Tr. J. Mathematics
23 (1999) , 89 – 115

Proceedings of 6^{th} Gökova
Geometry-Topology Conference,

Canonical framings for 3-manifolds

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0. Introduction

A framing of an oriented trivial bundle is a homotopy class of sections of the associated oriented frame bundle. This paper is a study of the framings of the tangent bundle τ_M of a smooth closed oriented 3-manifold M, often referred to simply as *framings* of M.[1] We shall also discuss *stable framings* and 2-*framings* of M, that is framings of $\varepsilon^1 \oplus \tau_M$ (where ε^1 is an *oriented* line bundle) and $2\tau_M = \tau_M \oplus \tau_M$.

The notion of a *canonical* 2-*framing* of M was introduced by Atiyah [1]. Motivated by Witten's paper [27] generalizing the Jones polynomial to links in M, Atiyah observed that Witten's invariant contained a phase factor specified by the choice of a 2-framing on M, and thus was an invariant of links in a 2-framed 3-manifold. Independent calculations by Reshetikhin and Turaev [23] for a related invariant, defined from a framed link description of M, did not appear to depend on a 2-framing. As Atiyah noted, however, the framed link description naturally gave a 2-framing of M, explained further by Freed and Gompf in [6, §2], and so Reshetikhin and Turaev were in fact calculating Witten's invariant for M with this framing times a phase factor depending on the difference between this framing and the canonical 2-framing, i.e. Witten's invariant for M with its canonical 2-framing.

In this paper we give a leisurely exposition of framings, stable framings and 2-framings of M, including some of the material in [1] and [6]. Our principal objective is to define the notion of a *canonical (stable) framing* within each spin structure on M. This is the content of §2. The set of possible framings φ for a given spin structure form an affine space $\mathbb{Z}$, corresponding to $\pi_3(SO_3)$, and we choose a canonical framing in this space by minimizing the absolute value of the "Hirzebruch defect" $h(\varphi)$ (defined in §1). More generally, there are $\mathbb{Z} \oplus \mathbb{Z} = \pi_3(SO_4)$ possible stable framings ϕ, and here we must also minimize a certain "degree" $d(\phi)$ associated with ϕ.

The typical application may be to calculate the difference between a naturally occurring framing and the canonical one, and this is carried out in a number of instances. In §3 we consider framings on quotients of S^3 by finite subgroups, where the calculations use signature defects and the G-signature theorem, and on certain circle bundles over surfaces. Natural framings also arise by restriction from framings on 4-manifolds bounded by M, and this situation is taken up in §4. In particular we discuss surgery on an even framed link L in S^3, which connects with the work of Freed and Gompf.

[1] Recall that τ_M is always trivial; see [21, p.148], or [13, p.46] for an elementary geometric proof.

Some of this work was done around the time of Atiyah's paper [1], whose appearance removed our impetus to publish in a timely fashion. We wish to thank Selman Akbulut, Turgut Onder and our Turkish hosts for a splendid conference in Gokova which provided the motivation to complete this work.

1. Preliminaries

We work throughout in the smooth category. It is assumed that the reader is familiar with the elementary notions of handlebody theory, as treated in Chapter I of [13], and with the basic theory of fibre bundles and characteristic classes, as presented for example in the beautiful books of Steenrod [25] and Milnor-Stasheff [21].

Our orientation conventions are as follows. It is always assumed implicitly that the framings of an oriented manifold M are consistent with its orientation. Reversal of orientations will be indicated with an overbar. In particular a framing or stable framing ϕ is reversed by negating the last vector in each frame, producing $\bar{\phi}$.

We use the "outward normal first" convention for compatibly orienting a manifold and its boundary. In particular, if M is the oriented boundary of an oriented 4-manifold W, then the oriented bundles $\varepsilon^1 \oplus \tau_M$ and $\tau_W|M$ are naturally isomorphic by identifying a framing ν of ε^1 with the outward pointing normal. This identification will be used implicitly when we discuss the problem of extending stable framings of M across W.

The degree and Hirzebruch defect

Each framing φ of a 3-manifold M can be identified with the stable framing $\phi = \nu \oplus \varphi$, where ν is a framing of ε^1 as above. (Note the difference between the symbols φ for the "honest" framing and ϕ for the stable framing.) Of course not all stable framings arise in this way. The obstruction to "destabilizing" a stable framing ϕ is measured by the *degree* $d(\phi)$ of the map $M \to S^3$ which assigns to each point of M the position of the outward normal in the 3-sphere determined ϕ,

$$d(\phi) = \deg(\,\nu\colon M \to S^3\,).$$

Other properties of the degree are discussed in §2.

In addition to the degree, the key invariant that will be brought to bear on the study of framings is the *Hirzebruch defect*, defined for a framing or stable framing ϕ of M by

$$h(\phi) = p_1(W, \phi) - 3\sigma(W)$$

where W is any compact oriented 4-manifold bounded by M. Here $p_1(W, \phi)$ is the relative first Pontrjagin number of W (explained carefully in Appendix A), and $\sigma(W)$ is the signature of W. It follows from the Hirzebruch signature theorem and Novikov additivity of the signature that this definition is independent of the choice of W. For 2-framings ${}^2\phi$ of M we define $h({}^2\phi) = p_1(W, {}^2\phi) - 6\sigma(W)$ [1]. These definitions are motivated and explained more thoroughly in Appendix B, which includes a general discussion of signature theorems and "defect" invariants of 3-manifolds.

It will be seen in the next section that the degree and Hirzebruch defect form a complete set of integer invariants for the stable framings within any given spin structure on M, and serve to identify this set of framings with an affine lattice in the the dh-plane. The *canonical* framing(s) will then be defined as the one(s) closest to the origin in this plane, with respect to a suitable norm. Before embarking on this program, however, we identify two well known elements of $\pi_3(SO_4)$ that will be used frequently in what follows.

The generators ρ and σ

The study of framings leads naturally to obstruction theory with coefficients in the homotopy groups $\pi_3(SO_n)$ and $\pi_3(SU_n)$. Recall that $\pi_3(SO_4) \cong \mathbb{Z} \oplus \mathbb{Z}$, $\pi_3(SO_n) \cong \mathbb{Z}$ for all other $n \geq 3$, and $\pi_3(SU_n) \cong \mathbb{Z}$ for all $n \geq 2$. Following Steenrod [25], we give explicit generators for all of these groups.

View S^3 as the unit sphere in the quaternions $\mathbb{H}$ (oriented by the ordered basis $1, i, j, k$) and SO_4 as the rotation group of $\mathbb{H}$. Then the maps ρ and $\sigma\colon S^3 \to SO_4$ defined by

$$\rho(q)x = qxq^{-1} \qquad \sigma(q)x = qx$$

represent generators of $\pi_3(SO_4)$ [25, p.117]. (By abuse of notation, we shall not distinguish between these maps and their homotopy classes.) This identifies $\pi_3(SO_4)$ with $\mathbb{Z} \oplus \mathbb{Z}$, where ρ, σ correspond to the standard ordered basis.

By restricting to the subspace $\mathbb{P}$ of pure (imaginary) quaternions, ρ also represents an element (in fact a generator) of $\pi_3(SO_3)$, and these two ρ's correspond under the natural map $\pi_3(SO_3) \to \pi_3(SO_4)$ induced by the inclusion $SO_3 \subset SO_4$. Furthermore, $\sigma \in \pi_3(SO_4)$ is carried under the natural maps $\pi_3(SO_4) \to \pi_3(SO_5) \to \cdots \to \pi_3(SO)$ onto generators, also denoted by σ, of each of the subsequent groups; the first of these maps also carries ρ to 2σ. These facts are all proved in Steenrod, and serve to identify all of these groups with $\mathbb{Z}$.

Similarly σ represents a generator of $\pi_3(SU_2)$ by identifying $\mathbb{H}$ with $\mathbb{C}^2$. (Note that for σ to induce a *unitary* action, the vector (u, v) in $\mathbb{C}^2$ must be identified with the quaternion $u + jv$ rather than $u + vj$; see also Remark 2.4 below.) This generator is then carried onto generators σ for all the subsequent groups in the natural sequence $\pi_3(SU_2) \to \pi_3(SU_3) \to \cdots \to \pi_3(SU)$, thereby identifying these groups with $\mathbb{Z}$.

Finally observe that the tautological map

$$\iota\colon \pi_3(SO) \to \pi_3(SU)$$

is multiplication by -2, that is $\iota(\sigma) = -2\sigma$. Indeed, this map was classically computed up to sign (see e.g. the proof of Lemma 2 in [20]), and an explicit formula giving the sign was written down by John Hughes in his thesis [11, §24], essentially as follows:

Fix a unit quaternion $q = a + bi + cj + dk = u + jv$, with $u = a + bi$ and $v = c - di$, and let P and Q denote the matrices in SO_4 and SU_2 corresponding to $\sigma(q)$,

$$P = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}.$$

Now observe that P is conjugate in SU_4 to the block sum $T = Q^{-1} \oplus Q^{-1}$ of two copies of the inverse of Q. Explicitly $RPR^{-1} = T$ for

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -i \\ 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & -i & 1 \end{pmatrix}.$$

Since SU_4 is connected, it follows that the maps π and $\tau\colon S^3 \to SU_4$ given by $\pi(q) = P$ and $\tau(q) = T$ are homotopic, and evidently π represents $\iota(\sigma)$ and τ represents -2σ.

The geometry of ρ and σ

There is a simple geometric description for the maps ρ and $\sigma \in \pi_3(SO_4)$ if one views S^3 as the unit 3-ball B^3 in $\mathbb{P}$ with its boundary collapsed to a point. (The great half circles in S^3 from -1 to 1 correspond to the radii of B^3.) In particular $\rho(0) = I$ and $\sigma(0) = -I$. If $x \in B^3$ is nonzero, then $\rho(x)$ is the rotation of $\mathbb{P}$ about x by $2\pi|x|$ radians, while $\sigma(x)$ is the simultaneous rotation by $\pi|x|$ radians of the oriented plane P_x (normal to x in $\mathbb{P}$) and its complement Q_x (with ordered basis 1, x). In other words as the diameter through x is traversed, ρ rotates P_x two full *right*-handed turns while fixing Q_x, and σ rotates both planes by one full right-handed turn.

2. Canonical framings

Let M be a closed connected oriented 3-manifold. We wish to study the framings φ, stable framings ϕ, and 2-framings ${}^2\phi$ of M. To organize this study, it is convenient to fix a spin structure Σ on M, which can be viewed as a framing of τ_M over the 2-skeleton of M (see [19] and the discussion in Chapter IV of [13]).

Framings extending Σ

Consider the set F_Σ of framings of M which are compatible with Σ. The difference of two such framings is specified by an element of $H^3(M; \pi_3(SO_3)) = \mathbb{Z}$, by obstruction theory. In other words F_Σ is an affine space with translation group $\pi_3(SO_3) = \mathbb{Z}$. The action of this group can be visualized geometrically using the description in §1 of its generator ρ. In particular ρ acts on a framing φ, producing a new framing $\varphi + \rho$, by rotating the frames along each diameter of a small 3-ball in M by two full twists. It will be seen below (Lemma 2.3a) that this corresponds to a shift by 4 in the affine space F_Σ.

Canonical framings

As an affine space, F_Σ has no apriori choice of basepoint. Our goal is to pick such a basepoint, i.e. a preferred or *canonical* framing in the given spin structure on M. We shall not attempt to pick a canonical framing within the set F of *all* framings on M, or what amounts to the same thing, a canonical spin structure on M. Indeed we do not know how to make such a choice in general. Recall that by obstruction theory, as above, the spin structures on M form an affine space over $\mathbb{Z}_2$ with translation group $H^1(M;\mathbb{Z}_2)$, and so $\mathrm{F} = \cup \mathrm{F}_\Sigma$ is (non-canonically) isomorphic to $H^1(M;\mathbb{Z}_2) \oplus \mathbb{Z}$.

Definition 2.1. A framing φ of the 3-manifold M is *canonical* for the spin structure Σ if it is compatible with Σ and $|h(\varphi)| \leq |h(\psi)|$ for all other framings ψ which are compatible with Σ. In other words, φ is a minimum for the invariant $|h|$ on F_Σ.

Since ρ translates F_Σ by 4 (Lemma 2.3a below), this minimum is at most 2. If it is 0 or 1, then the canonical framing is unique (in F_Σ) and in fact minimizes $|h|$ globally (in F). This is the case for "most" spin structures on most manifolds. The only exceptions are the spin structures for which $\mu \equiv 2\mathrm{rk}\, H_1(M;\mathbb{Z}_2) \pmod 4$ (e.g. the unique spin structure on a homology sphere). In these cases there are two canonical framings with $h = \pm 2$; one could of course select the positive one, but we have chosen not to out of respect for our left-handed colleagues.

Note that if one were to define a canonical framing to be one that minimized $|h|$ globally, then some spin structures might not have *any* canonical framings; indeed examples are easily given of manifolds in which the minimum of $|h|$ varies with the spin structure (e.g. the connected sum of two copies of real projective 3-space).

To justify the preceding remarks, it is necessary to analyze the behavior of the defect $h(\varphi)$ under the action of $\pi_3(SO_3)$ on the set of framings. This is not difficult, and is discussed below in the more general context of stable framings.

Stable framings extending Σ

Let $\mathbb{F}$ be the set of all stable framings of M, and $\mathbb{F}_\Sigma$ be the subset of those which extend the spin structure $\nu \oplus \Sigma$ on $\varepsilon^1 \oplus \tau_M$, where ν is a framing of ε^1 (restricted to the 2-skeleton of M). As above, $\mathbb{F}_\Sigma$ is an affine space with translation group $\pi_3(SO_4) = \mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{F} \cong H^1(M;\mathbb{Z}_2) \oplus \mathbb{Z} \oplus \mathbb{Z}$ (non-canonically). The action of $\pi_3(SO_4)$ on $\mathbb{F}$ can be understood as before using the geometric descriptions of ρ and σ (with ν playing the role of 1): $\phi + m\rho + n\sigma$ is obtained from ϕ by rotating the frames along each oriented diameter of a 3-ball by $2m + n$ full twists in the normal plane, perpendicular to the diameter, and by n full twists in the "conormal" plane, spanned by ν and the diameter.

The degree

Observe that the "honest" framings F_Σ can be viewed as a subset of the stable framings $\mathbb{F}_\Sigma$ by identifying φ with $\phi = \nu \oplus \varphi$. Thus we will freely use φ and ϕ interchangeably with the understanding that this identification is to be made. There is a simple invariant that

detects whether a stable framing ϕ corresponds to an honest framing in this way, namely the *degree* of ϕ

$$d(\phi) = \deg(\,\nu\colon M \to S^3\,).$$

Here S^3 is the unit sphere in ϕ, and the map $\nu\colon M \to S^3$ is defined in the obvious way: ν is the framing of ε^1, which at each point of M determines a point in S^3.

The degree satisfies the following properties.

Theorem 2.2. *Let ϕ be a stable framing of M. Then $d(\phi) = 0$ if and only if ϕ is of the form $\nu \oplus \varphi$ for some honest framing φ. Furthermore,*

a) (action) *$d(\phi + \rho) = d(\phi)$ and $d(\phi + \sigma) = d(\phi) - 1$.*
b) (boundary) *If ϕ extends to a framing of a compact 4-manifold W bounded by M, then $d(\phi) = \chi(W)$, where χ is the Euler characteristic.*
c) (covering) *If $(\tilde{M}, \tilde{\phi}) \to (M, \phi)$ is an r-fold covering map with compatible stable framings, then $d(\tilde{\phi}) = r\, d(\phi)$.*
d) (orientation) *$d(\bar{\phi}) = d(\phi)$, where $^-$ denotes orientation reversal.*

Proof. The first statement is immediate from the fact that the degree of a map $M \to S^3$ is zero if and only if the map is homotopic to a constant (Hopf's Theorem). Property a) can be verified using the geometric description of the action of ρ and σ on $\mathbb{F}$, and b) follows from the fact that the Euler characteristic is equal to the obstruction to extending ν across W, since the latter can be identified with $d(\phi)$ by the homological invariance of degree. (Thus $d(\phi)$ can be interpreted as the relative Euler class of (W, ϕ).) Property c) is obvious. For d) we have $d(\bar{\phi}) = \deg(\,\nu\colon \bar{M} \to \bar{S}^3\,) = d(\phi)$. □

The Pontrjagin number

Next we investigate the relative first Pontrjagin number $p_1(W, \phi)$, where W is a compact 4-manifold bounded by M. This is a key ingredient in the definition of the Hirzebruch defect $h(\phi)$, and was defined in Appendix A as the obstruction to extending ϕ (with the last vector dropped) across the complexified tangent bundle of W.

If W has a spin structure compatible with ϕ, then p_1 can be defined in a simpler way without complexifying τ_W. For in this case ϕ extends to a framing of W in the complement of a point. Indeed the spin structure gives an extension over the relative 2-skeleton of $(W, \partial W)$, and this further extends over the 3-skeleton since $\pi_2(SO_4) = 0$. Now the obstruction to extending ϕ across the last point is an element θ in $\pi_3(SO_4) = \mathbb{Z} \oplus \mathbb{Z}$. This element defines a 4-plane bundle ξ_θ over S^4 whose first Pontrjagin number $p_1(\xi_\theta)$ evidently coincides with $p_1(W, \phi)$. Alternatively, one may consider the image $\mathfrak{o}$ of θ in $\pi_3(SO) = \mathbb{Z}$, the "stable obstruction" to extending ϕ, and it is easily verified that

$$p_1(W, \phi) = 2\mathfrak{o}$$

since $\pi_3(SO) \to \pi_3(SU)$ is multiplication by -2 and $p_1 = -c_2$.

The relative first Pontrjagin number satisfies the following properties.

Lemma 2.3. *Let ϕ be a stable framing of $M = \partial W$. Then*

a) (action) $p_1(W, \phi + \rho) = p_1(W, \phi) + 4$ *and* $p_1(W, \phi + \sigma) = p_1(W, \phi) + 2$.
b) (boundary) *If ϕ is compatible with a spin structure on W, then ϕ extends to a framing of W if and only if $p_1(W, \phi) = 0$ and $d(\phi) = \chi(W)$.*
c) (covering) *If $(\tilde{W}, \tilde{\phi}) \to (W, \phi)$ is an r-fold covering map with compatible stable framings on the boundary, then $p_1(\tilde{W}, \tilde{\phi}) = r p_1(W, \phi)$.*
d) (orientation) $p_1(\bar{W}, \bar{\phi}) = -p_1(W, \phi)$, *where $^-$ denotes orientation reversal.*

Proof. The action of any $\theta \in \pi_3(SO_4)$ on ϕ is local, only changing ϕ in a 3-ball in M, and so $p_1(W, \phi + \theta) = p_1(W, \phi) + p_1(B^4, \phi_0 + \theta)$, where ϕ_0 is the restriction to S^3 of the unique framing of B^4. But $p_1(B^4, \phi_0 + \theta) = 2\mathfrak{o}$, where $\mathfrak{o}$ is the image of θ in $\pi_3(SO) = \mathbb{Z}$. Evidently $\mathfrak{o} = 1$ or 2 for $\theta = \sigma$ or ρ, repectively, and the property a) follows.

The forward implication in b) is immediate from Theorem 2.2b and the definition of p_1. For the reverse implication, suppose that $\phi \in \mathbb{F}_\Sigma$. Observe that any spin structure on W which extends Σ can be further extended to a framing of W, by obstruction theory. Let ψ denote the restriction of any such framing to M. Then $\psi = \phi + m\rho + n\sigma$, for some m and n, since both stable framings extend Σ. By hypothesis $p_1(W, \phi) = p_1(W, \psi) = 0$, and so $2m + n = 0$ by a). Also $d(\phi) = d(\psi) = \chi(W)$ and so $n = 0$ by Theorem 2.2a. Thus $m = n = 0$, and so $\phi = \psi$. This establishes the b). Properties c) and d) are immediate from the definition of p_1 as an obstruction (see Appendix A). □

Remark 2.4. Lemma 2.3a can also be established using the formula $p_1(B^4, \phi_0 + \theta) = p_1(\xi_\theta)$ (see the discussion preceding the lemma). Indeed the *oriented* bundle $\bar{\xi}_\sigma$ can be given the structure of a complex 2-plane bundle ω over S^4, by right multiplication, and so

$$p_1(\xi_\sigma) = c_1^2(\omega) - 2c_2(\omega) = -2e(\bar{\xi}_\sigma) = 2.$$

Note that the natural orientation on ξ_σ coming from $\mathbb{H}$ is inconsistent with the orientation arising from the complex structure. This is the reason for using the oppositely oriented bundle $\bar{\xi}_\sigma$, with Euler class -1, to compute $c_2(\omega)$ (cf. the computations in [5, p.673] and [13, p.43] where this sign is overlooked). Now since $\tau_{S^4} = \xi_{2\sigma-\rho}$ ([25, §23.6,27.3]) and $p_1(\tau_{S^4}) = p_1(\varepsilon^1 \oplus \tau_{S^4}) = p_1(\varepsilon^5) = 0$, it follows that $p_1(\xi_\rho) = 4$.

The Hirzebruch defect

Using the previous lemma, together with some facts about the signature and signature defects (see Appendix B), we deduce the following properties of the Hirzebruch defect $h(\phi)$. (Recall from §1 that $h(\phi) = p_1(W, \phi) - 3\sigma(W)$ for any compact oriented 4-manifold bounded by M.)

Theorem 2.5. *Let ϕ be a stable framing of M. Then*

a) (action) $h(\phi+\rho) = h(\phi)+4$ *and* $h(\phi+\sigma) = h(\phi)+2$.
b) (boundary) *If ϕ extends to a framing of a compact 4-manifold W bounded by M, then $h(\phi) = -3\sigma(W)$, where $\sigma(W)$ is the signature of W.*
c) (covering) *If $\pi\colon (\tilde{M},\tilde{\phi}) \to (M,\phi)$ is an r-fold cover with compatible stable framings, then $h(\phi) = (h(\tilde{\phi}) - 3\sigma(\pi))/r$, where $\sigma(\pi)$ is the signature defect of π.*
d) (orientation) $h(\bar{\phi}) = -h(\phi)$, *where $^-$ denotes orientation reversal.*

Proof. Properties a) and b) are immediate from the analogous properties for p_1 in Lemma 2.3. Property c) is proved in Appendix B, Lemma 1. The last property follows from Lemma 2.3d and the fact that $\sigma(\bar{W}) = -\sigma(W)$. □

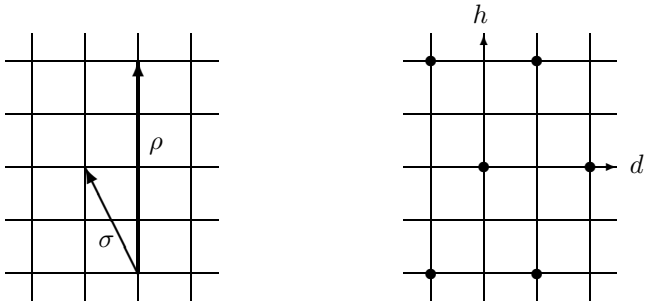
The total defect

The preceding results give a complete picture of the affine space $\mathbb{F}_\Sigma$ of stable framings which extend the spin structure Σ on M. Indeed Theorems 2.2a and 2.5a show that d and h together give an embedding $H : \mathbb{F}_\Sigma \hookrightarrow \mathbb{Z}\oplus\mathbb{Z}$ into the dh-plane, defined by

$$H(\phi) = (d(\phi), h(\phi)).$$

This embedding will be called the *total defect*. The (vertical) h-axis corresponds to the honest framings φ, and in line with the natural inclusion $\mathrm{F}_\Sigma \subset \mathbb{F}_\Sigma$ we often write $H(\varphi)$ for $H(\nu\oplus\varphi)$. Orientation reversal $\phi \mapsto \bar{\phi}$ corresponds to "conjugation" $(d,h)\mapsto(d,-h)$ (by 2.2d and 2.5d).

From the action of $\pi_3(SO_4)$ on the dh-plane (shown in Figure 1a), the image of $\mathbb{F}_\Sigma$ is seen to be an affine lattice of index 4 in $\mathbb{Z}\oplus\mathbb{Z}$. In particular it is a coset of the subgroup Λ_0 generated by $(0,4)$ and $(-1,2)$ (shown in Figure 1b), namely one of the four affine lattices $\Lambda_k = \Lambda_0 + (0,k)$ for $k\in\mathbb{Z}_4$.



a) the action of $\pi_3(SO_4)$ b) the lattice Λ_0

Figure 1: the dh-plane

To determine which one, consider the $\mathbb{Z}_4$-valued invariant $\lambda = 2d+h \pmod 4$ of the spin structure Σ on M, that is

$$\lambda(\Sigma) \;=\; 2d(\phi) + h(\phi) \pmod 4$$

for any stable framing ϕ in $\mathbb{F}_\Sigma$. Then $H(\mathbb{F}_\Sigma) = \Lambda_{\lambda(\Sigma)}$, and the next result expresses the invariant $\lambda(\Sigma)$, and its mod 2 reduction, in terms of the mu invariant $\mu(\Sigma)$ and homological invariants of M. (Recall that $\mu(\Sigma) = \sigma(W) \pmod{16}$ for any compact 4-manifold W bounded by M over which Σ extends; this is well defined by Rohlin's theorem on the signature of closed spin 4-manifolds [24] [13, pp.64–65].)

Theorem 2.6. *Let Σ be a spin structure on M. Then*

$$\begin{aligned}\lambda(\Sigma) &\equiv 2(1 + r(M)) + \mu(\Sigma) \pmod 4\\ &\equiv s(M) \pmod 2\end{aligned}$$

where $r(M) = \mathrm{rk}(H_1(M) \otimes \mathbb{Z}_2)$ and $s(M) = \mathrm{rk}(\mathrm{Tor}H_1(M) \otimes \mathbb{Z}_2)$.

Note that $s(M)$ is the number of 2-primary summands in $H_1(M)$, and so $r(M) - s(M)$ is the first Betti number $b_1(M) = \mathrm{rk}(H_1(M))$.

Proof. Choose any simply-connected spin 4-manifold W with spin boundary (M, Σ), for example constructed by attaching 2-handles to B^4 along an even framed link [12]. To compute $\lambda(\Sigma)$, we use the restriction ϕ to M of the unique framing of W.

First observe that

$$\chi(W) \equiv 1 + r(M) \pmod 2.$$

Indeed $\chi(W) = 1 + b_2(W)$, where $b_2(W) = \mathrm{rk}(H_2(W))$ can be expressed as the sum of the nullity and rank of the mod 2 intersection form on W. But the nullity of this form is equal to $r(M)$ since the intersection matrix is a presentation matrix for $H_1(M; \mathbb{Z}_2)$ (by Poincaré duality), and the rank is even since W is spin (whence the nonsingular part of the form is a sum of two-dimensional hyperbolic forms).

Now using Theorems 2.2b and 2.5b, we see that $\lambda(\Sigma)$ is equal to

$$2d(\phi) + h(\phi) \equiv 2\chi(W) - 3\sigma(W) \equiv 2(1 + r(M)) + \mu(\Sigma) \pmod 4.$$

Reducing mod 2 gives $\mu(\Sigma) \equiv \sigma(W) \equiv b_2(W) - b_1(M) \equiv r(M) - b_1(M) \equiv s(M)$. □

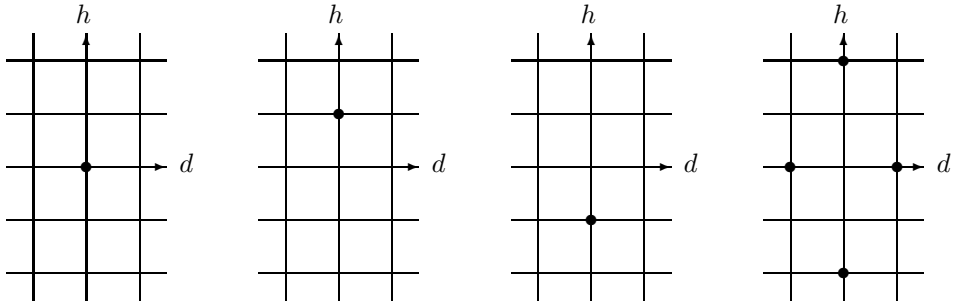
Canonical stable framings

The description of $\mathbb{F}_\Sigma$ above suggests the following generalization of the notion of canonical framings (Definition 2.1) using the norm on the dh-plane given by $|(d, h)| = 2|d| + |h|$.

Definition 2.7. A stable framing ϕ of the 3-manifold M is *canonical* for the spin structure Σ if it is compatible with Σ and $|H(\phi)| \leq |H(\psi)|$ for all other stable framings ψ which are compatible with Σ. In other words, ϕ is a minimum for the invariant $2|d| + |h|$ on $\mathbb{F}_\Sigma$.

It follows from this definition that $\mathbb{F}_\Sigma$ has a *unique* stable canonical framing when $\lambda(\Sigma) \not\equiv 2$, corresponding to the point $(0, 0)$ or $(0, \pm 1)$ in the dh-plane, according to whether $\lambda(\Sigma) \equiv 0$ or ± 1 (see Figure 2a-c). These are all honest framings, and so give the unique canonical framings in F_Σ as well. If $\lambda(\Sigma) \equiv 2$ (e.g. for the unique spin structure

on any homology sphere) then there are *four* canonical framings, corresponding to the points $(\pm 1, 0)$ and $(0, \pm 2)$ (see Figure 2d). The last two are honest framings.



a) $\lambda(\Sigma) = 0$ b) $\lambda(\Sigma) = 1$ c) $\lambda(\Sigma) = -1$ d) $\lambda(\Sigma) = \pm 2$

Figure 2: canonical stable framing(s) extending Σ

Remark 2.8. Canonical framings can be constructed from any given stable framing ϕ by adding suitable multiples of ρ and σ. For example, if ϕ has total defect (d, h), then the canonical framing with total defect $(0, \lambda)$, where $\lambda \equiv 2d + h \pmod 4$ and $|\lambda| \leq 2$, is given by

$$\phi + d\,\sigma - \tfrac{1}{4}(2d + h - \lambda)\,\rho.$$

If ϕ can be described geometrically, then so can the canonical framing, using the local geometric picture for the action of ρ and σ discussed above.

Examples of canonical framings

Example 2.9. (Lie groups) Consider the three 3-dimensional compact connected Lie groups – the 3-sphere S^3, the rotation group SO_3, and the 3-torus T^3 – which have natural framings arising from their group structure. In particular S^3 and SO_3 each have two Lie framings $\varphi_\pm$, obtained by left or right multiplication from a fixed frame at the identity, where the $+$ sign corresponds to left multiplication.[2] Since T^3 is abelian, it has only one Lie framing φ_1.

Now S^3 has a unique spin structure, while SO_3 and T^3 have $|H^1(SO_3; \mathbb{Z}_2)| = 2$ and $|H^1(T^3; \mathbb{Z}_2)| = 8$ spin structures, respectively. We will show how to construct *all* the canonical framings in these spin structures, and in particular show that the Lie framings are canonical in theirs. To emphasize the underlying manifold M of a framing φ, we will sometimes write $H(M, \varphi)$ for the total defect $H(\varphi)$.

[2]The reason for this sign convention in S^3 is that *left* multiplication yields the *right*-handed Hopf framing, in which the integral curves of any vectorfield in the framing are Hopf circles with $+1$ pairwise linking, along which the other two vectorfields spin once in a positive sense. This is easily seen from the geometric description for σ in §1. The framing φ_+ of SO_3 is likewise intrinsically right-handed, since it can be viewed as the quotient of the corresponding framing of S^3 under the projection $S^3 \to SO_3 = C_2 \backslash S^3$. Similar statements apply to the *left*-handed Hopf framing φ_- on S^3 and its quotient on SO_3.

a) For S^3, the Lie framings $\varphi_\pm$ have Hirzebruch defect ± 2. This can be seen in a variety of ways. For example, it is obvious that the obstruction to extending φ_+ over B^4 is σ, and so $p_1(B^4, \varphi_+) = p_1(\xi_\sigma) = 2$ (as shown in Remark 2.4). Thus $h(\varphi_+) = 2$ since $\sigma(B^4) = 0$. Now observe that $\varphi_- = \varphi_+ - \rho$, since right multiplication by q is the composition of left multiplication by q with conjugation by $\bar{q}$ (i.e. $\bar{q}(qx)q = xq$) and conjugation by $\bar{q}$ represents $-\rho \in \pi_3(SO_4)$. It follows that $h(\varphi_-) = -2$, by 2.5a.

The same result can be obtained another way, using the "canonical" stable framing δ of S^3 which is the restriction of the unique framing of B^4. Clearly $\varphi_+ = \delta + \sigma$. (This provides a sign check for Theorem 2.2a, that adding σ lowers the degree by one: δ has degree 1 since $\chi(B^4) = 1$, while the honest framing φ_+ has degree 0.) Thus $h(\varphi_+) = h(\delta + \sigma) = 2$ by Theorem 2.5a, since $h(\delta) = 0$ by 2.5b. Now $h(\varphi_-) = -2$ can be deduced as above, or using Theorem 2.5d and the observation that $(S^3, \varphi_-) = (\bar{S}^3, \bar{\varphi}_+)$ (indeed any orientation reversing automorphism of S^3 induces an orientation preserving diffeomorphism $S^3 \to \bar{S}^3$ which identifies φ_- with $\bar{\varphi}_+$, up to homotopy).

Summarizing, we have computed the total defects $H = (d, h)$ of the Lie framings $\varphi_\pm$ of S^3, and along the way the canonical stable framing δ coming from B^4, to be

$$H(S^3, \varphi_\pm) = (0, \pm 2) \qquad \text{and} \qquad H(S^3, \delta) = (1, 0).$$

Therefore these (stable) framings represent, by definition, three out of four of the canonical stable framings in the unique spin structure on S^3. The fourth, represented by the point $(-1, 0)$ in the dh-plane, can also be constructed in a natural way as the restriction δ_- of *any* framing of $B^4 \# S^1 \times S^3$. Indeed this 4-manifold has Euler characteristic -1 and signature 0, and so $H(S^3, \delta_-) = (-1, 0)$ (by Theorems 2.2b and 2.5b).

b) For SO_3, note that $(S^3, \varphi_\pm)$ double covers $(SO_3, \varphi_\pm)$ with zero signature defect (See Example 3 in Appendix B). It follows from Theorem 2.5c that the Lie framings $\varphi_\pm$ on SO_3 have Hirzebruch defect ± 1, and so total defects

$$H(SO_3, \varphi_\pm) = (0, \pm 1).$$

Noting that the two spin structures $\Sigma_\pm$ on SO_3 have μ-invariants ± 1, it follows from Theorem 2.6 that $\varphi_\pm \in \mathbb{F}_{\Sigma_\pm}$, and so we have identified the unique canonical framings in both spin structures. (These spin structures are equivalent under any orientation reversing automorphism of SO_3, and in fact $(SO_3, \varphi_-) = (\bar{SO}_3, \bar{\varphi}_+)$.)

c) For T^3, observe that the Lie framing φ_1 is amphicheiral, i.e. $(T^3, \varphi_1) = (\bar{T}^3, \bar{\varphi}_1)$. (To see this explicitly, note that φ_1 assigns the standard frame i, j, k in $\mathbb{R}^3$ to each point $(x, y, z) \in T^3 = \mathbb{R}^3/\mathbb{Z}^3$, and the reflection $(x, y, z) \mapsto (x, y, -z)$ carries this frame onto $i, j, -k$.) It follows by 2.2d and 2.5d that the total defect vanishes.[3] With each of the other seven spin structures, T^3 is diffeomorphic to the boundary of the spin 4-manifold

[3]There is another way see this: φ_1 obviously has degree zero, since it is an honest framing, and its Hirzebruch defect can be computed by viewing (T^3, φ_1) as the boundary of the spin 4-manifold N obtained by removing an open tubular neighborhood of a regular fiber in the rational elliptic surface of signature -8 and Euler class 12. (This manifold is discussed in detail in Chapter V of [13].) Indeed it can be shown that $p_1(N, \varphi_1) = -2\chi(N)$, and so $h(\varphi_1) = -2\chi(N) - 3\sigma(N) = 0$.

$W = T^2 \times B^2$ (with the Lie spin structure on T^2) – note that the diffeomorphism might not be the obvious one. Now there is a natural framing on W, namely the product of the Lie framing on T^2 with the constant framing on B^2. The restriction of this framing to the boundary is a stable framing ϕ_0 of T^3 with vanishing total defect, since $\chi(W) = \sigma(W) = 0$. In particular ϕ_0 is the stabilization of the honest framing φ_0 which assigns the frame $\cos(2\pi z)\, i + \cos(2\pi z)\, j + k$ to the point $(x, y, z) \in T^3$.

Summarizing, we have shown that

$$H(T^3, \varphi_0) = H(T^3, \varphi_1) = (0, 0).$$

This identifies the unique canonical framings in the eight spin structures on T^3, since φ_0 represents each of the seven non-Lie structures under a suitable diffeomorphism. In particular, note that $\lambda(\Sigma) = 0$ for every spin structure on T^3 (see Figure 2a), which is consistent with Theorem 2.6 since $r(T^3) = 3$, and $\mu(\Sigma) = 8$ for the Lie spin structures and 0 for the rest.

Example 2.10. (Products) Let $M_g = F \times S^1$, where F is a closed orientable surface of genus g. Then M_g has $2^{2g+1} = |H^1(M; \mathbb{Z}_2)|$ spin structures. Up to a diffeomorphism, however, there are only four when $g > 1$, two when $g = 1$, and one when $g = 0$. To see this, assign to each spin structure Σ on M_g the bordism invariant

$$(\alpha, \beta) \in \Omega_2^{\text{spin}} \oplus \Omega_1^{\text{spin}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

determined by the restrictions of Σ to the two factors of $F \times S^1$ (α is the Arf invariant of $\Sigma|F$; see Chapter IV in [13]). Then an analysis of the action of the diffeomorphism group of M_g on $H^1(M_g; \mathbb{Z}_2)$ shows that any other spin structure with the same bordism invariant is equivalent to Σ, and conversely for $g > 1$, any spin structure equivalent to Σ will have the same bordism invariant. For $g = 0$ or 1, the internal symmetry of the manifold M_g further restricts the number of diffeomorphism classes of spin structures. We consider only the cases $g = 0$ and $g > 1$, since $M_1 = T^3$ was treated in the last example. (See also Theorem 3.3 for certain circle bundles over F.)

First observe by Theorem 2.6 that $\lambda(\Sigma) = 0$ for every spin structure Σ on M_g. Indeed $r(M_g) = 2g + 1$, and so this is equivalent to the fact that $\mu(\Sigma) \equiv 0 \pmod 4$. In fact an elementary induction shows that $\mu(\Sigma) \equiv 0 \pmod 8$, since M_g can be viewed as a "fiber connected sum" of M_{g-1} and M_1 (i.e. remove tubular neighborhoods of circle fibers in both and identify the boundaries) and the mu invariants clearly add. It follows that the total defect of any canonical stable framing on M_g is $(0, 0)$.

a) For $g = 0$ the manifold is $M_0 = S^2 \times S^1$, which has two spin structures with bordism invariants $(\alpha, \beta) = (0, 0)$ and $(0, 1)$. In fact these spin structures are equivalent under the automorphism τ of $S^2 \times S^1$ which spins the S^2 once along the S^1 factor. Thus it suffices to consider the spin structure with bordism invariant $(0, 1)$, for which there is an obvious stable framing ϕ_1 obtained by restricting the product framing on $B^3 \times S^1$ (the constant framing on B^3 crossed with the tangent framing on S^1) to the boundary. Since the Euler characteristic and signature of $B^3 \times S^1$ vanish, the total defect of this framing vanishes

as well. Therefore $\phi_0 = \tau^*\phi$ and ϕ_1 are the unique canonical framings for the two spin structures on $S^2 \times S^1$.

Note that since the stable framings ϕ_i have degree zero, they are stabilizations of honest framings φ_i on $S^2 \times S^1$, obtained from ϕ_i as follows: Rotate each frame so that the last vector, which is initially tangent to S^1, becomes the inward normal to B^3, and then drop this vector from the frame.

b) For $g > 1$ we only treat the case when the Arf invariant $\alpha = 0$. Then M_g is the spin boundary of a product 4-manifold $N \times S^1$. Now if $\beta = 1$, then the canonical framing in the associated spin structure Σ_1 is just the restriction ϕ_1 to M_g of any product framing on $N \times S^1$. The total defect of this framing vanishes since the Euler characteristic and signature of $N \times S^1$ are zero. For the spin structure Σ_0 with $\beta = 0$, we modify ϕ_1 by putting a full twist in the plane of the first two vectors in each frame while traversing the S^1 factor. This yields the canonical stable framing ϕ_0 compatible with Σ_0. Of course both ϕ_0 and ϕ_1 can be homotoped to honest framings as in the genus 0 case.

2-*framings*

There is a canonical spin structure on $2\tau_M$ coming from the diagonal embedding of τ_M in $2\tau_M$, namely $\Sigma \oplus \Sigma$ for any spin structure Σ on M. It is easy to see that this spin structure is in fact independent of the choice of Σ. Following Atiyah, we consider only the 2-framings which extend this canonical spin structure (although the general case is not much harder). These 2-framings form an affine space ${}^2\mathbb{F}$ with translation group $\pi_3(SO_6) = \mathbb{Z}$. In particular the generator σ acts in the usual way on the first four vectors in any 2-framing ${}^2\phi$, and trivially on the last two, and it follows that

$$h({}^2\phi + \sigma) = h({}^2\phi) + 2.$$

Recalling from §1 that $h(2\varphi)$ is even for any honest framing φ of M, we see that ${}^2\mathbb{F}$ can be identified with the even integers in $\mathbb{Z} = (h)$. It follows that there exists a *unique* 2-framing in ${}^2\mathbb{F}$ with zero Hirzebruch defect. This is Atiyah's *canonical* 2-framing.

Definition 2.11. (Atiyah) The *canonical* 2-framing on M is the unique 2-framing ${}^2\phi$ compatible with the canonical spin structure on $2\tau_M$ for which $h({}^2\phi) = 0$.

The canonical 2-framing ${}^2\phi$ on M is clearly given by ${}^2\phi = 2\varphi - h(\varphi)\sigma$ for *any* honest framing φ. In particular ${}^2\phi$ is of the form 2φ for some framing on M if and only if φ is the canonical framing in a spin structure Σ with $\lambda(\Sigma) = 0$. By Theorem 2.6, this will occur if and only M has a spin structure Σ with $\mu(\Sigma) \equiv 2(1 + b_1(M)) \pmod 4$, where b_1 is the first Betti number. This condition is satisfied, for example, for all products $F \times S^1$ (see Example 2.10), but fails for all homology spheres. It holds for the lens spaces $L(n, 1)$ if and only if $n \equiv \text{sign}(n) \pmod 4$.

More generally, one can ask whether the canonical framing can be written as a Whitney sum $\varphi_1 \oplus \varphi_2$ of two framings (necessarily) in the same spin structure Σ. This is clearly equivalent to having $\lambda(\Sigma) = 0$ or 2, which by Theorem 2.6 occurs if and only if the number $s(M)$ of 2-primary components in $H_1(M)$ is even. For example this condition is satisfied

for all homology spheres (e.g. the canonical 2-framing on S^3 is the sum of the two Hopf framings) and all odd lens spaces, but fails for all even lens spaces.

3. Other natural framings

In this section we consider a variety of naturally arising framings of 3-manifolds, and show how to compute their total defects. As in previous examples, we may write (M, ϕ) for (ϕ) to highlight the underlying manifold M on which the framing ϕ is defined.

Homogeneous spaces

Let G be a finite subgroup of $S^3 = SU_2$. Then the right handed Hopf framing φ_+ on S^3, which is defined by left translation of a frame at the identity, induces an honest framing on the homogenous space $G\backslash S^3$ of right cosets of G in S^3. This framing will also be denoted by φ_+, with associated spin structure Σ_+, and its Hirzebruch defect can be computed from the formula in Theorem 2.5c to be

$$h(G\backslash S^3, \varphi_+) = (2 - \sigma(G))/|G|$$

where $\sigma(G)$ is three times the signature defect of the universal covering $S^3 \to G\backslash S^3$. By Examples 3 and 4 in Appendix B, $\sigma(G)$ is equal to $m^2 - 3m + 2$, $4m^2 + 2$, 98, 242 or 722, according to whether $G = C_m$ (for $m \geq 1$), D^*_m (for $m \geq 2$), T^*, O^* or I^*. (The star indicates the double cover of the relevant subgroup of SO_3 [28].) The corresponding Hirzebruch defects are $3 - m$, $-m$, -4, -5 and -6, respectively. Note that φ_+ is canonical only in the cyclic case for $m \leq 5$, and in the dihedral case for $m = 2$ when G is the quaternion group Q_8.

In particular $C_m\backslash S^3$ is the lens space $L(m, 1)$ (for $m > 0$). The quotient framing φ_+ with Hirzebruch defect

$$h(L(m, 1), \varphi_+) = 3 - m$$

can then be modified to give a canonical framing $\varphi_+ + \lfloor (m-1)/4 \rfloor \rho$ for the associated spin structure Σ_+ on $L(m, 1)$. If m is even, then $L(m, 1)$ has an another spin structure, and it will be seen below how to construct an associated framing.[4]

For the Poincaré homology sphere $P^3 = I^*\backslash S^3$ we have

$$h(P^3, \varphi_+) = -6.$$

Thus $\varphi_+ + \rho$ is a canonical framing (with defect -2) for the unique spin structure on P^3. The other canonical (stable) framings are obtained by adding ρ, σ or $\rho - \sigma$ to this one.

[4]The case $m = 2$ has already been fully treated in 2.9, since $L(2, 1)$ $(= \mathbb{R}P^3) = SO_3$. Moreover, since $h(\varphi_+) = 1$, the quotient framing φ_+ coincides with the right-handed Lie framing. Similarly the quotient framing φ_- for the left coset space S^3/C_2 coincides with the left handed Lie framing.

Circle bundles

Consider an oriented circle bundle E with Euler class n over a closed, oriented surface F of genus g. Set $\chi = 2 - 2g$, the Euler characteristic of F. There is an obvious vector field τ on E, tangent to the oriented circle fibers. Any framing of E which extends τ will be called *fiber-preserving*. The following lemma and discussion can be compared with Gompf's discussion of fields of 2-planes on 3-manifolds in section 4 of [7].

Lemma 3.1. *E has fiber-preserving framings if and only if n divides χ.*

Proof. Let $\tau^\perp$ denote the oriented 2-plane bundle over E which is orthogonal to τ. Clearly τ extends to a framing of E if and only if $\tau^\perp$ has a nonvanishing section. The obstruction to finding such a section is given by the Euler class $e(\tau^\perp) \in H^2(E) = \mathbb{Z}^{2g} \oplus \mathbb{Z}_n$. Since the projection $p\colon E \to F$ is covered by a bundle map $\tau^\perp \to \tau_F$, it follows that $e(\tau^\perp) = p^*(e(\tau_F)) = (0, \chi)$, and so $e(\tau^\perp) = 0$ if and only if $\chi \equiv 0 \pmod{n}$. □

The fiber-preserving framings of E can be described explicitly as follows. Let D be a disk in F and let F_0 be the complement of the interior of D. Then E is trivial over F_0, so we frame its tangent bundle by the product of τ with a (tangential) framing of F_0. Note that the framing of F_0 is not unique, for it can be changed by elements of $H^1(F)$, but the 2-frame on ∂F_0 is unique up to homotopy, and it spins $1 - \chi$ times compared to the stabilized tangent framing of ∂F_0. When the trivial circle bundle over D is attached to the trivial circle bundle over F_0 to get a bundle with Euler class n, we pull back a framing on $\partial D \times S^1 = T^2$. On a $(1, n)$ curve in T^2 (the image of $\partial F_0 \times$point), the framing has one vector equal to τ and the other two vectors spin $\chi - 1$ times compared to a tangent vector to the $(1, n)$ curve. Thus, if we frame $D \times S^1$ by τ and a pair of vectors which are constant on each copy of D, but rotate χ/n times as the S^1 is traversed, then this framing matches up with the one induced from the framing on $F_0 \times S^1$.

We illustrate this construction for the case $g = 0$ and $n = 1$, where the fiber-preserving framing corresponds to the right handed Lie framing on S^3, and show how to use it to calculate the first Pontrjagin number of the disk bundle Δ associated with E.

Example 3.2. The boundary S^3 of the normal bundle Δ of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ is the Hopf circle bundle over $\mathbb{C}P^1$, of Euler class 1. The right handed Hopf framing φ_+ on S^3 corresponds in the above discussion to $\chi/n = 2/1 = 2$ rotations in the framing on $D \times S^1$. This gives a stable framing ϕ_+ of τ_{S^3}, the outward normal plus τ for one complex line, and the framing of $\tau^\perp$ for the other complex line. This stable framing does not extend over $\mathbb{C}P^1$, but it does provide a trivialization of the *complex* bundle τ_Δ over the boundary S^3, and so

$$p_1(\Delta, \phi_+) = c_1^2(\Delta, \phi_+) - 2c_2(\Delta, \phi_+)$$

(see Appendix A). Now c_1 is the obstruction to extending this trivialization over the 2-skeleton of Δ, that is, over a fiber of Δ. There is one full twist in the complex line spanned by the outward normal and τ, and two full twists in the orthogonal complex line as noted above, so $3\,\mathbb{C}P^1$ is the Poincaré dual to c_1. Thus c_1^2 is the self intersection of $3\,\mathbb{C}P^1$. Of course c_2 is the Euler class, so $p_1(\Delta, \phi_+) = 9 - 4 = 5$.

Note that this calculation is consistent with previous calculations of the Hirzebruch defect: $h(\phi_+) = p_1(\Delta, \phi_+) - 3\sigma(\Delta) = 5 - 3 = 2$. It also gives direct confirmation of the Hirzebruch signature formula for $\mathbb{C}P^2$. Indeed, for the remaining 4-handle of $\mathbb{C}P^2$ we calculate $p_1(\bar{B}^4, \bar{\phi}_+) = -2$, so $p_1(\mathbb{C}P^2) = 5 - 2 = 3$.

Now in general, observe that the flexibility in the initial choice of the framing of F_0 in the construction above, arising from the action of $H^1(F)$, leads to a fiber-preserving framing in each spin structure on E which is the pull back $p^*\Sigma$ of some spin structure Σ on F. These will be called the *bundle spin structures* on E.

Theorem 3.3. *If n divides χ, then there is a unique fiber-preserving framing φ in each bundle spin structure on E, and $h(\varphi) = n + \chi^2/n - 3\text{sign}(n)$.*

Proof. The existence of φ was shown above, and the uniqueness (up to not necessarily fiber-preserving homotopy) is clear. Calculating the relative Pontrjagin number for the disk bundle Δ over F bounded by E, as in the example, we have

$$p_1(\Delta, \varphi) = (1 + \chi/n)F \cdot (1 + \chi/n)F - 2\chi = (1 + \chi/n)^2 n - 2\chi = n + \chi^2/n.$$

The Hirzebruch defect is then gotten by subracting $3\sigma(W) = 3\text{sign}(n)$. □

4. Boundaries and Surgery

In this section we discuss natural stable framings of a 3-manifold M that arise when viewing it as the boundary of a simply-connected spin 4-manifold W. For simplicity, we consider only the case when W can be built without 1- or 3-handles, and so M can also be viewed as surgery on a link in S^3.

Boundaries

Let M_L denote the boundary of the 4-manifold W_L obtained by adding 2-handles to B^4 along a (normally) framed link L in S^3 [13]. It is a classical theorem of Lickorish and Wallace that every closed connected oriented 3-manifold is diffeomorphic to some M_L.

If the framings on L are all even, then W_L is parallelizable with a unique framing (since $H_1(W_L) = 0$). This framing restricts to a stable framing δ_L of M_L, with associated spin structure Σ_L. By Theorems 2.2b and 2.5b, the total defect $H = (d, h)$ of this stable framing is

$$H(\delta_L) = (\chi_L, -3\sigma_L)$$

where χ_L and σ_L denote the Euler characteristic and signature of W_L. From δ_L we get an honest framing $\varepsilon_L = \delta_L + \chi_L\,\sigma$ with Hirzebruch defect

$$h(\varepsilon_L) = \lambda_L = 2\chi_L - 3\sigma_L.$$

Note that λ_L reduces mod 4 to the invariant $\lambda(\Sigma_L)$ that identifies the affine lattice in the dh-plane associated with the set $\mathbb{F}_L$ of stable framings compatible with Σ_L (see above Theorem 2.6). Now subtracting a suitable multiple of ρ from ε_L gives a canonical framing for Σ_L with total defect $(0, \lambda)$, for $|\lambda| \leq 2$ and $\lambda \equiv \lambda_L$. Since every spin 3-manifold (M, Σ) is spin diffeomorphic to some (M_L, Σ_L), and there is an effective algorithm for finding

L [12], this provides a general construction for all the canonical framings of any given 3-manifold.

Example 4.1. The lens space $L(m,1)$ for $m>0$ can be described either as the boundary of W_K where K is the unknot with framing $-m$ (the minus sign can be seen geometrically from a careful look at the quaternions), or as the boundary of W_L where L consists of a simple chain of $m-1$ simply linked unknots all with framings $+2$. Now $L(m,1)$ has exactly two spin structures if m is even, given by Σ_K and Σ_L, and a unique spin structure if m is odd, given by Σ_L, with mu invariants $\mu_K \equiv -1$ and $\mu_L \equiv m-1$.[5] The corresponding stable framings δ_K (for even m) and δ_L (in general) have total defects

$$H(L(m,1),\delta_K) = (2,3) \qquad \text{and} \qquad H(L(m,1),\delta_L) = (m, 3-3m)$$

since $\chi_K = 2$, $\sigma_K = -1$, $\chi_L = m$ and $\sigma_L = m-1$. From this (or using Theorem 2.6 and the mu invariant calculation above) we deduce that $\lambda_K \equiv -1$ and $\lambda_L \equiv 3-m \pmod 4$. The associated canonical framings are $\delta_K + 2\sigma - 2\rho$ and $\delta_L + m\sigma + \frac{1}{4}(m-3+\lambda)\rho$, for $|\lambda|<2$ and $\lambda \equiv 3-m \pmod 4$.

The stable framing δ_L can also be expressed in terms of the quotient φ_+ of the right-handed Lie framing on S^3, discussed in the previous section. Indeed the defect calculations above show that $\delta_L = \phi_+ - m\sigma$, where ϕ_+ is the stabilization $\nu \oplus \varphi_+$. Implicit in this statement is the assumption that ϕ_+ is compatible with the spin structure Σ_L. This is automatic if m is odd, when there is only one spin structure on $L(m,1)$, and is also clear for $m \equiv 2 \pmod 4$, since $\lambda(\Sigma_L)$ then distinguishes the two spin structures.

If $m \equiv 0 \pmod 4$, then the situation is more subtle. In this case the affine lattices for $\mathbb{F}_K$ and $\mathbb{F}_L$ coincide, both being equal to Λ_{-1} (see §2). It follows from Theorems 2.2 and 2.5 applied to the universal cover $\pi : S^3 \to L(m,1)$ that π^* embeds both lattices onto the same sublattice $m\Lambda_0 + (0,2)$ of Λ_2 (of index m^2). This implies that there exist *homotopic* representatives of the Lie framing φ_+ on S^3, equivariant with respect to the covering transformations of π, which project to framings φ_+^K and φ_+^L of defect $3-m$ in the respective spin structures Σ_K and Σ_L on $L(m,1)$. Thus the precise statement when m is a multiple of 4 is that $\delta_K = \phi_+^K + (1+m/4)\rho - 2\sigma$ and $\delta_L = \phi_+^L - m\sigma$.

Example 4.2. The Poincaré homology sphere P^3 is the boundary of W_L where L is the E_8-link with framings $+2$. Since $\chi_L = 9$ and $\sigma_L = 8$, we have

$$H(P^3,\delta_L) = (9,-24)$$

and so $\delta_L + 9\sigma + r\rho$ is canonical for $r = 1$ or 2. Also note that $\delta_L + 9\sigma = \phi_+$, the stabilized quotient of the right-handed Lie framing on S^3.

[5] This is a standard exercise in the calculus of framed links: Each spin structure Σ on a 3-manifold M_L is uniquely specified by a *characteristic* sublink C of L, with $\mu(\Sigma) \equiv \sigma_L - C\cdot C + 8\mathrm{Arf}(C) \pmod{16}$ (see [14, Appendix C]). The algorithm in [12] shows how to *blow down* C to obtain an even framed link representing Σ as above.

Surgery

Starting with a framed link L in S^3 as above, there is a natural way to construct a 2-framing ${}^2\phi_L$ of $M_L = \partial W_L$, discovered by Freed and Gompf [6], which proceeds by extending the canonical 2-framing on the complement of L over the solid tori which are glued in under surgery on L. They showed that

$$h({}^2\phi_L) = 2\tau_L - 6\sigma_L$$

where $\tau_L = \sum a_i$, the sum of the framings a_i of the components of L, and $\sigma_L = \sigma(W_L)$.

Using the same philosophy, we show how to construct framings and stable framings whose Hirzebruch defects satisfy similar formulas. For simplicity we assume that the (normal) framings on the components of L are all *even*, and consider only framings of M_L which are compatible with the spin structure $\Sigma_L$coming from W_L. As above, χ_L will denote the Euler characteristic of W_L. Note that $\chi_L = \ell + 1$, where ℓ is the number of components in L.

Our constructions are based on the following gluing principle for 4-manifolds.

Lemma 4.3. (Gluing) *Let W_1 and W_2 be compact oriented 4-manifolds with stable framings ϕ_1 and ϕ_2 on the boundary, and $f : M_2 \to M_1$ be an orientation reversing diffeomorphism between codimension zero submanifolds $M_i \subset \partial W_i$ such that $f^*\phi_1 = \bar{\phi}_2$. Then there is a natural stable framing ϕ on the boundary of the 4-manifold $W = W_1 \cup_f W_2$ such that $d(\phi) = \sum d(\phi_i) - \chi$, where χ is the Euler characteristic of (either) M_i, and $p_1(W, \phi) = \sum p_1(W_i, \phi_i)$.*[6]

Proof. After a homotopy, we can assume that $\phi_1|M_1$ and $\bar{\phi}_2|\bar{M}_2$ are identified under f. Extend each ϕ_i to a framing of a collar neighborhood of ∂W_i inside W_i. The result of gluing W_1 and W_2 together using f is a priori a 4-manifold with *corners*, which we denote by V, and ϕ_1 and ϕ_2 clearly combine to give a framing Φ of a collar neighborhhood of ∂V. Now W can be viewed as a retracted copy of V with smooth boundary in this collar, and the restriction of Φ to ∂W is the desired stable framing ϕ.

Note that W is the union of (copies of) W_1, W_2, and a "thin" 4-manifold Θ bounded by $\partial W \cup \partial\bar{W}_1 \cup \partial\bar{W}_2$. By Theorem 2.2b $d(\phi) - d(\phi_1) - d(\phi_2) = \chi(\Theta) = -\chi$, and the additivity of p_1 is obvious. □

Now apply the lemma to the handlebody W_L, obtained from B^4 (the 0-handle) by attaching 2-handles along the even framed link L. Note that the attaching regions are solid tori, with $\chi = 0$, and so the degrees add. We begin with the stable case.

Construction 4.4. (Natural stable framings of M_L extending Σ_L.) Because the framings on L are even, any choice of stable framing on $S^3 = \partial B^4$ will extend across the 2-handles. If we choose $\delta + n\sigma$ (where δ is the canonical stable framing coming from B^4)

[6]The behavior of the signature under gluing, discussed in [26], reduces to the computation of $\sigma(\Theta)$ since $\sigma(W) = \sum \sigma(W_i) + \sigma(\Theta)$ by Novikov additivity.

and frame each 2-handle with δ, then the associated stable framing of M_L, denoted ϕ_L^n, will have total defect

$$H(\phi_L^n) = (\chi_L - n, 2n - 3\sigma_L).$$

In particular the case $n = 0$ gives the restriction δ_L of the unique framing on W_L, and $n = \chi_L$ gives the honest framing ε_L discussed above.

Another natural choice is $n = \tau_L/2 = \sum b_i$ (where the even framings on the components of L are $a_i = 2b_i$), and we write ϕ_L for the corresponding stable framing. This stable framing, with Hirzebruch defect

$$h(\phi_L) = \tau_L - 3\sigma_L$$

reminiscent of the Freed-Gompf 2-framing, arises in an effort to constuct an "explicit" framing of M_L, one that can be visualized without appealing to homotopy theory. The construction is in two steps:

- *Tilting:* Choose a framing representing δ which reflects the topology of L.
- *Matching:* Adjust the framing to take the framings on L into account.

Tilting. For simplicity we assume that L consists of a single knot K; the process that we describe can be repeated for each component of L. Start with an oriented embedding of K in a coordinate chart $\mathbb{R}^3$ in which δ is the standard constant framing (outward normal "1" followed by i, j, k), and assume that the projection of K onto the ij-plane is generic. Let w be the writhe (i.e. the sum of the crossing signs) and d be the Whitney degree (i.e. the degree of the Gauss map) of this projection. Then there is a canonical shortest isotopy of the framing near K (called "tilting") which rotates k to the oriented tangent vector of K. The vectors i and j will then provide a (normal) framing of K which spins $w - d$ times relative to the 0-framing.

Note that $w - d$ is unchanged by the second and third Reidemeister moves, but can change by ± 2 by the first move: when a kink (there are four types depending on the sign of the crossing and the orientation) is added, w and d both change by ± 1, and so $w - d$ changes by $-2, 0$ or 2. Thus a projection of K can be chosen so that the quantity $w - d$ takes on any prescribed *odd* value (since it is odd for the round unknot, and crossing changes preserves its parity). We choose a projection with $w - d = 1$. This yields a stable framing on S^3 whose second vector is tangent to K and whose last two provide the $+1$-framing of K in S^3

Matching. The framing on the 2-handle $H = B^2 \times B^2$ can be taken to be the product of the constant framings on the two factors, denoted by u_1, v_1 and u_2, v_2. It is more useful however to have the framing on the attaching circle $K = S^1 \times B^2$ consist of one vector tangent to K and one pointing into H. This can be achieved by an isotopy of the framing on H with support near K which rotates u_1 to the oriented tangent vector to K, rotates v_1 to point inward, and necessarily moves u_2 and v_2 so that they rotate -1 time as an attaching circle is traversed. (The last statement follows from the well known fact that the natural homomorphism $\pi_1(U_1) \oplus \pi_1(U_1) \to \pi_1(U_2)$ maps (p, q) to $p + q$.)

First suppose that K has the 0-framing. Then the framing on K in the 0-handle perfectly matches its framing in the 2-handle, with $\{u_1, u_2, v_1, v_2\}$ corresponding in order to $\{i, 1, j, k\}$; note the transposition of 1 and i which corresponds to the fact that one orientation is negated when two manifolds are joined along part of their boundaries. Also note that the framing along K is $+1$ compared to the 0-framing of K, whereas the normal bundle of the attaching circle has framing -1 in the boundary of the 2-handle; this is correct because of the orientation change.[7] Since the framings agree on K, they are close on a neighborhood of K and therefore canonically isotopic on this neighborhood, and so the gluing lemma applies.

Now suppose that K has framing $a = 2b$. Recall from §2 that adding σ to a stable framing puts a full right twist in both the normal and conormal planes along each diameter of a small 3-ball. In fact this local description can be modified by a homotopy so that along one diameter, the normal plane rotates two full right twists while the conormal plane does not rotate at all. Using the latter description, modify the stable framing δ near K by adding $b = (\tau_L/2)$ copies of σ in a ball intersecting K in a diameter. This puts $2b = a$ twists in the framing along K in the 0-handle, and so the framings along K in the 0- and 2-handles match.

Construction 4.5. (Natural framings of M_L extending Σ_L.) We apply Lemma 4.3 as in the stable case, framing the boundary of the 0-handle with $\varphi_\pm + n\rho$ (where $\varphi_\pm$ are the canonical Lie framings) and each 2-handle with $\varphi_\pm$. The associated framings $\varphi^n_{\pm L}$ have Hirzebruch defects

$$h(\varphi^n_{\pm L}) = 4n \pm 2\chi_L - 3\sigma_L$$

and so in particular $\varphi^0_{+L} = \varepsilon_L$. As for stable framings, the case $n = \tau_L/2 = \sum b_i$ (where the $a_i = 2b_i$ are the framings on L) is also a natural choice, and we write $\varphi_{\pm L}$ for the corresponding framings of M_L. These framings have Hirzebruch defects

$$h(\varphi_{\pm L}) = 2\tau_L - 3\sigma_L \pm 2\chi_L$$

and can be constructed by a "tilting/matching" scheme as above. The only difference is that when it becomes necessary to modify the framing near L in S^3 to match the 2-handle framings, we use ρ (which also puts two full twists in the framing along a diameter) instead of σ. Note that τ_L is replaced with $2\tau_L$, since adding ρ adds 4 to p_1, and thus to h, rather than 2.

Construction 4.6. (Natural 2-framings on M_L.) Finally for 2-framings, one way to proceed is to take the Whitney sum of a pair of natural framings (just described) on each factor of τ_M. In particular, the natural 2-framing of Freed and Gompf [6] can be expressed as

$$^2\phi_L = \varphi_{+L} \oplus \varphi^0_{-L} = \varphi^0_{+L} \oplus \varphi_{-L}.$$

[7] The easiest way to be convinced of this is to consider a Hopf link in S^3, which is the equator of S^4. Orient the Hopf link, and let it bound 2-balls in both hemispheres of S^4. The two 2-balls intersect $+1$ in one hemisphere and -1 in the other, since the two 2-spheres intersect algebraically zero, and so the linking number of the oriented Hopf link is $+1$ in one hemisphere and -1 in the other.

Alternatively ${}^2\phi_L$ can be described in a natural way using an analogue of Lemma 4.3. For the reader's convenience, we also recall the description of ${}^2\phi_L$ in [6]: The first step is to isotope the canonical 2-framing on S^3 so that restricted to the boundary torus T of the attaching region V for each 2-handle, it is the Whitney sum of two copies of a Lie framing on T plus the normal vector. There is a choice in how this is done, since $\pi_3(\mathrm{Spin}_6) = \mathbb{Z}$, but a different choice would change the 2-framing on $S^3 - V$ by some $\alpha \in \pi_3(\mathrm{Spin}_6)$ while changing the 2-framing on V by $-\alpha$, and so these changes cancel on M_L after regluing. Now the solid torus is removed and glued back in by an element A of $SL_2(\mathbb{Z})$, but the 2-framings on T can be made to match by choosing a shortest path in $SL_2(\mathbb{R})$ from A to the identity. This gives ${}^2\phi_L$, and Freed and Gompf prove directly that $h({}^2\phi_L) = 2\tau_L - 6\sigma_L$.

Appendix A. Pontrjagin numbers

This appendix is a brief review of some aspects of the theory of characteristic classes from the obstruction point of view. The focus is on the relative first Pontrjagin number of a compact 4-manifold, which along with the signature is used to define the Hirzebruch defect of a framing of the boundary 3-manifold. This material is of course well known, although we do not know where to find an elementary discussion of the relative theory from the obstruction point of view.

Absolute characteristic classes

Let X be a closed oriented 4-manifold. The kth Chern class $c_k(\omega)$ of a complex n-plane bundle ω over X can be identified with the obstruction to finding an $(n-k+1)$-*field* on ω (i.e. $n-k+1$ linearly independent sections of ω) over the $2k$-skeleton of X [21, p.171]. In particular the second Chern class $c_2(\omega) \in H^4(X;\mathbb{Z})$ is the obstruction to finding an $(n-1)$-field on ω over all of X. (This obstruction is a priori an element of $H^4(X;\{\pi_3(U_n/U_1)\})$, but the coefficient groups are canonically identified with $\pi_3(SU_n) = \mathbb{Z}$ as explained in §1.) Evaluating on the fundamental class $[X]$ gives the second Chern number of ω, an integer which by abuse of notation will also be denoted by $c_2(\omega)$. For example, if $n = 2$ then $c_2(\omega) = e(\omega_{\mathbb{R}})$, the Euler class of the underlying oriented real bundle or equivalently the self intersection of the zero-section of ω. In particular $c_2(\tau_X) = \chi(X)$ if X is a complex surface, where χ is the Euler characteristic.

The first Pontrjagin class of a real n-plane bundle ξ over X is defined in terms of the complexified bundle $\xi_{\mathbb{C}}$ by

$$p_1(\xi) = -c_2(\xi_{\mathbb{C}})$$

[21, p.174]. If ξ admits a complex structure, then p_1 can also be computed using the formula $p_1(\omega_{\mathbb{R}}) = c_1^2(\omega) - 2c_2(\omega)$ [21, p.177]. For example $p_1(\tau_X) = -2\chi(X)$ if X is a complex elliptic surface, since $c_1^2 = 0$ for elliptic surfaces [17].

Taking $\xi = \tau_X$ and evaluating on the fundamental class gives the first Pontrjagin number

$$p_1(X) = p_1(\tau_X)[X].$$

Thus $p_1(X)$ is seen to be the integer obstruction to the existence of a 3-field on the complexified tangent bundle of X.

Relative characteristic classes

Let W be a compact oriented 4-manifold with nonempty boundary. Then of course $c_2(\omega)$ and $p_1(\xi)$ vanish for all complex bundles ω and real bundles ξ over W, since $H^4(W) = 0$, and so we consider *relative* characteristic classes instead. In particular for any $(n-1)$-field ϕ on $\omega|\partial W$, the relative second Chern class $c_2(\omega, \phi) \in H^4(W, \partial W; \mathbb{Z})$ can be defined as the obstruction to extending ϕ over W. One then defines the relative first Pontrjagin class

$$p_1(\xi, \phi) = -c_2(\xi_{\mathbb{C}}, \phi_{\mathbb{C}}),$$

where ϕ is an $(n-1)$-field on $\xi|\partial W$ and $\phi_{\mathbb{C}}$ is the induced $(n-1)$-field on $\xi_{\mathbb{C}}|\partial W$. For complex bundles ω one also has $p_1(\omega_{\mathbb{R}}, \phi) = c_1^2(\omega, \phi) - 2c_2(\omega, \phi)$, as in the absolute case.

Taking $\xi = \tau_W$ and evaluating on the fundamental class gives a relative first Pontrjagin number

$$p_1(W, \phi) = p_1(\tau_W, \phi)[W, \partial W]$$

associated to any given tangential 3-field ϕ over ∂W. Thus $p_1(W, \phi)$ is an integer invariant which measures the obstruction to extending ϕ to a 3-field on the complexified tangent bundle of W. If ϕ is a stable framing, then the notation $p_1(W, \phi)$ should be interpreted to mean $p_1(W, \varphi)$, where φ is obtained by dropping the first vector of ϕ. (Note that φ determines ϕ, using the orientation of W.)

Similarly, taking $\xi = 2\tau_W$, one defines $p_1(W, \phi) = p_1(2\tau_W, \phi)[W, \partial W]$ for any 7-field ϕ on $\tau_W \oplus \tau_W$ over ∂W. For example, ϕ can be taken to be of the form $\nu \oplus {}^2\phi$, where ν is the outward normal vector field in the first factor of τ_W and ${}^2\phi$ is a 2-framing of ∂W. In this case, this invariant is also denoted $p_1(W, {}^2\phi)$. Note that $p_1(W, \varphi_1 \oplus \varphi_2) = p_1(W, \varphi_1) + p_1(W, \varphi_2)$ for any two framings φ_1 and φ_2 of ∂W, by the product formula for Pontrjagin classes [21, p.175].

Appendix B. Signature theorems

This appendix contains a discussion of the signature theorem for closed 4-manifolds (classically due to Hirzebruch, and in its equivariant form, to Atiyah and Singer) and various "defect" invariants of 3-manifolds that result from the failure of this theorem for 4-manifolds with boundary. In particular, we give a formula which relates the behavior of the Hirzebruch defect of framed 3-manifolds under a covering projection, with the signature defect of the covering (see Lemma B.1).

The Hirzebruch signature theorem

The first Pontrjagin number of a closed oriented 4-manifold X is related to the signature $\sigma(X)$ of X (the difference of the number of positive and negative eigenvalues of the intersection form on $H_2(X)$) by Hirzebruch's signature formula

$$p_1(X) = 3\sigma(X)$$

[9, p.86] (see also [10, 21, 13]). This obviously fails for manifolds with boundary, since then p_1 is always zero while σ need not be. In fact it even fails when p_1 is replaced with the appropriate relative version, as will be seen below (cf. [2]). Note that replacing τ_X by $2\tau_X$ gives $p_1(2\tau_X)[X] = 2p_1(X) = 6\sigma(X)$ (see [21, p.175]).

One important consequence of the signature theorem is the multiplicativity of the signature under finite covers: if $\tilde{X} \to X$ is an r-fold covering map of closed oriented 4-manifolds, then

$$\sigma(\tilde{X}) = r\,\sigma(X).$$

This follows from the multiplicativity of p_1, which is evident from the obstruction point of view. Again this fails for bounded manifolds.

Defects

The failure of the signature theorem for bounded 4-manifolds gives rise to an integer invariant of framed 3-manifolds (cf. [1, 2]). This is what we have called the Hirzebruch defect in §1, and is the key invariant used in defining canonical framings in §2. We recall the definition from §1: The *Hirzebruch defect* of a framing or stable framing ϕ of a closed oriented 3-manifold M is defined by

$$h(\phi) = p_1(W, \phi) - 3\sigma(W)$$

for any compact oriented 4-manifold W with oriented boundary M. (This is seen to be independent of the choice of W by Novikov additivity of the signature and the signature formula for closed manifolds.) Similarly define

$$h(^2\phi) = p_1(W, {}^2\phi) - 6\sigma(W)$$

for any 2-framing $^2\phi$ of M, in light of the observation above that $p_1(2\tau_X)[X] = 6\sigma(X)$ for closed manifolds. Note that $h(\varphi_1 \oplus \varphi_2) = h(\varphi_1) + h(\varphi_2)$ for any two framings φ_1 and φ_2 of M, since the first Pontrjagin numbers add (see Appendix A).

Similarly the failure of the multiplicativity of the signature for finite covering spaces of bounded 4-manifolds leads to an invariant for 3-dimensional coverings. First define the *signature defect* of an r-fold covering map $\Pi\colon \tilde{W} \to W$ of compact oriented 4-manifolds to be $\sigma(\Pi) = r\sigma(W) - \sigma(\tilde{W})$. Of course this vanishes if the manifolds are closed, but is in general nonzero. From this one defines the *signature defect* $\sigma(\pi)$ of an r-fold covering map $\pi : \tilde{M} \to M$ of closed oriented 3-manifolds, as follows. If π is a regular cover, then it corresponds to a finite index normal subgroup H of $\pi_1(M)$, or equivalently to a homomorphism $\pi_1(M) \to G = \pi_1(M)/H$. Since 3-dimensional bordism over a finite group G is torsion [4], some finite number m of copies of π bounds over G. In other words, there exists a finite covering $\Pi\colon \tilde{W} \to W$ of compact oriented 4-manifolds with $m\pi = \partial\Pi$ (= the restriction $\Pi|\partial\tilde{W} \to \partial W$). Now set

$$\sigma(\pi) \;=\; \frac{1}{m}\sigma(\Pi) \;=\; \frac{1}{m}(r\sigma(W) - \sigma(\tilde{W})).$$

This is clearly well-defined by Novikov additivity and the multiplicativity of the signature for closed manifolds. If π is irregular, then the core $\hat{H}$ of H (i.e. the intersection of all

the conjugates of H in $\pi_1(M)$) is normal and of finite index in both $\pi_1(M)$ and H, and so corresponds to a pair of finite regular covers $\hat{\pi}\colon \hat{M} \to M$ and $\tilde{\pi}\colon \hat{M} \to \tilde{M}$. In this case the signature defect is defined by $s\sigma(\pi) = \sigma(\hat{\pi}) - \sigma(\tilde{\pi})$, where s is the index of $\hat{H}$ in H.

In the same spirit one can define the *Pontrjagin defect* $p_1(\Pi)$ of an r-fold covering map $\Pi\colon (\tilde{W}, \tilde{\phi}) \to (W, \phi)$ of compact bounded 4-manifolds by $p_1(\Pi) = rp_1(W, \phi) - p_1(\tilde{W}, \tilde{\phi})$. Here ϕ and $\tilde{\phi}$ are framings or stable framings on the boundary which are assumed to be compatible (i.e. $\tilde{\phi} = \Pi^*\phi$, the pull back of ϕ under the covering map). It is clear from the characterization of p_1 as an obstruction that this invariant is in fact *identically zero.*

Finally the *Hirzebruch defect* $h(\pi)$ of an r-fold cover $\pi : (\tilde{M}, \tilde{\phi}) \to (M, \phi)$ of closed 3-manifolds, with compatible (stable) framings, is defined by

$$h(\pi) = rh(\phi) - h(\tilde{M}, \tilde{\phi}).$$

Observe that for $\pi = \partial\Pi$, we have by definition $h(\pi) = p_1(\Pi) - 3\sigma(\Pi) = -3\sigma(\pi)$. In fact the Hirzebruch and signature defects are always related in this way.

Lemma B.1. *Let $\pi : (\tilde{M}, \tilde{\phi}) \to (M, \phi)$ be an r-fold covering map of closed oriented 3-manifolds with compatible (stable) framings. Then $h(\pi) = -3\sigma(\pi)$. In other words*

$$h(\tilde{M}, \tilde{\phi}) = r\, h(\phi) + 3\, \sigma(\pi)$$

where $\sigma(\pi)$ is the signature defect of π. □

Proof. If π is regular, then some multiple of π bounds, $m\pi = \partial\Pi$. It follows that $mh(\pi) = h(m\pi) = h(\partial\Pi) = -3\sigma(\Pi)$. Dividing by m gives $h(\pi) = -3\sigma(\Pi)/m = -3\sigma(\pi)$. For irregular π there exist regular covers $\hat{\pi}$ and $\tilde{\pi}$ of degrees rs and s such that $\hat{\pi} = \pi\tilde{\pi}$, as discussed above. Then by definition $sh(\pi) = h(\hat{\pi}) - h(\tilde{\pi}) = 3(\sigma(\tilde{\pi}) - \sigma(\hat{\pi}))$ (since $\hat{\pi}$ and $\tilde{\pi}$ are regular). Dividing by s gives $h(\pi) = 3(\sigma(\tilde{\pi}) - \sigma(\hat{\pi}))/s = -3\sigma(\pi)$. □

To exploit this result, one must find ways to compute the signature defect. The most powerful tool for this purpose is the equivariant version of the signature theorem, due to Atiyah and Singer [3] (see also [10] and [8]).

The Atiyah-Singer G-signature theorem

Let G be a finite group of order r acting effectively on a closed oriented 4-manifold $\tilde{X}$ by orientation preserving diffeomorphisms. Since G is finite, the orbit space $X = \tilde{X}/G$ is an oriented rational homology manifold with signature $\sigma(X)$ defined in the usual way. Furthermore, each element $g \in G$ has an associated *g-signature* $\sigma(g)$ (= $\mathrm{sign}(g, \tilde{X})$ in the literature) given by the difference of the traces of g acting on the positive and negative definite subspaces of $H^2(\tilde{X}; \mathbb{R})$ with the cup product form. An elementary representation theory argument shows that the sum of all the g-signatures is equal to $r\sigma(X)$, and so

$$r\sigma(X) - \sigma(\tilde{X}) = \sum_{g \neq 1} \sigma(g)$$

since clearly $\sigma(1) = \sigma(\tilde{X})$.

As one might suspect, this equality fails if $\tilde{X}$ is allowed to have boundary. Thus one obtains in the usual way an invariant for free actions of G on closed oriented 3-manifolds $\tilde{M}$, or equivalently covering spaces $\pi : \tilde{M} \to M = \tilde{M}/G$. Namely, choose any compact oriented 4-manifold $\tilde{W}$ bounded by $\tilde{M}$ over which the action extends, and write W for the orbit space of this extended action. Then define

$$\sigma(\pi) = r\sigma(W) - \sigma(\tilde{W}) - \sum_{g \neq 1} \sigma(g).$$

This invariant clearly coincides with the signature defect of π defined above.

Now the G-signature theorem [3] gives local formulas for the g-signatures $\sigma(g)$ when $g \neq 1$ in terms of the infinitesimal action of g on its fixed point set. In particular, this fixed point set consists of a finite union of points x_i and connected surfaces F_j, and if g acts on the tangent space at x_i by rotating a pair of orthogonal planes through angles α_i and β_i, and on the tangent space at each point on F_j by rotating the normal plane to F_j through an angle γ_j, then

$$\sigma(g) = -\sum \cot(\alpha_i/2)\cot(\beta_i/2) + \sum F_j \cdot F_j \csc^2(\gamma_j/2).$$

These formulas provide a way to compute signature defects in many situations.

For example Hirzebruch [10] used this approach to compute the defects of (the universal covers of) lens spaces in terms of classical Dedekind sums (see also [15]). This was accomplished by viewing $L(m, n)$ as the orbit space of a linear action of the cyclic group C_m on $S^3 = \partial B^4$. When $n = \pm 1$ the picture is particularly simple since C_m can be taken to be a subgroup of S^3 acting by multiplication. We explain this case below, along with the slightly more complicated case of the binary icosahedral group I^* of order 120. The quotient $I^* \backslash S^3$ is the well known Poincaré homology sphere P^3.

Example B.2. The lens space $L(m, 1)$ for $m > 0$ has signature defect $(m-1)(m-2)/3$. To see this, view S^3 as the unit quaternions and $L(m, 1)$ as the homogeneous space $C_m \backslash S^3$ of right cosets of any of the cyclic subgroups C_m of S^3 of order m (they are all conjugate). The action of C_m by left multiplication extends by coning to $W = B^4$, with quotient $\tilde{W} =$ the cone on $L(m, 1)$. The signatures of W and $\tilde{W}$ vanish, as both spaces are contractible, and so the defect consists only of the contributions from the fixed points.

Now each $u \neq 1$ in C_m has a unique fixed point at the origin, where the action is given by left multiplication by u. This rotates the plane spanned by 1 and u through the angle $\arg(u)$ $(= 2k\pi/m$ for some $k)$ and the orthogonal plane through the *same* angle (it would be the opposite angle if the action were on the right), and so contributes $\cot^2(\arg(u)/2)$ to the defect. Since the elements of C_m are equally spaced on a great circle in S^3 through ± 1 the total contribution is

$$\sum_{k=1}^{m-1} \cot^2(k\pi/m) = (m-1)(m-2)/3.$$

(The closed form for the sum is classical, cf. [10, p.19].)

Example B.3. The Poincaré homology sphere $P^3 = I^*\backslash S^3$ has signature defect 722/3. Indeed, proceeding as in the previous example, each $u \neq 1$ in C_p contributes $\cot^2(\arg(u)/2)$ to the defect. Note in particular that -1 makes no contribution, since $\cot(\pi/2) = 0$. Now observe that the icosahedral group I can be expressed as the union of fifteen cyclic subgroups of order 2 (given by π-rotations around the fifteen axes through opposite edges of the icosahedron), ten of order 3 (given by $2k\pi/3$-rotations around the ten axes through opposite faces), and six of order 5 (given by $2k\pi/5$-rotations around the six axes between opposite vertices). These lift to cover I^* with fifteen cyclic subgroups of order 4, ten of order 6, and six of order 10. Since any two of these subgroups intersect in ± 1, the defect is just the sum of their contributions. From the previous example we know that a subgroup of order m contributes $(m-1)(m-2)/3$, and so the defect is $15 \cdot 6/3 + 10 \cdot 20/3 + 6 \cdot 72/3 = 722/3$.

A similar calculation gives 98/3 and 242/3 for the defects of quotients $T^*\backslash S^3$ and $O^*\backslash S^3$ by the binary tetrahedral and octahedral groups, and $(4m^2+2)/3$ for the defects of the prism manifolds $D^*_m\backslash S^3$. (See for example [28, §2.6] for a discussion of the finite subgroups of S^3.)

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