Tr. J. of Mathematics
22 (1998) , 369 – 377.

NORMAL SUBGROUPS OF HECKE GROUPS ON SPHERE AND TORUS*

İsmail Naci Cangül & Osman Bizim

Abstract

We use regular map theory to obtain all normal subgroups of Hecke groups of genus 0 and 1. The existence of a regular map corresponding uniquely to every normal subgroup of Hecke groups $H(\lambda_q)$ is a result of Jones and Singerman, and it is frequently used here to obtain normal subgroups. It is found that when q is even, $H(\lambda_q)$ has infinitely many normal subgroups on the sphere, while for odd q, this number is finite. The total number of normal subgroups of $H(\lambda_q)$ on a torus is found to be either 0 or infinite. The latter case appears iff q is a multiple of 4. Finally, a result of Rosenberger and Kern-Isberner is reproved here.

Keywords: Hecke groups, genus, regular maps

1. Introduction

Hecke groups $H(\lambda_q)$ are the discrete subgroups of PSL(2, $\mathbf{R}$) generated by two linear fractional transformations $R(z) = -1/z$ and $T(z) = z+\lambda_q$, where $\lambda_q = 2\cos(\pi/q), q \in \mathbf{N}$, $q \geq 3$. Let S(z)=RT(z). Then R and S are elliptic elements (rotations) of orders 2 and q, respectively, and T is parabolic. $H(\lambda_q)$ is a Fuchsian group of the first kind with signature (0; 2, q, ∞), and therefore can be considered as a triangle group with a parabolic generator.

* This work is partly based on the first author's PhD Thesis

The study of subgroups of Hecke groups, specially, of those which are normal, has been done in [1]. One of the ways of studying them is to make use of regular map theory. *A map M* is an embedding (without crossings) of a finite connected graph G into a compact connected surface S without boundary such that $S-G$ is a union of 2-cells. If m and n are the ℓ.c.m.s of the valencies of the faces and vertices, respectively, we then say M has *type* $\{m,\ n\}$. An *automorphism of M* is an orientation-preserving homeomorphism of S preserving the incidence relations. If the set of automorphisms of M is transitive on the set of edges, then M is called *regular.* All regular maps of genus ≤ 7 are known (see [2], [3] and [4]).

In [5], Jones and Singerman proved the existence of a 1:1 correspondence between regular maps and normal subgroups of certain triangle groups including Hecke groups. If N is a normal subgroup of $H(\lambda_q)$ corresponding to a regular map of type {m, n}, then n corresponds to the level of N, i.e. the least positive integer so that $T^n \in N$. This correspondence can be used to prove many results concerning normal subgroups of $H(\lambda_q)$. For example, as all regular maps of genus less than 8 are classified, we can easily find all normal subgroups of genus < 8 of $H(\lambda_q)$. Another nice application of regular map theory is the determination of the normal subgroups of g=0 and g=1 of Hecke groups. First we consider the case of g=0:

2. Normal Genus 0 Subgroups of $H(\lambda_q)$

Let N be a normal subgroup of genus 0 in $H(\lambda_q)$. Then $H(\lambda_q)/N$ is a group of automorphisms of $\hat{U}/N$ where $\hat{U} = U \cup \mathbf{Q} \cup \{\infty\}$. This gives a regular map on the sphere so that $H(\lambda_q)/N$ is isomorphic to one of the finite triangle groups. These are known to be isomorphic to A_4, S_4, A_5 C_n and D_n, for $n \in \mathbf{N}$. Now considering each of these groups as a quotient group of $H(\lambda_q)$, whenever possible, we can find all genus 0 normal subgroups of Hecke groups.

Let us begin with the cyclic group C_n of order n. If we map R to identity and S to the generator α of C_n where $n|q$, we obtain a homomorphism of $H(\lambda_q)$ to C_n. For each such n, we can obtain a normal subgroup N of genus 0 as the kernel of this homomorphism. By the permutation method, [1], N has the signature (0; $2^{(n)}, q/n, \infty$), i.e. it is isomorphic to the free product of n cyclic groups of order 2 and a cyclic group of order q/n. Corresponding regular maps are called star maps. They consist of a vertex surrounded by a number of edges (see Figure 1).

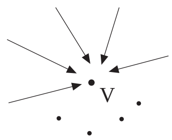

Figure 1.

Similarly, for each $n|q$, we obtain a normal subgroup N of genus 0 in $H(\lambda_q)$ such hat $H(\lambda_q)/N \cong D_n$, the dihedral group of order 2n. In that case, N has signature (0; q/n, q/n, $\infty^{(n)}$) and therefore is isomorphic to the free product of two cyclic groups of order q/n with n-1 infinite cyclic groups. Corresponding regular maps are regular polygons on the sphere.

The above classes of normal genus 0 subgroups of $H(\lambda_q)$ occur for any q. There are some more normal genus 0 subgroups whose existence completely depend on q. Recall that $A_4 \cong (2,3,3)$, $S_4 \cong (2,3,4)$ and $A_5 \cong (2,3,5)$. Now, if $3|q$, then $H(\lambda_q) \cong (2,q,\infty)$ can be mapped to A_4, S_4 and A_5 homomorphically. If $4|q$ ($5|q$, respectively), it can only be mapped to $S_4(A_5$, respectively). When $H(\lambda_q)/N$ is isomorphic to A_4, then the corresponding regular map is a tetrahedron. If $H(\lambda_q)/N \cong (2,3,4)$ we obtain an octahedron and when it is (2, 4, 3), a cube is obtained. Finally, when it is isomorphic to (2, 3, 5) ((2, 5, 3), respectively) an icosahedron (dodecahedron) is obtained.

There are also duals of regular polygons on the sphere corresponding to dihedral quotients $H(\lambda_q)/N \cong D_n \cong (2,2,n)$. This class of genus 0 normal subgroups is obtained only when q is even. As for each $n \in \mathbf{N}$, we can obtain a normal subgroup, this class contains infinitely many normal genus 0 subgroups of $H(\lambda_q), q$ even. Therefore we have the following theorem.

Theorem 1. **(i)** *If q is odd then $H(\lambda_q)$ has finitely many normal subgroups of genus 0; their number is given by*

$$(a)\ 2d(q) \qquad \textit{if } (q,15)=1,$$

(b) $2d(q)+3$ *if* $(q,15)=3,$

(c) $2d(q)+1$ *if* $(q,15)=5,$

(d) $2d(q)+4$ *if* $(q,15)=15,$

where $d(q)$ *denotes the number of divisors of* q.

(ii) *If q is even then* $H(\lambda_q)$ *has infinitely many normal genus 0 subgroups.*

Proof. We only prove (i) (b), i.e. we let q be odd such that (q,15)=3. As $H(\lambda_q)$ has signature $(2,q,\infty)$ it is only possible to map it to (1, m, n) or (2, m, n) where $m|q$ and $n \in \mathbf{N}$. Former ones give degenerate regular maps and therefore are omitted. As $m|q$, and as we are interested in the normal subgroups of genus 0, m is 3. This is because when q is odd we can only map $H(\lambda_q)$ to A_4, S_4 or A_5 to obtain normal subgroups corresponding to non-degenerate regular maps. This gives 3 normal genus 0 subgroups. There is also d(q) of them corresponding to cyclic quotients and d(q) of them corresponding to dihedral quotients. Therefore the result follows.

Other parts of the proof are similar and therefore omitted. □

3. Normal Genus 1 Subgroups of $H(\lambda_q)$

It is well-known that all regular maps of genus one are those of type {4,4}, {3,6} or {6, 3}. They are classified in [2] and [5] as $\{4,\ 4\}_{r,s}$, $\{3,\ 6\}_{r,s}$ and $\{6,\ 3\}_{r,s}$ for non-negative integers r and s, not both 0.

In [7], Kern-Isberner and Rosenberger proved some results on genus 1 normal subgroups of certain free products. Their method of proof was number theoretical. As Hecke groups are free products, Theorems 2 to 6 of this section can also be deduced from Theorems 1, 2 and 3 of this paper. Here we use the 1:1 correspondence mentioned and used above to find all genus 1 normal subgroups of Hecke groups. The proofs are direct results of this correspondence. We first have the following theorems.

Theorem 2. $H(\lambda_q)$ *has a normal subgroup of genus 1 iff* $q \equiv 0 \mod 3$ *or* $q \equiv 0 \mod 4$.

Proof. The existence of such a subgroup completely depends on the divisibility of q by 3, 4 and 6, as all regular maps on a torus are of type {4, 4}, {3, 6} and {6, 3}. For example, if $4|q$, then $H(\lambda_q)$ has normal genus 1 subgroups corresponding to regular maps of type $\{4, 4\}$. Now if N is a normal subgroup of genus 1 in $H(\lambda_q)$, then $3|q$, $4|q$ or $6|q$ implying $q \equiv 0 \mod 3$ or q≡0 mod 4. Conversely, if $q \equiv 0 \mod 3$ or $q \equiv 0 \mod 4$, then either 3 or 4 divides q. In the first (second) case $H(\lambda_q)$ has a normal genus 1 subgroup corresponding to a regular map of type {3,6} ({4, 4}, respectively). □

Theorem 3. *The total number of normal genus 1 subgroups of $H(\lambda_q)$ is either 0 or ∞.*

Proof. If $H(\lambda_q)$ has a normal genus 1 subgroup, then it corresponds to either $\{4, 4\}_{r,s}$, $\{3, 6\}_{r,s}$ or $\{6, 3\}_{r,s}$ for a pair of non-negative integers r and s. As r and s can be chosen in infinitely many ways, the result follows. □

We can characterize the freeness of a normal genus 1 subgroup of $H(\lambda_q)$ as follows:

Theorem 4. **(i)** *All normal subgroups of genus 1 of $H(\lambda_q)$ are free iff q=3 or 4.*
(ii) *The only values of q such that $H(\lambda_q)$ has a normal free subgroup of genus 1 are 3, 4 and 6.*

Proof. To have a free normal subgroup, we must map $(2, q, \infty)$ to a subgroup of the triangle group (2, q, n), where $n \in \mathbf{N}$. In that case the corresponding regular map will be type {q,n}. Because of g=1, q or n could only be 3, 4 or 6. Then result follows. □

Now we want to calculate the number of normal genus 1 subgroups of $H(\lambda_q)$. Note that if N is a normal genus 1 subgroup of $H(\lambda_4)$, then it is also a normal genus 1 subgroup of every $H(\lambda_{4k})$, $k \in \mathbf{N}$. Similarly, if M is a normal genus 1 subgroup of $H(\lambda_6)$, then it is also a normal genus 1 subgroup of every $H(\lambda_{6\ell})$, $\ell \in \mathbf{N}$. Therefore if we can calculate the number of normal genus 1 subgroups of $H(\lambda_4)$ and $H(\lambda_6)$, then we can calculate this number for all q. Theorem 2 implies that if $(q, 12) < 3$ then $H(\lambda_q)$ has no normal genus 1 subgroups. Let us begin with q=4 case. If N is a normal genus 1 subgroup of $H(\lambda_4)$, then it corresponds, in a 1:1 way, to a regular map $M = \{4, 4\}_{r,s}$. Also,

$$|AutM| = |H(\lambda_4) : N| = \mu = 4(r^2 + s^2).$$

Note that we obtain N by mapping $H(\lambda_4)$ to a normal subgroup of (2, 4, 4). This implies that the level of N is 4. Therefore

$$\mu = 4t,$$

where t is the parabolic class number of N. This implies that

$$t = r^2 + s^2.$$

Clearly, more than one pair (r, s) satisfy the last equation. We define an equivalence relation on these pairs: $(r_1, s_1) \approx (r_2, s_2)$ if $|r_1| = |r_2|$ or $|r_1| = |s_2|$, and $|s_1| = |r_2|$ or $|s_1| = |s_2|$ i.e. two pairs are equivalent if the entries of one can be transformed to the entries of the other by changing the signs and/or order. Now given $\mu = 4.t$, $H(\lambda_4)$ has as many normal subgroups of genus 1 with index μ as the number of possible "non-equivalent" pairs (r, s) such that $r^2 + s^2 = t$. Note that for each pair (r, s) there are 3 more equivalent pairs. As each set of equivalent pairs gives a normal genus 1 subgroup N, we obtain the following result:

Theorem 5. *The number $N_4(\mu)$ of normal genus 1 subgroups of $H(\lambda_4)$ of index μ is*

$$N_4(\mu) = 1/4.\#\{(r, s) : r, s \in \mathbf{Z},\ r^2 + s^2 = t\},$$

i.e. $N_4(\mu)$ is equal to a quarter of the number of representations of $t = \mu/4$ as the sum of two squares in $\mathbf{Z}$.

Remark 1. *This number has been found in [7] using the multiplicativity of $N_4(\mu)$.*

Using a well-known number-theoretical result (see e.g. [6]), we can determine $N_4(\mu)$ more explicity:

Theorem 6. *Let $t = 2^\alpha \prod_b P_b^{\ell_b} \prod_c q_c^{m_c}$ be the prime power decomposition of t, where $p_b \equiv 1$ (mod 4) and $q_c \equiv 3$ (mod 4). Then*

$$N_4(\mu) = r(t)/4,$$

where r(t) is the number of integer solutions of the Diophantine equation $x^2 + y^2 = t$ *given by r(t)=0 if one of the* m_c *is odd, and by*

$$r(t) = 4\prod_b (\ell_b + 1)$$

if all m_c *are even.*

Remark 2. *The first few values of* $N_4(\mu)$ *are given in the following table:*

t	1	2	3	4	5	6	7	8	9	10
μ	4	8	12	16	20	24	28	32	36	40
$N_4(\mu)$	1	1	0	1	2	0	0	1	1	2

In a similar way to the case q=4 discussed above, we can find the number of normal genus 1 subgroups of $H(\lambda_6)$ *as follows (there is ahint in [7] about how to prove Theorems 7 to 9. But the proofs we give here are completely different):*

Theorem 7. **(i)** *The number* $N_6(\mu)$ *of all normal subgroups of genus 1 of* $H(\lambda_6)$ *of index* μ *is*

$$N_6(\mu) = 1/3.\#\{(r,s) : r, S \in \mathbf{Z}, r^2 + rs + s^2 = t/2\}.$$

(ii) $N_6(\mu)/2$ *is the number of normal torsion (or torsion-free) subgroups of genus 1 of* $H(\lambda_6)$ *having index* μ.

Again using a well-known result (see e.g. [6]), we can express $N_6(\mu)$ *more explicitly:*

Theorem 8. $N_6(\mu) = 2\varepsilon$, *where* ε *is the number of divisors of* $\mu/6$ *of the form 3a+1 subtracting the number of divisors of the form 3b+2.*

Remark 3. **(i)** *The folloing table gives the first few values of* $N_6(\mu)$*:*

t	2	4	6	8	10	12	14	16	18	20
$\mu = 3t$	6	12	18	24	30	36	42	48	54	60
$N_6(\mu)$	2	0	2	2	0	0	4	0	2	0

(ii) $N_6(\mu)$ *is always even.*

We can now generalize all these to all values of q:

Theorem 9. *The number $N_q(\mu)$ of normal genus 1 subgroups of $H(\lambda_q)$ of index μ is*

$$N_q(\mu) = \begin{cases} 0 & \text{if } (q,12)=1 \text{ or } 12 \\ \beta/2 & \text{if } 3|q, q \text{ odd and } \mu = 3t_2 \\ 0 & \text{if } 3|q, q \text{ odd and } \mu \neq 3t_2 \\ \alpha & \text{if } 4|q, 3|q \text{ and } \mu = 4t_1 \\ 0 & \text{if } 4|q, 3|q \text{ and } \mu \neq 4t_1 \\ \beta & \text{if } 6|q, 4|q \text{ and } \mu = 3t_2 \\ 0 & \text{if } 6|q, 4|q \text{ and } \mu \neq 3t_2 \\ \alpha+\beta & \text{if } 12|q \text{ and } \mu = 3t_2 = 4t_1 \\ \alpha & \text{if } 12|q \text{ and } \mu = 3t_2 \neq 4t_1 \\ \beta & \text{if } 12|q \text{ and } \mu = 4t_1 \neq 3t_2 \\ 0 & \text{if } 12|q \text{ and } 3t_2 \neq \mu \neq 4t_1 \end{cases}$$

where t_1 and t_2 are such that $t_1 = r_1^2 + s_1^2$ and $t_2 = 2(r_2^2 + r_2 s_2 + s_2^2)$ and also

$$\alpha = 1/4.\#\{(r_1,s_1) : r_1, s_1 \in Z, t_1 = r_1^2 + s_1^2\}$$

and

$$\beta = 1/3.\#\{(r_2,s_2) : r_2, s_2 \in Z, 2(r_2^2 + r_2 s_2 + s_2^2\})$$

Example 1. *Let $q = 84$. As $12|q$, total number of normal genus 1 subgroups is either $0, \alpha, \beta$ or $\alpha+\beta$. The first few values of $N_{84}(\mu)$ are given in the following table:*

μ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$N_{84}(\mu)$	0	0	0	1	0	2	0	1	0	0	0	0	0	0	0

μ	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$N_{84}(\mu)$	1	0	2	0	2	0	0	0	2	0	0	0	0	0	0

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İsmail Naci CANGÜL
Uludağ University Art and Science Faculty
Dept. of Maths., Görükle 16059 Bursa-TURKEY
Osman BİZİM
Uludağ University Art and Science Faculty
Dept. of Maths., Görükle 16059 Bursa-TURKEY

Received 02.03.1999