal of K is contained in some $\mathcal{D}$-maximal $\mathcal{D}$-ideal. In particular, there is a $\mathcal{D}$-maximal $\mathcal{D}$-ideal of K.*

A proof is standard.

Proposition 1.3. *Let K be a $\mathcal{D}$-ring such that $K^2=K$. Then any $\mathcal{D}$-maximal $\mathcal{D}$-ideal K is $\mathcal{D}$-prime.*

Proof. Let M be $\mathcal{D}$-maximal $\mathcal{D}$-ideal of K. Supporse that $A\cdot B\subseteq M$, $A,B\in Id(K)$. If $A\not\subseteq M$, then $A\vee M=K$. Therefore

$$K \cdot K = (A \vee M) \cdot (B \vee M) = A \cdot B \vee A \cdot M \vee M \cdot B \vee M \cdot M \subseteq M \subseteq K.$$

We obtain $M = K$. This is a contradiction. □

Remark *The condition of Proposition 1.3 fulfils for $\mathcal{D}$-rings with 1.*

Proposition 1.4 *Let K be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. Then the following conditions are equivalent:*
(1) $K^2 = K$;
(2) Every $\mathcal{D}$-maximal $\mathcal{D}$-ideal of K is $\mathcal{D}$-prime.

Proof. $(1) \Rightarrow (2)$ follows from proposition 1.3.
$(1) \Rightarrow (2)$: Assume that $K^2 \neq K$. By proposition 1.2 there exists a $\mathcal{D}$-maximal M of K such that $K^2 \subseteq M$. It is contradiction since M is $\mathcal{D}$-prime. □

For an element $a \in K$ denote by [a] the smallest $\mathcal{D}$-ideal containing a.

Everey sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$, where $x_0, x_{n+1} \in [x_n]^2$, will be called a $\mathcal{D}$-sequence of the element a.

Definition *An element $a \in K$ is strongly $\mathcal{D}$-nilpotent if every its $\mathcal{D}$-sequence is ultimately zero.*

Remark *This definition is a generalization of differential rings of the similar definition in ([5], p.55; [1], p.574).*

Denote by $n(0)$ the set of all strongly $\mathcal{D}$-nilpotent elements of K, where 0 is zero ideal of K.

Theorem 1.5 *For any $\mathcal{D}$-ring K the equalities $n(0) = r^s(0) = r^w(0) = R(0)$ hold.*

Proof. First we prove that $n(0) \subseteq r^s(0)$. If there are no $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideals then $r^s(0) = K$. Hence $n(0) \subseteq r^s(0)$. Assume that there is a $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal. Let

$a \in n(0)$ and S be and $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal. Prove that $a \in S$. Assume that $a \notin S$. Then $[x_0] \nsubseteq S$, where $x_0 = a$. There exists $x_1 \in [x_0]^2$ such that $x_1 \notin S$. Continuing in this manner we obtain a $\mathcal{D}$-sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$ of the element a such that $x_n \notin S$ for all n. But it is a contradiction since every $\mathcal{D}$-sequence of the element a is ultimetely zero.

Thus $a \in S$ and $a \in r^s(0)$ since S is any $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal. Hence $n(0) \subseteq r^s(0) \subseteq r^w(0) \subseteq R(0)$.

Prove that $R(0) \subseteq n(0)$. If $n(0) = K$ then $n(0) = r^s(0) = r^w(0) = R(0) = K$. Let $n(0) \neq K$. Let $b \in K$ such that $b \notin n(0)$. Then there exists a $\mathcal{D}$-sequence $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ of the element b such that $X \cap 0 = \emptyset$, where 0 is the zero ideal of K.

Denote by $\sum$ the set of $\mathcal{D}$-ideals M of K such that $X \cap M = \emptyset$. Then $\sum$ is not empty since $0 \in \sum$. We can apply Zorn's lemma to the set $\sum$; so there exists a maximal element P of $\sum$. Show that P is $\mathcal{D}$-prime.

First, P is proper since $b \in P$. Let $B, C \in Id(K)$, $B \nsubseteq P$, $C \nsubseteq P$. Then $P \vee B \neq P$ and $P \vee C \neq P$. By the maximality of P in $\sum$ we have $P \vee B \notin \sum$ and $P \vee C \notin \sum$. Hence there exist $x_m \in X$, $x_q \in X$ such that $x_m \in P \vee B$, $x_q \in P \vee C$. Then

$$[x_m] \subseteq P \vee M, \quad [x_q] \subseteq P \vee C.$$

Hence

$$x_{m+1} \in [x_m]^2 \subseteq P \vee B, \quad x_{q+1} \in [x_q]^2 \subseteq P \vee C.$$

Continuing in this manner we find that

$$x_{m+t} \in P \vee B, \quad x_{q+t} \in P \vee C$$

for all t. Put $n = \max(m, q)$. Then

$$x_n \in P \vee B, \quad x_n \in P \vee C.$$

Hence

$$x_{n+1} \in [x_n]^2 \subseteq (P \vee B)(P \vee C) \subseteq P \vee B \cdot C,$$

by Proposition 1.1. But $x_{n+1} \notin P$. Hence $B \cdot C \not\subseteq P$. Therefore P is $\mathcal{D}$-prime. Thus there exists a $\mathcal{D}$-prime $\mathcal{D}$-ideal P such that $b \notin P$. Then $n(0) = r^s(0) = r^w(0) = R(0)$. □

The $\mathcal{D}$-ideal $R(0)$ will be called differential prime radical of K and will be denoted by $\mathcal{D}Pr.rad(K)$.

Definition *$\mathcal{D}$-ideal H of K is $\mathcal{D}$-radical if $H = R(H)$.*

For $A \in Id(K)$ denote by $n(A)$ the set of all elements $x \in K$ that every $\mathcal{D}$-sequence of x meets A.

Corollary 1 *For any $A \in Id(K)$ the following equalities hold:*

$$n(A) = r^s(A) = r^w(A) = R(A).$$

In particular, every $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal is $\mathcal{D}$-radical.

Proof. Applying theorem 1.5 to the quotient $\mathcal{D}$-ring K/A, we obtain $n(A) = r^s(A) = r^w(A) = R(A) = \mathcal{D}Pr.rad(K/A)$. □

Corollary 2. *For a $\mathcal{D}$-ring K the following conditions are equivalent:*

1. Every $\mathcal{D}$-ideal of K is $\mathcal{D}$-radical;

2. $A \cdot B = A \cap B$ for all $A, B \in Id(K)$;

3. $[a]^2 = [a]$ for all $a \in K$.

Proof. We use the following lemma: □

Lemma *Let K be a $\mathcal{D}$-ring. Then*

$$R(A \cdot B) = R(A \cap B) = R(A) \cap R(B)$$

for any $A, B \in Id(K)$. The proof of this lemma follows from proposition 1.6 in [8],

$(1) \Rightarrow (2)$: *If every $\mathcal{D}$-ideal of K is $\mathcal{D}$-radical then using the lemma we obtain*

$$A \cdot B = R(A \cdot B) = R(A) \cap R(B) = A \cap B.$$

$(2) \Rightarrow (3)$: *Let $A \cdot B = A \cap B$ for any $A, B \in Id(K)$. Then $A^2 = A$ for any $A \in Id(K)$.*

$(3) \Rightarrow (1)$: *Prove that every $\mathcal{D}$-ideal of K is $\mathcal{D}_s$-semiprime. Let A be a $\mathcal{D}$-ideal of K. Then $A = \vee_{a \in A}[a]$. Using proposition 1.1 we have*

$$\begin{aligned} A^2 &= (\vee_{a \in A}[a])^2 = (\vee_{a \in A}[a]^2) \vee (\vee_{a,b \in A}[a] \cdot [b]) = \\ &\quad (\vee_{a \in A}[a]) \vee (\vee_{a,b \in A}[a] \cdot [b]) = \vee_{a \in A}[a] = A, \end{aligned}$$

since $[a] \cdot [b] \subseteq [a] \cap [b]$ for any $a, b \in A$. Thus $A^2 = A$ for any $A \in Id(K)$. Let $B^2 \subseteq A$, $B \in Id(K)$. Then $B = B^2 \subseteq A$. Therefore every $\mathcal{D}$-ideal A is $\mathcal{D}_s$-semiprime. By Corollary 1 A is $\mathcal{D}$-radical.

Remark *This corollary is a generalization of the similar theorem in ([7], ch.4, §5).*

Let $A \in Id(K)$. Put $A^{(0)} = A$, $A^{(n+1)} = (A^{(n)})^2$.

Corollary 3 *For a $\mathcal{D}$-ring K the following conditions are equivalent:*

(1) $\mathcal{D}Pr.rad(K) = 0$

(2) If $A^{(n)} = 0$, $A \in Id(K)$, for some n then $A = 0$;

(3) If $A^2 = 0$, $A \in Id(K)$, then A=0.

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ is obvious.

Prove that $(3) \Rightarrow (1)$. Condition (3) implies that $r^s(0) = 0$. By theorem 1.5 we see $Pr.rad(K) = r^s(0) = 0$. □

Definition *A $\mathcal{D}$-ideal A of K is s-nilpotent if $A^{(n)} = 0$ for some n.*

Proposition 1.6 *Let K be a $\mathcal{D}$-rign and $A, B \in Id(K)$, $A \subseteq B$. If A is s-nilpotent and B/A is s-nilpotent in K/A then B is s-nilpotent in K.*

Proof. Since B/A is s-nilpotent the $B^{(n)} \subseteq A$ for some n. Then $B^{(n+m)} = (B^{(n)})^{(m)} = 0$ since A is s-nilpotent. □

Theorem 1.7 *Let K be a $\mathcal{D}$-rign satisfying the ascending chain condition for $\mathcal{D}$-ideals. The $\mathcal{D}Pr.rad(K)$ is s-nilpotent.*

Proof. Denote by $\sum$ the set of s-nilpotent $\mathcal{D}$-ideals of K. $\sum$ is not empty since $0 \in \sum$. There exists a maximal element P in $\sum$. By proposition 1.6 the $\mathcal{D}$-ring K/P have the following property: if $(A/P)^2 = 0$, $A \in Id(K)$, $P \subseteq A$, then $A/P = 0$. By corollary 3 of theorem 1.5 we have $\mathcal{D}Pr.rad(K/P) = 0$. Thes means that $\mathcal{D}Pr.rad(K) \subseteq P$. But $P \subseteq \mathcal{D}Pr.rad(K)$ since P is s-nilpotent. Therefore $\mathcal{D}Pr.rad(K) = P$. □

Corollary *Let K be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. Then the followings are equivalent:*

(1) $K^{(n)} = 0$ for some n.

(2) K has not a $\mathcal{D}$-prime $\mathcal{D}$-ideal;

(3) K has not a $\mathcal{D}_s$-semiprime $\mathcal{D}$-ideal.

Denote by $Id_r(K)$ the set all $\mathcal{D}$-radical $\mathcal{D}$-ideals of K. $Id_r(K)$ is a complete lattice with respect to the inclusion relation. Denote by $\vee$ and $\wedge$ the lattice operations in $Id_r(K)$.

Theorem 1.8 *Let K be a $\mathcal{D}$-ring. Then the lattice $Id_r(K)$ satisfies the infinte $\wedge$-distributive condition:*

$$A \wedge (\vee_{\tau \in T} B_\tau) = \vee_{\tau \in T} (A \wedge B_\tau)$$

for any $A, B \in Id_r(K)$. In particular, $Id_r(K)$ is distributive.

A proof follows from Theorem 1.3 in [8].

Theorem 1.9 *Let K be a $\mathcal{D}$-ring satisfying the ascending chain condition for $\mathcal{D}$-ideals. Then any $\mathcal{D}$-radical $\mathcal{D}$-ideal A is an intersection of finite $\mathcal{D}$-prime $\mathcal{D}$-ideals and a such representation of A is unique.*

Proof. First prove the following. □

Lemma *$A \in Id_r(K)$ is a $\mathcal{D}$-prime $\mathcal{D}$-ideal iff A is an $\wedge$-indecomposable element of the lattice $Id_r(K)$.*

Proof. Let A be a $\mathcal{D}$-prime $\mathcal{D}$-ideal of K and $A = A_1 \wedge A_2$, $A_1, A_2 \in Id_r(K)$. Then

$$A_1 \cdot A_2 \subseteq A_1 \cap A_2 \subseteq R(A_1 \cap A_2) = A_1 \wedge A_2 = A.$$

Hence $A_1 \subseteq A$ or $A_2 \subseteq A$ since A is $\mathcal{D}$-prime. Then $A = A_1$ or $A = A_2$.

Let A be an $\wedge$-indecomposable element of the lattice $Id_r(K)$ and $B \cdot C \subseteq A$, $B, C \in Id_r(K)$. Then $R(B \cdot C) \subseteq A$. By lemma 1.6 in [8] we have $R(B) \wedge R(C) = R(B \cdot C) \subseteq A$. We obtain

$$A = A \vee (R(B) \wedge R(C)) = (A \vee R(B)) \wedge (A \vee R(C))$$

since $Id_r(K)$ is distributive. Hence $A = A \vee R(B)$ or $A = A \wedge R(C)$ since A is an $\wedge$-indecomposable. This means that $B \subseteq R(B) \subseteq A$ or $C \subseteq R(C) \subseteq A$. Thus A is $\mathcal{D}$-prime. The lemma is proved.

By the lemma and the corollary in ([4], p.183), we obtain the every $\mathcal{D}$-radical $\mathcal{D}$-ideal of K is an intersection of finite $\mathcal{D}$-prime $\mathcal{D}$-ideals of K and such a represantation is unique. □

Remark *This theorem is a generalization of the similar statement from the theory of associative rings.*

Let $A \in Id(K)$. Put $N_0(A) = A$. Denote by $N_1(A)$ the supremum of all $B \in Id(K)$ such that $B^{(n)} \subseteq A$ for some n (n depends from B). Put $N_\alpha(A) = N_1(N_\beta(A))$ for $\alpha = \beta + 1$ and $N_\alpha(A) = \vee_{\beta<\alpha} N_\beta(A)$ for α a limit ordinal

Put $L(K, \mathcal{D}) = N_\alpha(0)$ for any ordinal α of cardinality $\geq |K|$.

Theorem 1.10 *$L(K,\mathcal{D}) = \mathcal{D}Pr.rad(K)$ for $\mathcal{D}$-ring K.*

Proof. By theorem 1.5 it is enough to prove that $L(K,\mathcal{D})r^s(0)$.

It is clear that $N_1(0) \subseteq r^s(0)$. By transfinite induction we obtain $N_\alpha(0) \subseteq r^s(0)$ for any ordinal α. Hence $L(K,\mathcal{D}) \subseteq r^s(0)$.

Prove that $L(K,\mathcal{D})$ is $\mathcal{D}_s$-semiprime. Assume that $B^2 \subseteq L(K,\mathcal{D})$. Then there exists an ordinal α such that $B^2 \subseteq N_\alpha(0)$. Hence $B \subseteq N_{\alpha+1}(0)$ by definition of $N_{\alpha+1}(0)$. Therefore $B \subseteq L(K,\mathcal{D})$ and $L(K,\mathcal{D})$ is $\mathcal{D}_s$-semiprime.
By the definition of $r^s(0)$ we have $L(K,\mathcal{D}) = r^s(0)$. □

2. $\mathcal{D}$-algebras over the field of rational numbers

Let $\mathcal{Q}$ be the field of rational numbers. Put $\mathcal{D}Pr.rad(K) = Pr.rad(K)$ and $L(K,\mathcal{D}) = L(K)$ if $\mathcal{D} = \emptyset$.

$\mathcal{D}$-sequence of $a \in K$ will be called n-sequence of a if $\mathcal{D} = \emptyset$.

For $a \in K$ we denote its n-sequence $\{x_0(a) = a, x_1, \ldots, x_m, \ldots\}$ in the form:

$$\{x_0(a) = a, x_1(a), \ldots, x_m(a), \ldots\}.$$

If R is associative then every element $x_m(a) \in [x_{m+1}(a)]^2 \subseteq [a]^{2m}$ is a finite sum:

$$x_m(a) = \sum f_1 f_2 \ldots f_s,$$

where $s = 2^m$ and every f_i has the form $f_i = r_{1i} a r_{2i}$. Thus $x_m(a)$ is a homogeneous polynomial of a a degree 2^m with coefficients from R.

If R is nonassociative then every element $x_m(a)$ is a homogeneous nonassociative polynomial of degree 2^m with coefficients from R.

Theorem 2.1 *If R is $\mathcal{Q}$-algebra with 1 and δ is a differentiation of R, then*

$$\delta(Pr.rad(R)) \subseteq Pr.rad(R).$$

Proof. Let $a \in Pr.rad(R)$. We consider any n-sequence of the element δa:

$$\{x_0(\delta a) = \delta a, x_1(\delta a), \ldots, x_m(\delta a), \ldots, \}.$$

We have

$$\delta^{2m}(x_m(a)) \in q \cdot x_m(\delta a) + [a],$$

where $q \neq 0$ is an integer. Then $x_m(\delta a) \in \frac{1}{q}\delta^{2m}(x_m(a)) + [a]$. Since $a \in Pr.rad(R)$ then there exist m_0 such that $x_m(a) = 0$ for all $m \geq m_0$. Then the following sequence

$$\{y_0(a) = a, y_1(a), \ldots, y_k(a), \ldots\},$$

where $y_k(a) = x_{m_0+k}(\delta a)$, is an n-sequence of the element a. Therefore there exists k_0 such that $y_k(a) = 0$ for all $k \geq k_0$. This means that any n-sequence of the form

$$\{x_0(\delta a) = \delta a, x_1(\delta a), \ldots, x_m(\delta a), \ldots, \}$$

of element δa is ultimately zero. Thus $\delta a \in Pr.rad(R)$. □

Remark *This theorem is the Ritt's theorem if R is a commutative associative $\mathcal{Q}$-algebra ([6], p.12). If R is an associative $\mathcal{Q}$-algebra then there is only the formulation of this theorem and putline of its prof in ([10], p.207).*

Bu this instruction is not correct (the member n in proposition 2.6.28 in ([10], p.207) depends for r_{1i}, r_{2i}). By this instruction the theorem may be proved for only Noetherian rings.

Further we investigate connections between radical $\mathcal{D}$-ideals and $\mathcal{D}$-radical $\mathcal{D}$-ideals in $\mathcal{D}$-rings.

If $\mathcal{D} = \emptyset$ then in §1 we obtain results for usual rings.

Corollary *Let K be a differential $\mathcal{Q}$-algebra with 1. Then radical of any $\mathcal{D}$-ideal of K is a $\mathcal{D}$-ideal of K.*

Proof. Let H be a $\mathcal{D}$-ideal of K. Then by Theorem 2.1 $Pr.rad(K/H)$ is a $\mathcal{D}$-ideal. This means that $rad(H)$ is a $\mathcal{D}$-ideal of K. □

Let K be a ring. For $a \in K$ denote by (a) the intersection of all ideals of K containing a.

For ideals A, B of K denote by $A \star B$ the intersection of all ideals of K containing the set $\{x \in K : x = a \cdot b,\ a \in A,\ b \in B\}$.

Proposition 2.2 *Let K be a ring and B be an ideal of K. Then B is prime iff for every $t_1, t_2 \in K \backslash B$ such that $\in (t_1) \star (t_2)$.*

A proof is obvious.

Theorem 2.3 *Let H be a radical $\mathcal{D}$-ideal of a $\mathcal{D}$-rign K. Then H is an intersection of $\mathcal{D}$-prime $\mathcal{D}$-ideals.*

Proof. Let $x \notin H$. Then there exists a prime ideal B of K such that $H \subseteq B$ and $x \notin B$.

Denote by $\sum$ the set of $\mathcal{D}$-ideals A of K such that $H \subseteq A$ and $A \cap (K \backslash B) = \emptyset$. $\sum \neq \emptyset$ since $H \in \sum$. By Zorn's lemma there exists a maximal element P in $\sum$. Prove that P is a $\mathcal{D}$-prime $\mathcal{D}$-ideal. P is proper since $x \notin P$.

Let $A_1, A_2 \in Id(K)$ and $A_1 \nsubseteq P$, $A_2 \nsubseteq P$. Then $P \vee A_1 \neq P$, $P \vee A_2 \neq P$. There exist $t_1 \in K \backslash B$, $t_2 \in K \backslash$ such that $t_1 \in P \vee A_1$, $t_2 \in P \vee A_2$. Then

$$(t_1) \subseteq P \vee A_1, \quad (t_2) \subset P \vee A_2.$$

By proposition 1.1 we have

$$(t_1) \star (t_2) \subseteq (P \vee A_1) \cdot (P \vee A_2) \subseteq P \vee A_1 \cdot A_2.$$

By proposition 2.2 for $t_1, t_2 \in K \backslash B$ there exists $t \in K \backslash B$ such that $t \in (t_1) \star (t_2)$. Then $t \in P \vee A_1 \cdot A_2$. This means that $A_1 \cdot A_2 \nsubseteq P$. Therefore P is a $\mathcal{D}$-prime $\mathcal{D}$-ideal and $H \subseteq P$. Thus for any $x \notin H$ there exists a $\mathcal{D}$-prime $\mathcal{D}$-ideal P such that $H \subseteq P$ and $x \notin P$. This means that H is an intersection of all $\mathcal{D}$-prime $\mathcal{D}$-ideals containing H. □

Theorem 2.4 *Let K be a differential $\mathcal{Q}$-algebra with 1. Then $Pr.rad(K)$ is an intersection of all $\mathcal{D}$-prime $\mathcal{D}$-ideals of K containing $Pr.rad(K)$ and $\mathcal{D}Pr.rad(K) \subseteq Pr.rad(K)$.*

Proof. By theorem 2.1 $Pr.rad(K)$ is a $\mathcal{D}$-ideal of K. By theorem 2.3 $Pr.rad(K)$ is an intersection of all $\mathcal{D}$-prime $\mathcal{D}$-ideals of K containing $Pr.rad(K)$. Hence $\mathcal{D}Pr.rad(K) \subseteq Pr.rad(K)$. □

Theorem 2.5 *Let K be a differential $\mathcal{Q}$-algebra with 1. Assume that K satisfies the ascending chain condition for ideals. Then $\mathcal{D}Pr.rad(K) = Pr.rad(K)$.*

Proof. In this case $Pr.rad(K)$ is s-nilpotent by theorem 1.7 $\mathcal{D} = \emptyset$. Therefore $Pr.rad(K))^{(n)} = 0$ for some n. Since $Pr.rad(K)$ is a $\mathcal{D}$-ideal $[Pr.rad(K)]^{(r)} = (Pr.rad(K))^{(n)} = 0$. Then by theorem 1.10 $Pr.rad(K) \subseteq \mathcal{D}Pr.rad(K)$. Thus $\mathcal{D}Pr.rad(K) = Pr.rad(K)$. □

Corollary *Let K be a differential $\mathcal{Q}$-algebra with 1. Assume that K satisfies the ascending chain condition for ideals. Then every $\mathcal{D}$-radical $\mathcal{D}$-ideal of K is radical.*

Proof. The statement follows from theorem 2.5. □

Remark *Theorems 2.3-2.5 are known for commutative differential rings [1].*

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Djavvat KHADJIEV
Tashkent State University
Mech. Math. Fac.
700095 Tashkent-UZBEKİSTAN
Fethi ÇALLIALP
Tech. University of İstanbul
Faculty of Arts and Sciences
Department of Math. İstanbul-TURKEY

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