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Seiberg-Witten invariants of mapping tori, symplectic fixed points, and Lefschetz numbers

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1. Introduction

Let Y be a compact oriented smooth 3-manifold with nonzero first Betti number. Two nonzero vector fields on Y are called **homologous** if they are homotopic over the complement of a ball in Y. An **Euler structure** on Y is an equivalence class of homologous vector fields (see Turaev [33]). Let $\mathcal{E}(Y)$ denote the space of Euler structures on Y. If Y carries a Riemannian metric then an Euler structure can also be defined as a cohomology class $e \in H^2(SY;\mathbb{Z})$ on the unit sphere bundle SY in TY which restricts to a positive generator on each fiber (with the orientation given by the complex structure $\eta \mapsto v \times \eta$). The correspondence assigns to each unit vector field $v : Y \to SY$ the Euler structure

$$e_v = \mathrm{PD}(v_*[Y]) \in H^2(SY;\mathbb{Z}).$$

With the second description it follows that there is a free and transitive action of $H^2(Y;\mathbb{Z})$ on the space of Euler structures, given by

$$H^2(Y;\mathbb{Z}) \times \mathcal{E}(Y) \to \mathcal{E}(Y) : (h,e) \mapsto h \cdot e = e + \pi^* h.$$

Moreover there is a natural map

$$\mathcal{E}(Y) \to H^2(Y;\mathbb{Z}) : e \mapsto c(e)$$

which assigns to $e = \mathrm{PD}([v])$ the Euler class of the normal bundle $v^\perp$. These maps are related by $c(h \cdot e) = c(e) + 2h$. Turaev introduces a torsion invariant

$$\mathcal{T} : \mathcal{E}(Y) \to \mathbb{Z}$$

which is a kind of refinement of the Reidemeister-Milnor torsion. In the case $b_1(Y) = 1$ this function depends on a choice of orientation of $H_1(Y)$.

A unit vector field $v : Y \to SY$ also determines a spin^c structure γ_v on Y (see Example 3.1 below). Turaev [33] observes that two such spin^c structures γ_{v_0} and γ_{v_1} are isomorphic if and only if the vector fields v_0 and v_1 are homologous, and hence there is a natural bijection between $\mathcal{E}(Y)$ and the set $\mathcal{S}^c(Y)$ of isomorphism classes of spin^c structures on Y (see also [26]). Now the Seiberg-Witten invariants of Y take the form of a function

$$\mathrm{SW} : \mathcal{S}^c(Y) \to \mathbb{Z}$$

As above, this function depends on a choice of orientation of $H_1(Y)$ whenever $b_1(Y) = 1$. In [33] Turaev conjectures that the Seiberg-Witten invariants and the torsion invariants of Y should agree under the natural identification of $\mathcal{E}(Y)$ with $\mathcal{S}^c(Y)$. The purpose of this paper is to outline a proof of this conjecture for mapping tori.[1]

Theorem 1.1. *Let Σ be a compact oriented Riemann surface and $f : \Sigma \to \Sigma$ be an orientation preserving diffeomorphism. Denote by Y_f the mapping torus of f. Then*

$$\mathrm{SW}(Y_f, \gamma_v) = \mathcal{T}(Y_f, e_v)$$

for every nonzero vector field v on Y_f.

The horizontal vector field $\partial/\partial t$ determines a canonical Euler structure $e_f \in \mathcal{E}(Y_f)$. Likewise, there is a canonical spin^c structure $\gamma_f \in \mathcal{S}^c(Y_f)$ which corresponds to e_f under the isomorphism $\mathcal{E}(Y_f) \cong \mathcal{S}^c(Y_f)$. Hence both $\mathcal{E}(Y_f)$ and $\mathcal{S}^c(Y_f)$ can be naturally identified with $H^2(Y_f;\mathbb{Z})$. A cohomology class in $H^2(Y_f;\mathbb{Z})$ can be represented as the first Chern class of a complex line bundle over Y_f. Every such line bundle is isomorphic to the mapping torus of a lift $\tilde{f} : E \to E$ of f to a unitary bundle automorphism of a Hermitian line bundle over Σ:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ \Sigma \arrow[r, "f"] & \Sigma \end{tikzcd}.$$

Let $d = \deg(E) := \langle c_1(E), [\Sigma] \rangle$ and denote by $e_{d,\tilde{f}} \in \mathcal{E}(Y_f)$ and $\gamma_{d,\tilde{f}} \in \mathcal{S}^c(Y_f)$ the Euler and spin^c structures induced by $\tilde{f}$. Then the assertion of Theorem 1.1 can be restated in the form

$$\mathrm{SW}(Y_f, \gamma_{d,\tilde{f}}) = \mathcal{T}(Y_f, e_{d,\tilde{f}})$$

for every Hermitian line bundle $E \to \Sigma$ and every automorphism $\tilde{f} : E \to E$ that descends to f.

2. Lefschetz numbers

Let M be a compact smooth manifold and $\phi : M \to M$ be a continuous map. Denote by Ω_ϕ the space of continuous paths $x : \mathbb{R} \to M$ such that $x(t+1) = \phi(x(t))$. For $x \in \Omega_\phi$ denote by $[x] \in \pi_0(\Omega_\phi)$ the homotopy class of the path. Two pairs $(\phi_0, \mathcal{P}_0)$ and $(\phi_1, \mathcal{P}_1)$ with $\mathcal{P}_i \in \pi_0(\Omega_{\phi_i})$ are called **conjugate** if there exists a homeomorphism $\psi : M \to M$ such that $\phi_1 = \psi^{-1} \circ \phi_0 \circ \psi$ and $\mathcal{P}_1 = \psi^*\mathcal{P}_0$. They are called **homotopic** if there exist a homotopy $s \mapsto \phi_s$ from ϕ_0 to ϕ_1 and a continuous map $[0,1] \times \mathbb{R} \to M : (s,t) \mapsto x_s(t)$ such that $x_s \in \Omega_{\phi_s}$ for all s and $[x_0] = \mathcal{P}_0$, $[x_1] = \mathcal{P}_1$. Every fixed point $x = \phi(x)$ determines a constant path in Ω_ϕ. For $\mathcal{P} \in \pi_0(\Omega_\phi)$ let $\mathrm{Fix}(\phi, \mathcal{P})$ denote the set of all

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[1] While this paper was written the author received a message that Turaev had proved the conjecture for general 3-manifolds [34]. Turaev's proof is based on the work by Meng-Taubes [20].

fixed points $x \in \mathrm{Fix}(\phi)$ with $[x] = \mathcal{P}$. If ϕ is smooth then a fixed point $x \in \mathrm{Fix}(\phi)$ is called **nondegenerate** if $\det(\mathbb{1} - d\phi(x)) \neq 0$. In this case the number

$$\mathrm{ind}(x, \phi) = \mathrm{sign}\ \det(\mathbb{1} - d\phi(x))$$

is called the **fixed point index** of x.

The **Lefschetz invariant** assigns an integer to every pair $(\phi, \mathcal{P})$ where $\phi : M \to M$ is a continuous map and $\mathcal{P} \in \pi_0(\Omega_\phi)$. It is characterized by the following axioms.

(Fixed point index): If ϕ is smooth and the fixed points in $\mathrm{Fix}(\phi, \mathcal{P})$ are all nondegenerate then

$$L(\phi, \mathcal{P}) = \sum_{x \in \mathrm{Fix}(\phi, \mathcal{P})} \mathrm{ind}(x, \phi).$$

(Homotopy): Homotopic pairs have the same Lefschetz invariant.
(Naturality): Conjugate pairs have the same Lefschetz invariant.
(Trace formula): The **Lefschetz number of** ϕ is given by

$$L(\phi) := \sum_{\mathcal{P} \in \pi_0(\Omega_\phi)} L(\phi, \mathcal{P}) = \sum_i (-1)^i \mathrm{trace}(\phi_* : H_i(M) \to H_i(M)).$$

(Zeta function): The **zeta function** of ϕ is given by

$$\begin{aligned} \zeta_\phi(t) &:= \exp\left(\sum_{k=1}^{\infty} \frac{t^k}{k} L(\phi^k)\right) \\ &= \prod_{i=0}^{\dim M} \det(\mathbb{1} - tH_i(\phi))^{(-1)^{i+1}} \qquad (1) \\ &= \sum_{d=0}^{\infty} t^d L(S^d \phi). \end{aligned}$$

Here ϕ^k denotes the k-th iterate of ϕ and $S^d\phi : S^dM \to S^dM$ denotes the homeomorphism of the d-fold symmetric product S^dM induced by ϕ.

(Product formula): If the periodic points of ϕ are all nondegenerate then

$$\zeta_\phi(t) = \prod_{k=1}^{\infty} \prod_{\bar{x} \in \mathcal{P}(\phi,k)/\mathbb{Z}_k} (1 - \varepsilon(x, \phi^k) t^k)^{-\varepsilon(x,\phi^k)\mathrm{ind}(x,\phi^k)}.$$

Here $\varepsilon(x, \phi^k) = \mathrm{sign}\det(\mathbb{1} + d\phi^k(x))$ and $\mathcal{P}(\phi, k)$ denotes the set of periodic points of minimal period k.

The Lefschetz invariant is uniquely determined by the *"homotopy"* and *"fixed point index"* axioms. The *"trace formula"* is the Lefschetz fixed point theorem. The *"product formula"* is due to Ionel–Parker [16] and also plays a crucial role in the work of Hutchings–Lee [14, 15].

Proof of (1) and the product formula. The second equation in (1) follows from the trace formula and the identity

$$\det(\mathbb{1}-A)^{-1} = \exp\left(\sum_{k\geq 1}\frac{A^k}{k}\right).$$

The third equation follows from the identities

$$L(S^d\phi) = \sum_{j=0}^{d}(-1)^j \mathrm{trace}(\Lambda^j H_{\mathrm{odd}}(\phi))\mathrm{trace}(S^{d-j}H_{\mathrm{ev}}(\phi)),$$

$$\det(\mathbb{1}-A) = \sum_{j\geq 0}(-1)^j\mathrm{trace}(\Lambda^j A), \quad \det(\mathbb{1}-A)^{-1} = \sum_{k\geq 0}(-1)^k\mathrm{trace}(S^k A).$$

To prove the product formula note that the indices of the iterated periodic points are given by

$$\mathrm{ind}(x,\phi^{k\ell}) = \mathrm{ind}(x,\phi^k)\varepsilon(x,\phi^k)^{\ell-1}.$$

Let $p^\pm(\phi,k)$ denote the sum of the fixed point indices of the periodic orbits in $\mathcal{P}(\phi,k)/\mathbb{Z}_k$ which satisfy $\varepsilon(x,\phi^k)=\pm 1$. Then

$$L(\phi^k) = \sum_{n|k}\frac{k}{n}\left(p^+(\phi,k/n) + (-1)^{n-1}p^-(\phi,k/n)\right).$$

This implies

$$\sum_{k=1}^{\infty}t^kL(\phi^k) = \sum_{k=1}^{\infty}\left(p^+(\phi,k)\frac{kt^k}{1-t^k} + p^-(\phi,k)\frac{kt^k}{1+t^k}\right).$$

Divide by t, integrate, and exponentiate to obtain the product formula. □

Let us now return to the case of a diffeomorphism $f:\Sigma\to\Sigma$ of a Riemann surface and a lift $\tilde{f}:E\to E$ to an automorphism of a line bundle of degree d. For $d\geq 2$ such a lift determines a homotopy class $\mathcal{P}_{d,\tilde{f}}\in\pi_0(\Omega_{S^df})$ (Lemma 7.1). If $d=1$ then $\mathcal{P}_{1,\tilde{f}}$ denotes a union of connected components of Ω_{S^df}.

Theorem 2.1. *Let Σ be a compact oriented Riemann surface, $E\to\Sigma$ be a Hermitian line bundle of degree d, $f:\Sigma\to\Sigma$ be an orientation preserving diffeomorphism and $\tilde{f}:E\to E$ be an automorphism that descends to f. Then*

$$\mathrm{SW}(Y_f,\gamma_{d,\tilde{f}}) = L(S^df,\mathcal{P}_{d,\tilde{f}}).$$

The proof of Theorem 2.1 is outlined below. Full details will appear elsewhere.

Theorem 2.1 implies Theorem 1.1. In [14, 15] Hutchings and Lee proved that

$$\mathcal{T}(Y_f,e_{d,\tilde{f}}) = L(S^df,\mathcal{P}_{d,\tilde{f}}).$$

Their proof is based on a comparison between the topological torsion and the torsion of the Morse complex of a closed 1-form α, twisted by a suitable Novikov ring. The quotient is the zeta function given by counting the periodic solutions of the gradient flow of α. In the case of mapping tori the proof can be thought of as an interpolation between a representative of α without periodic solutions (giving the torsion invariant) and one without critical points (giving the Lefschetz invariant). $\square$

Corollary 2.2. *Let Σ be a compact oriented Riemann surface of genus g and $f : \Sigma \to \Sigma$ be an orientation preserving diffeomorphism. Then*

$$\sum_{\gamma \in \mathcal{S}^c(Y_f)} \mathrm{SW}(Y_f, \gamma) t^{c(\gamma)\cdot\Sigma/2} = t^{1-g}\zeta_f(t),$$

Proof. The characteristic class of the spin^c structure $\gamma_{d,\tilde{f}}$ satisfies $c(\gamma_{d,\tilde{f}})\cdot\Sigma = 2d+2-2g$. Hence the result follows from Theorem 2.1 and (1). $\square$

Note that ζ_f is a polynomial if and only if 1 is an eigenvalue of the automorphism $f^* : H^1(\Sigma) \to H^1(\Sigma)$ or, equivalently, $b_1(Y_f) \geq 2$.

3. Seiberg-Witten invariants

Fix a Riemannian metric on Y. A spin^c structure on Y is a pair (W, γ) where $W \to Y$ is a Hermitian rank-2 bundle and $\gamma : TY \to \mathrm{End}(W)$ is a bundle homomorphism which satisfies

$$\gamma(v)\gamma(w) = \gamma(v \times w) - \langle v, w\rangle \mathbb{1}$$

for $v, w \in T_yY$. The **characteristic class** of γ is defined by $c(\gamma) = c_1(W) \in H^2(Y;\mathbb{Z})$.

Example 3.1. A unit vector field $v : Y \to TY$ determines a spin^c structure (W_v, γ_v) where $W_v = \mathbb{C} \oplus v^\perp$ and

$$\gamma_v(\eta)\begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} -i\langle\eta, v\rangle\theta_0 + \langle\eta, \theta_1\rangle + i\langle v \times \eta, \theta_1\rangle \\ \langle\eta, v\rangle v \times \theta_1 - (\mathrm{Re}\,\theta_0)(\eta - \langle\eta, v\rangle v) - (\mathrm{Im}\,\theta_0) v \times \eta \end{pmatrix}$$

for $\theta_0 \in \mathbb{C}$, $\theta_1 \in v^\perp$, and $\eta \in TY$. The characteristic class of this structure is $c(\gamma_v) = c_1(v^\perp)$.

Let $\mathcal{A}(\gamma)$ denote the space of connections on the square root $\det(W)^{1/2}$ of the determinant bundle of W. Every connection $A \in \mathcal{A}(\gamma)$ determines a spin^c connection ∇_A on W which is compatible with the Levi-Civita connection on TY. The Seiberg-Witten equations on Y take the form

$$\mathcal{D}_A\Theta = 0, \qquad \gamma(*F_A + *\eta) = (\Theta\Theta^*)_0, \tag{2}$$

for $A \in \mathcal{A}(\gamma)$ and $\Theta \in C^\infty(Y, W)$. Here $\mathcal{D}_A : C^\infty(Y, W) \to C^\infty(Y, W)$ denotes the Dirac operator induced by ∇_A, $F_A \in \Omega^2(Y, i\mathbb{R})$ denotes the curvature form of A, and $(\Theta\Theta^*)_0 \in C^\infty(Y, \mathrm{End}(W))$ is defined by $(\Theta\Theta^*)_0\theta = \langle\Theta, \theta\rangle\Theta - |\Theta|^2\theta/2$ for $\theta \in C^\infty(Y, W)$. The metric identifies TY with T^*Y and so γ induces a bundle isomorphism between $T^*Y \otimes \mathbb{C}$ and the bundle $\mathrm{End}_0(W)$ of traceless endomorphisms of W. This isomorphism identifies the imaginary valued 1-forms with the traceless Hermitian endomorphisms of W. The 2-form $\eta \in \Omega^2(Y, i\mathbb{R})$ represents a perturbation. Since $d^*\gamma^{-1}((\Theta\Theta^*)_0) = i\mathrm{Im}\,\langle\mathcal{D}_A\Theta, \Theta\rangle$ equation (2) has no solutions unless η is closed.

Remark 3.2. **(i)** The solutions of (2) are the critical points of the Chern-Simons-Dirac functional $\mathcal{CSD}_\eta : \mathcal{A}(\gamma) \times C^\infty(Y, W) \to \mathbb{R}$ given by

$$\mathcal{CSD}_\eta(A, \Theta) = -\frac{1}{2}\int_Y (A - A_0) \wedge (F_A + F_{A_0} + 2\eta) - \frac{1}{2}\int_Y \mathrm{Re}\,\langle \mathcal{D}_A\Theta, \Theta\rangle \mathrm{dvol}.$$

(ii) Every solution (A, Θ) of (2) with $\Theta \not\equiv 0$ satisfies

$$\sup_Y |\Theta|^2 \le \sup_Y \left(2|\eta| - \frac{s}{2}\right),$$

where $s : Y \to \mathbb{R}$ denotes the scalar curvature [17]. This implies that the space of gauge equivalence classes of solutions of (2) is compact.

(iii) The augmented Hessian of the Chern-Simons-Dirac functional is the self-adjoint operator $\mathcal{H}_{A,\Theta}$ on the space $\Omega^0(Y, i\mathbb{R}) \oplus \Omega^1(Y, i\mathbb{R}) \oplus C^\infty(Y, W)$ given by

$$\mathcal{H}_{A,\Theta}\begin{pmatrix} \psi \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} d^*\alpha - i\mathrm{Im}\,\langle\Theta, \theta\rangle \\ d\psi + *d\alpha - \gamma^{-1}((\theta\Theta^* + \Theta\theta^*)_0) \\ -\mathcal{D}_A\theta - \gamma(\alpha)\Theta - \psi\Theta \end{pmatrix}.$$

If (A, Θ) is a solution of (2) with $\Theta \neq 0$ then

$$\mathcal{H}_{A,\Theta}\mathcal{H}_{A,\Theta}\begin{pmatrix} \psi \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} \Delta\psi + |\Phi|^2\psi \\ \Delta\alpha + |\Theta|^2\alpha - 2i\mathrm{Im}\,\langle\nabla_A\Theta, \theta\rangle \\ \mathcal{D}_A\mathcal{D}_A\theta + |\Theta|^2\theta - 2\nabla_{A,\alpha}\Theta \end{pmatrix}$$

(see [26]). Hence every triple $(\psi, \alpha, \theta) \in \ker \mathcal{H}_{A,\Theta}$ satisfies $\psi = 0$. It follows that the kernel of the augmented Hessian agrees with the kernel of the actual Hessian $d^2\mathcal{CSD}_\eta(A, \Theta)$ on the quotient $\Omega^1(Y, i\mathbb{R}) \times C^\infty(Y, W)/\{(d\xi, -\xi\Theta) \,|\, \xi \in \Omega^0(Y, i\mathbb{R})\}$.

A solution (A, Θ) of (2) with $\Theta \neq 0$ is called **nondegenerate** if $\mathcal{H}_{A,\Theta}$ is bijective. In [9] Froyshov proved that for a generic closed perturbation η the solutions of (2) are all nondegenerate, and hence form a finite set of gauge equivalence classes (see also [26]). Perturbations with this property are called **regular**. Let (A, Θ) be a nondegenerate solution of (2). Then the index $\mu^{\mathrm{SW}}(A, \Theta)$ is defined as the spectral flow of the operator family $[-1, 1] \ni s \mapsto \mathcal{H}_s$ where $\mathcal{H}_s = \mathcal{H}_{A,s\Theta}$ for $0 \le s \le 1$ and

$$\mathcal{H}_s = \begin{pmatrix} s\pi_0 & d^* & 0 \\ d & *d + s\pi_1 & 0 \\ 0 & 0 & \mathcal{D}_A \end{pmatrix}, \qquad -1 \le s \le 0.$$

This operator is injective for $s < 0$. (See [23] for an exposition of the spectral flow.) The index $\mu^{\mathrm{SW}}(A, \Theta)$ is well defined whenever the Hessian $\mathcal{H}_{A,\Theta}$ is injective. It satisfies

$$\mu^{\mathrm{SW}}(u^*A, u^{-1}\Theta) - \mu^{\mathrm{SW}}(A, \Theta) = \left[\frac{u^{-1}du}{2\pi i}\right] \cdot c_1(W)$$

for every gauge transformation $u : Y \to S^1$. This number is always even. The Seiberg-Witten invariant of (Y, γ) is defined by

$$\mathrm{SW}(Y, \gamma) = \sum_{[A,\Theta] \in \mathrm{Crit}(\mathcal{CSD}_\eta)} (-1)^{\mu^{\mathrm{SW}}(A,\Theta)} \tag{3}$$

for every regular perturbation η, where the sum runs over all gauge equivalence classes of solutions of (2). If $b_1(Y) > 1$ then the right hand side of (3) is independent of η and the metric and depends only on the isomorphism class of the spin^c structure γ (see [26] for details).

Remark 3.3. Care must be taken when $b_1(Y) = 1$. In this case the right hand side of (3) is not independent of η but may change when η passes through the codimension-1 subspace for which there are solutions of (2) with $\Theta = 0$. This is the case whenever

$$\left[\frac{i\eta}{\pi}\right] + c_1(W) = 0$$

(in deRham cohomology). To avoid this it is convenient to fix an orientation of $H^1(Y)$ and, for each metric g on Y, denote by $\alpha_g \in \Omega^1(Y)$ the unique harmonic 1-form which has norm 1 and represents the given orientation of $H^1(Y)$. Then we impose the condition

$$\varepsilon_\gamma(g, \eta) := -\int_Y \frac{i\eta}{\pi} \wedge \alpha_g - c_1(W) \cdot [\alpha_g] < 0$$

in the definition (3) of the Seiberg-Witten invariant.

4. Vortex equations

Let Σ be a compact oriented 2-manifold of genus g. Fix a volume form $\omega \in \Omega^2(\Sigma)$ and denote by $\mathcal{J}(\Sigma)$ the space of complex structures on Σ that are compatible with the orientation. Let $E \to \Sigma$ be a Hermitian line bundle of degree

$$d = \langle c_1(E), [\Sigma] \rangle$$

and denote by $\mathcal{A}(E)$ the space of Hermitian connections on E. For every $J \in \mathcal{J}(\Sigma)$ there is a natural bijection from $\mathcal{A}(E)$ to the space of Cauchy-Riemann operators on E. The Cauchy-Riemann operator associated to $A \in \mathcal{A}(E)$ and $J \in \mathcal{J}(\Sigma)$ will be denoted by $\bar\partial_{J,A} : C^\infty(\Sigma, E) \to \Omega^{0,1}_J(\Sigma, E)$. When the complex structure is understood from the context we shall drop the subscript J. The vortex equations take the form

$$\bar\partial_{J,A}\Theta_0 = 0, \qquad *iF_A + \frac{|\Theta_0|^2}{2} = \tau \tag{4}$$

for $A \in \mathcal{A}(E)$ and $\Theta_0 \in C^\infty(\Sigma, E)$. Here $\tau : \Sigma \to \mathbb{R}$ is a smooth function such that

$$\int_\Sigma \tau\omega > 2\pi d.$$

The space of gauge equivalence classes of solutions of (4) will be denoted by

$$\mathcal{M}(J,\tau) = \mathcal{M}_{\Sigma,d}(J,\tau) = \frac{\{(A,\Theta_0) \in \mathcal{A}(E) \times C^\infty(\Sigma,E) \,|\, (4)\}}{\mathrm{Map}(\Sigma,S^1)}.$$

This space can be interpreted as a symplectic quotient as follows. The space $\mathcal{A}(E) \times C^\infty(\Sigma,E)$ carries a symplectic form Ω given by

$$\Omega((\alpha,\theta_0),(\alpha',\theta_0')) = -\int_\Sigma \alpha\wedge\alpha' + \int_\Sigma \mathrm{Im}\,\langle\theta_0,\theta_0'\rangle\omega \tag{5}$$

and a compatible complex structure $(\alpha,\theta_0) \mapsto (*\alpha, i\theta_0)$. The gauge group $\mathcal{G} = \mathrm{Map}(\Sigma,S^1)$ acts by Hamiltonian symplectomorphisms and it is a simple matter to check that the moment map is given by

$$\mathcal{A}(E)\times C^\infty(\Sigma,E) \to C^\infty(\Sigma) : (A,\Theta_0) \mapsto *iF_B + |\Theta_0|^2/2.$$

Now the space

$$\mathcal{X}_J = \big\{(A,\Theta_0) \,|\, \bar\partial_A\Theta_0 = 0,\ \Theta_0 \not\equiv 0\big\}$$

is a complex submanifold of $\mathcal{A}(E)\times C^\infty(\Sigma,E)$ and is invariant under the action of $\mathcal{G}$. Hence the moduli space $\mathcal{M}(J,\tau)$ of solutions of (4) can be interpreted as the Marsden-Weinstein quotient $\mathcal{X}_J/\!\!/\mathcal{G}(\tau)$.

Remark 4.1. The tangent space of $\mathcal{M}_{\Sigma,d}(J,\tau)$ at (A,Θ_0) consists of all pairs $(\theta_0,\alpha_1) \in C^\infty(\Sigma,E)\times\Omega^{0,1}(\Sigma)$ that satisfy

$$\bar\partial_{J,A}\theta_0 + \alpha_1\Theta_0 = 0, \qquad \bar\partial_J{}^*\alpha_1 - \frac{1}{2}\langle\Theta_0,\theta_0\rangle = 0. \tag{6}$$

Here α_1 is the $(0,1)$-part of an infinitesimal connection $\alpha \in \Omega^1(\Sigma, i\mathbb{R})$. Since $2\bar\partial^*\alpha^{0,1} = d^*\alpha - *id\alpha$ (cf. [26, Corollary 3.28]) the second equation in (6) decomposes into $*id\alpha + \mathrm{Re}\,\langle\Theta_0,\theta_0\rangle = 0$ and $d^*\alpha - i\mathrm{Im}\,\langle\Theta_0,\theta_0\rangle = 0$. The first of these equations is the infinitesimal version of the second equation in (4) and the second is the local slice condition for the action of the gauge group. Now the left hand sides of the equations (6) determine an operator $\mathcal{D}_{A,\Theta_0}$ which satisfies $\mathcal{D}_{A,\Theta_0}{}^*\mathcal{D}_{A,\Theta_0} = \Delta_{\bar\partial} + |\Theta_0|^2/2$ and hence is surjective. This shows that the moduli space $\mathcal{M}(J,\tau)$ is smooth.

Remark 4.2. The Jacobian torus of E is the quotient

$$\mathrm{Jac}_{\Sigma,d}(J) := \frac{\mathcal{A}^\omega(E)}{\mathcal{G}} \cong \frac{\mathcal{A}(E)}{\mathcal{G}^c}, \qquad \mathcal{A}^\omega(E) = \left\{A \,|\, *iF_A = \frac{2\pi d}{\mathrm{Vol}(\Sigma)}\right\}.$$

Here the complexified gauge group $\mathcal{G}^c = \mathrm{Map}(\Sigma,\mathbb{C}^*)$ acts on $\mathcal{A}(E)$ by

$$u^*A = A + u^{-1}\bar\partial u - \bar u^{-1}\partial\bar u.$$

With $u = e^{-f} : \Sigma \to \mathbb{R}$ we obtain $u^*A = A + *idf$ and $*iF_{u^*A} - *iF_A = d^*df$. Hence $u^*A \in \mathcal{A}^\omega(E)$ if and only if $d^*df = 2\pi d/\mathrm{Vol}(\Sigma) - *iF_A$. This equation has a unique solution f with mean value zero. Hence each complex gauge orbit of $\mathcal{A}(E)$ intersects $\mathcal{A}^\omega(E)$ in precisely one unitary gauge orbit.

Remark 4.3. The moduli space $\mathcal{M}_{\Sigma,d}(J,\tau)$ can be identified with the GIT quotient $\mathcal{X}_J/\mathcal{G}^c$ (see García-Prada [11]). To see this let $u = e^{-f} : \Sigma \to \mathbb{R}$. Then, by Remark 4.2, $*iF_{u^*A} - *iF_A = d^*df$ and hence the pair $(u^*A, u^{-1}\Theta_0)$ satisfies the second equation in (4) if and only if

$$d^*df + e^{2f}\frac{|\Theta_0|^2}{2} = \tau - *iF_A.$$

This is the Kazdan–Warner equation and, since the right hand side has positive mean value, it has a unique solution $f : \Sigma \to \mathbb{R}$ [26, Appendix D]. This establishes the bijection

$$\mathcal{M}_{\Sigma,d}(J,\tau) = \mathcal{X}_J /\!\!/ \mathcal{G}(\tau) \cong \mathcal{X}_J/\mathcal{G}^c.$$

There is a holomorphic projection

$$\mathcal{M}_{\Sigma,d}(J,\tau) \to \mathrm{Jac}_{\Sigma,d}(J)$$

given by $[A,\Theta_0]^c \mapsto [A]^c$. This is an embedding whenever dim ker $\bar{\partial}_A \le 1$ for every $A \in \mathcal{A}(E)$.

Remark 4.4. The complex quotient $\mathcal{M}_{\Sigma,d}(J,\tau) \cong \mathcal{X}_J/\mathcal{G}^c$ is the set of effective divisors on Σ and can be identified with the symmetric product

$$\mathcal{M}_{\Sigma,d}(J,\tau) \cong S^d\Sigma = \frac{\Sigma \times \cdots \times \Sigma}{S_d}.$$

The projection $\mathcal{X}_J \to S^d\Sigma$ assigns to a pair (A,Θ_0) the set of zeros of Θ_0. Thus every complex structure $J \in \mathcal{J}(\Sigma)$ determines a smooth atlas on $S^d\Sigma$. For different choices of J the coordinate charts are not compatible but have only Lipschitz continuous transition maps.

5. The universal connection

The next theorem shows that the moduli spaces $\mathcal{M}_{\Sigma,d}(J,\tau)$ can be identified as symplectic manifolds, and that the symplectic structure depends only on the mean value of τ.

Theorem 5.1. *Let $[0,1] \to \mathcal{J}(\Sigma) \times C^\infty(\Sigma) : t \mapsto (J_t,\tau_t)$ be a smooth function such that $\int_\Sigma \dot\tau_t\omega = 0$ and choose $[0,1] \to \Omega^1(\Sigma) : t \mapsto \sigma_t$ such that $\dot\tau_t + *d\sigma_t = 0$. Then there is a symplectomorphism*

$$\psi = \psi_{\{J_t,\tau_t,\sigma_t\}} : \mathcal{M}(J_0,\tau_0) \to \mathcal{M}(J_1,\tau_1)$$

defined by $[A(0),\Theta_0(0)] \mapsto [A(1),\Theta_0(1)]$, where

$$i\dot A = \mathrm{Re}\,\langle\Theta_0,\Theta_1\rangle - \sigma, \qquad i\dot\Theta_0 = \bar{\partial}_{J,A}{}^*\Theta_1, \tag{7}$$

and $\Theta_1 = \Theta_1(t) \in \Omega^{0,1}_{J_t}(\Sigma,E)$ is the unique solution of the elliptic equation

$$\bar{\partial}_{J,A}\,\bar{\partial}_{J,A}{}^*\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 = \frac{1}{2}(\partial_{J,A}\Theta_0)\circ\dot J + \sigma^{0,1}\Theta_0. \tag{8}$$

If $J_0 = J_1$, $\tau_0 = \tau_1$, and $\int_0^1 \sigma_s\,ds = 0$ then ψ is Hamiltonian.

Choose $\sigma_t = *_t df_t$ where $f_t : \Sigma \to \mathbb{R}$ is the unique function of mean value zero which satisfies $\dot{\tau}_t = d^{*_t} df_t$. The resulting symplectomorphisms $\psi_{\{J_t,\tau_t\}} : \mathcal{M}(J_0, \tau_0) \to \mathcal{M}(J_1, \tau_1)$ determine a **universal Hamiltonian connection** on the fibre bundle over $\mathcal{J}(\Sigma) \times C^\infty_m(\Sigma)$ with fibres $\mathcal{M}(J, \tau)$. Here $C^\infty_m(\Sigma)$ denotes the space of functions with fixed mean value $m > 2\pi d$.

Remark 5.1. Suppose that $A(t)$, $\Theta_0(t)$, and $\Theta_1(t)$ satisfy

$$i(\dot{A} - d\Psi) = \mathrm{Re}\,\langle \Theta_0, \Theta_1 \rangle - \sigma, \qquad i(\dot{\Theta}_0 + \Psi\Theta_0) = \bar{\partial}_{J,A}{}^* \Theta_1, \tag{9}$$

and (8). Let $[0,1] \to \mathcal{G} : t \mapsto u(t)$ be a solution of the ordinary differential equation $u^{-1}\dot{u} + \Psi = 0$. Then the functions

$$\tilde{A} = A + u^{-1}du, \qquad \tilde{\Theta}_0 = u^{-1}\Theta_0, \qquad \tilde{\Theta}_1 = u^{-1}\Theta_1$$

satisfy (7) and (8).

Exercise 5.2. Suppose $J_t \equiv J$ and $\tau_t \equiv \tau$. Let $\psi_t : \mathcal{M}(J, \tau) \to \mathcal{M}(J, \tau)$ be defined by the solutions of (7) and (8). If $\sigma_t = dh_t$ prove that the ψ_t are generated by the Hamiltonian functions $H_t([A, \Theta_0]) = -\int_\Sigma ih_t F_A$. In general, prove that $\mathrm{Flux}(\{\psi_t\}) \in H^1(\mathcal{M}(J, \tau))$ is the cohomology class of the 1-form

$$T_{[A,\Theta_0]}\mathcal{M}(J, \tau) \to \mathbb{R} : (\alpha, \theta_0) \mapsto \int_\Sigma i\sigma \wedge \alpha, \qquad \sigma = \int_0^1 \sigma_s \, ds.$$

Prove that the flux is zero if and only if σ is exact.

To prove Theorem 5.1 it is useful to examine the spaces

$$\mathcal{X}_{J,\sigma} = \left\{ (A, \Theta_0) \in \mathcal{A}(E) \times C^\infty(X, E) \,|\, \bar{\partial}_{J,A+i\sigma}\Theta_0 = 0,\ \Theta_0 \not\equiv 0 \right\}$$

for $J \in \mathcal{J}(\Sigma)$ and $\sigma \in \Omega^1(\Sigma)$. Suitable Sobolev completions of these spaces are Banach manifolds.

Lemma 5.2. *For every $J \in \mathcal{J}(\Sigma)$ and every $\alpha \in \Omega^1(\Sigma)$ the space $\mathcal{X}_{J,\sigma}$ is a complex submanifold of $\mathcal{A}(E) \times C^\infty(X, E)$ with respect to the complex structure $(\alpha, \theta_0) \mapsto (*_J\alpha, i\theta_0)$.*

Proof. The tangent space of $\mathcal{X}_{J,\sigma}$ at the point (A, Θ_0) is the kernel of the operator $\mathcal{D}_{J,A+i\sigma,\Theta_0} : \Omega^1(\Sigma, i\mathbb{R}) \times C^\infty(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$ given by

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}(\alpha, \theta_0) = \bar{\partial}_{J,A+i\sigma}\theta_0 + \alpha^{0,1}\Theta_0.$$

The identity $(*_J\alpha)^{0,1} = i\alpha^{0,1}$ shows that this operator is complex linear. Its L^2-adjoint $\mathcal{D}_{J,A+i\sigma,\Theta_0}{}^* : \Omega^{0,1}(\Sigma, E) \to \Omega^1(\Sigma, i\mathbb{R}) \times C^\infty(\Sigma, E)$ is given by

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}{}^*\theta_1 = (i\mathrm{Im}\,\langle \Theta_0, \theta_1 \rangle,\ \bar{\partial}_{J,A+i\sigma}{}^*\theta_1).$$

Since $(i\mathrm{Im}\,\langle \Theta_0, \theta_1 \rangle)^{0,1} = \langle \Theta_0, \theta_1 \rangle / 2$ we obtain

$$\mathcal{D}_{J,A+i\sigma,\Theta_0}\mathcal{D}_{J,A+i\sigma,\Theta_0}{}^*\theta_1 = \bar{\partial}_{J,A+i\sigma}\,\bar{\partial}_{J,A+i\sigma}{}^*\theta_1 + \frac{1}{2}\left|\Theta_0\right|^2 \theta_1.$$

It follows from elliptic regularity that $\mathcal{D}_{J,A+i\sigma,\Theta_0}$ is surjective and hence $\mathcal{X}_{J,\sigma}$ is an infinite dimensional manifold. □

The required identification of the moduli spaces $\mathcal{M}(J,\tau)$ arises from a symplectic connection on the universal bundle

$$\mathcal{E} = \bigcup_{J,\sigma} \{(J,\sigma)\} \times \mathcal{X}_{J,\sigma} \longrightarrow \mathcal{J}(\Sigma) \times \Omega^1(\Sigma).$$

Think of $\mathcal{E}$ as a submanifold of the space $\mathcal{J}(\Sigma) \times \Omega^1(\Sigma) \times \mathcal{A}(E) \times C^\infty(\Sigma, E)$. The formula (5) defines a closed 2-form on $\mathcal{E}$ which restricts to the given symplectic form on each fibre. Hence it determines a symplectic connection on $\mathcal{E}$, where the horizontal subspace at (J,σ,A,Θ_0) is the Ω-complement of the vertical space $T_{(A,\Theta_0)}\mathcal{X}_J$. We call this the **universal symplectic connection on $\mathcal{E}$**. The next proposition gives an explicit formula for this connection.

Proposition 5.3. *A smooth path $[0,1] \to \mathcal{E} : t \mapsto (J(t),\sigma(t),B(t),\Theta_0(t))$ is horizontal with respect to the universal connection on $\mathcal{E}$ if and only if*

$$i\dot{A} = \mathrm{Re}\,\langle \Theta_0, \Theta_1\rangle, \qquad i\dot{\Theta}_0 = \bar{\partial}_{J,A+i\sigma}{}^*\Theta_1, \tag{10}$$

$$\bar{\partial}_{J,A+i\sigma}\,\bar{\partial}_{J,A+i\sigma}{}^*\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 = \frac{1}{2}(\partial_{J,A+i\sigma}\Theta_0)\circ \dot{J} + \dot{\sigma}^{0,1}\Theta_0. \tag{11}$$

Every horizontal path satisfies

$$\frac{d}{dt}\left(*iF_A + \frac{|\Theta_0|^2}{2}\right) = 0. \tag{12}$$

Proof. A path $t \mapsto (J(t),\sigma(t),A(t),\Theta_0(t))$ in $\mathcal{E}$ is horizontal with respect to the universal connection if and only if

$$(*_J\dot{A}, i\dot{\Theta}_0) \perp \ker \mathcal{D}_{J,A+i\sigma,\Theta_0}$$

for every t. By the proof of Lemma 5.2, this holds if and only if

$$(*_J\dot{A}, i\dot{\Theta}_0) \in \mathrm{im}\,\mathcal{D}_{J,A+i\sigma,\Theta_0}{}^*.$$

The formula for this operator in the proof of Lemma 5.2 shows that this means

$$*_J\dot{A} = i\mathrm{Im}\,\langle\Theta_0,\Theta_1\rangle, \qquad i\dot{\Theta}_0 = \bar{\partial}_{J,A+i\sigma}{}^*\Theta_1$$

for some $\Theta_1 \in \Omega^{0,1}(\Sigma,E)$. Since $*_J\mathrm{Im}\,\langle\Theta_0,\Theta_1\rangle = \mathrm{Im}\,\langle\Theta_0,i\Theta_1\rangle = \mathrm{Re}\,\langle\Theta_0,\Theta_1\rangle$, this is equivalent to (10). Since $(A,\Theta_0) \in \mathcal{X}_{J,\sigma}$ for every t we obtain

$$\begin{aligned}
0 &= \frac{d}{dt}\bar{\partial}_{J,A+i\sigma}\Theta_0 \\
&= \bar{\partial}_{J,A+i\sigma}\dot{\Theta}_0 + \dot{A}^{0,1}\Theta_0 + i\dot{\sigma}^{0,1}\Theta_0 + \frac{i}{2}(d_{A+i\sigma}\Theta_0)\circ\dot{J} \\
&= -i\bar{\partial}_{J,A+i\sigma}\,\bar{\partial}_{J,A+i\sigma}{}^*\Theta_1 - i\frac{|\Theta_0|^2}{2}\Theta_1 + \frac{i}{2}(\partial_{J,A+i\sigma}\Theta_0)\circ\dot{J} + i\dot{\sigma}^{0,1}\Theta_0
\end{aligned}$$

Hence Θ_1 is given by (11).

Conversely, suppose that the path $t \mapsto (J(t), \sigma(t), A(t), \Theta_0(t))$ satisfies (10) and (11) as well as $(A(0), \Theta_0(0)) \in \mathcal{X}_{J(0),\sigma(0)}$. Then the same argument as above shows that

$$\frac{d}{dt}\bar\partial_{J,A+i\sigma}\Theta_0 = \frac{i}{2}(\bar\partial_{J,A+i\sigma}\Theta_0) \circ \dot J$$

and hence $\bar\partial_{J,A+i\sigma}\Theta_0 = 0$ for all t. We prove directly that the path is horizontal. If $\bar\partial_{J,A+i\sigma}\theta_0 + \alpha^{0,1}\Theta_0 = 0$ then, since $*_J\dot A = i\mathrm{Re}\,\langle i\Theta_0, \Theta_1\rangle$,

$$\begin{aligned}
\Omega((\dot A, \dot\Theta_0), (\alpha, \theta_0)) &= \int_\Sigma \Big(\mathrm{Re}\,\langle *_J\dot A, \alpha\rangle + \mathrm{Re}\,\langle i\dot\Theta_0, \theta_0\rangle\Big)\,\omega \\
&= \int_\Sigma \big(\mathrm{Re}\,\langle i\mathrm{Re}\,\langle i\Theta_0, \Theta_1\rangle, \alpha\rangle + \mathrm{Re}\,\langle \bar\partial^*_{J,A+i\sigma}\Theta_1, \theta_0\rangle\big)\,\omega \\
&= \int_\Sigma \mathrm{Re}\,\langle \Theta_1, \bar\partial_{J,A+i\sigma}\theta_0 + \alpha^{0,1}\Theta_0\rangle\,\omega \\
&= 0.
\end{aligned}$$

To prove (12) note that $d^*\langle\Theta_0, \Theta_1\rangle = \langle\Theta_0, {\bar\partial_{J,A+i\sigma}}^*\Theta_1\rangle - \langle\bar\partial_{J,A+i\sigma}\Theta_0, \Theta_1\rangle$. Using $*id\dot A = d^* *_J i\dot A = d^*\mathrm{Re}\,\langle\Theta_0, i\Theta_1\rangle$ we obtain

$$\begin{aligned}
\frac{d}{dt}\left(*iF_A + \frac{|\Theta_0|^2}{2}\right) &= *id\dot A + \mathrm{Re}\,\langle\Theta_0, \dot\Theta_0\rangle \\
&= d^*\mathrm{Re}\,\langle\Theta_0, i\Theta_1\rangle - \mathrm{Re}\,\langle\Theta_0, i\,{\bar\partial_{J,A+i\sigma}}^*\Theta_1\rangle \\
&= -\mathrm{Re}\,\langle\bar\partial_{J,A+i\sigma}\Theta_0, i\Theta_1\rangle \\
&= 0.
\end{aligned}$$

This proves the proposition. □

Proof of Theorem 5.1. Define $A'(t) \in \mathcal{A}(E)$ and $\sigma'(t) \in \Omega^1(\Sigma, E)$ by

$$A'(t) = A(t) - i\sigma'(t), \qquad \sigma'(t) = \int_0^t \sigma_s\,ds.$$

Then the map $\mathcal{X}_{J(t)} \to \mathcal{X}_{J(t),\sigma'(t)} : (A(t), \Theta_0(t)) \mapsto (A'(t), \Theta_0(t))$ is a Kähler isomorphism. Now equations (7) and (8) show that

$$i\dot A' = i\dot A + \sigma = \mathrm{Re}\,\langle\Theta_0, \Theta_1\rangle, \qquad i\dot\Theta_0 = {\bar\partial_{J,A}}^*\Theta_1 = {\bar\partial_{J,A'+i\sigma'}}^*\Theta_1$$

and Θ_1 satisfies (11) with A and σ replaced by A' and σ'. Hence, by Proposition 5.3, the map $\mathcal{X}_{J(0),\sigma'(0)} \to \mathcal{X}_{J(1),\sigma'(1)} : (A'(0), \Theta_0(0)) \mapsto (A'(1), \Theta_0(1))$ defines a symplectomorphism which is Hamiltonian if the loop is closed (cf. McDuff–Salamon [19, Chapter 6]). Now use the identification of $\mathcal{X}_{J(t),\sigma'(t)}$ with $\mathcal{X}_{J(t)}$ to deduce that there is a well defined symplectomorphism

$$\mathcal{X}_{J(0)} \xrightarrow{\tilde\psi} \mathcal{X}_{J(1)} : (A(0), \Theta_0(0)) \mapsto (A(1), \Theta_0(1))$$

that is Hamiltonian whenever $J(0) = J(1)$ and $\sigma'(0) = \sigma'(1)$. Since

$$\frac{d}{dt}\left(*iF_{A'} + \frac{|\Theta_0|^2}{2}\right) = 0$$

we have

$$\frac{d}{dt}\left(\tau_t - *iF_A - \frac{|\Theta_0|^2}{2}\right) = \frac{d}{dt}(\tau_t + *id(A' - A)) = \frac{d}{dt}\tau_t + *d\sigma_t = 0,$$

and hence the symplectomorphism $\tilde{\psi}$ maps the solutions of (4) with $(J, \tau) = (J_0, \tau_0)$ to those with $(J, \tau) = (J_1, \tau_1)$. Let $\psi : \mathcal{M}(J_0, \tau_0) \to \mathcal{M}(J_1, \tau_1)$ denote the symplectomorphism induced by $\tilde{\psi}$. If $J(0) = J(1)$ and $\int_0^1 \sigma_s \, ds = 0$ then $\sigma'(0) = \sigma'(1) = 0$. In this case $\tilde{\psi}$ is a Hamiltonian symplectomorphism and hence, so is ψ. This proves the theorem. $\square$

6. Symmetric products

The rational cohomology of the symmetric product is well understood and can be computed in terms of symmetric differential forms on Σ^d. For $j \le d$ one obtains

$$H^j(S^d\Sigma) \cong \Lambda^j \oplus \Lambda^{j-2} \oplus \cdots,$$

where $\Lambda^j = \Lambda^j H^1(\Sigma)$. Hence

$$\chi(S^d\Sigma) = \sum_{j=0}^{d} (-1)^j (d + 1 - j) \binom{2g}{j} = (-1)^d \binom{2g-2}{d}.$$

This description of the cohomology is functorial with respect to the action of the mapping class group of Σ. Hence

$$L(S^d f) = \sum_{j=0}^{d} (-1)^j (d + 1 - j) \mathrm{trace}(\Lambda^j f^*)$$

where $S^d f$ denotes the induced map on $S^d\Sigma$ and f^* denotes the induced endomorphism of $H^1(\Sigma)$.

For $d = \deg(E) > 2g - 2$ the Riemann–Roch theorem asserts that the space of holomorphic 1-forms with values in any holomorphic line bundle E of degree d is zero. Hence the space $H^0(\Sigma, E)$ of holomorphic sections has complex dimension $d + 1 - g$. It follows that $S^d\Sigma$ is a fiber bundle over the Jacobian with fiber $\mathbb{P}H^0(\Sigma, E) \cong \mathbb{C}P^{d-g}$:

$$\mathbb{C}P^{d-g} \hookrightarrow S^d\Sigma \longrightarrow \mathrm{Jac}_{\Sigma,d}.$$

In particular, this shows that the first Chern class $c_1 = c_1(TS^d\Sigma)$ evaluates on the positive generator $A \in \pi_2(S^d\Sigma)$ by

$$c_1(A) = d + 1 - g$$

whenever $d \ge 2g - 1$. (This continues to hold for all $d \ge 2$.)

Proposition 6.1. *The space*

$$\tilde{\mathcal{M}}_{\Sigma,d} = \tilde{\mathcal{M}}_{\Sigma,d}(J,\tau) = \{(A,\Theta_0) \in \mathcal{A}(E) \times C^\infty(\Sigma,E) \,|\, (4)\}$$

is connected. If $d \geq 2$ then $\tilde{\mathcal{M}}_{\Sigma,d}$ is simply connected and

$$\pi_1(\mathcal{M}_{\Sigma,d}) = \pi_0(\mathcal{G}) = \mathbb{Z}^{2g}.$$

If $d = 1$ then $\mathcal{M}_{\Sigma,1} \cong \Sigma$ and $\pi_1(\tilde{\mathcal{M}}_{\Sigma,1}/S^1)$ is the Torelli group.

Proof. We prove that $\tilde{\mathcal{M}}_{\Sigma,d}$ is connected. To see this note that there is a fibration

$$\mathcal{G} \hookrightarrow \tilde{\mathcal{M}}_{\Sigma,d} \to \mathcal{M}_{\Sigma,d}. \tag{13}$$

Fix a point $(A,\Theta_0) \in \tilde{\mathcal{M}}_{\Sigma,d}$ such that Θ_0 has d distinct zeros. Since $\mathcal{M}_{\Sigma,d}$ is connected it suffices to prove that, for every $u \in \mathcal{G}$, the points (A,Θ_0) and $(u^*A, u^{-1}\Theta_0)$ can be connected by a path in $\tilde{\mathcal{M}}_{\Sigma,d}$. Moreover, it suffices to consider one gauge transformation from each of $2g$ components that generate $\pi_0(\mathcal{G})$. Choose a circle $C \subset \Sigma$ that contains precisely one zero of Θ_0 and choose a gauge transformation $u : \Sigma \to S^1$ such that $u = 1$ in the complement of a small neighbourhood of C and

$$\left[\frac{u^{-1}du}{2\pi i}\right] = \mathrm{PD}([C]).$$

Then the required path from (A,Θ_0) to $(u^*A, u^{-1}\Theta_0)$ can be obtained by sliding the zero of Θ_0 once around C. This shows that $\tilde{\mathcal{M}}_{\Sigma,d}$ is connected.

We prove that, for $d \geq 2$,

$$\pi_1(S^d\Sigma) \cong H_1(\Sigma;\mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

(This is well known and the first identity extends to symmetric products of any compact manifold. We include a proof for the sake of completeness.) Fix a base point $c \in \Sigma$ and note that every loop in $S^d\Sigma$ has the form $[\gamma_1,\ldots,\gamma_d] : S^1 \to S^d\Sigma$ for d based loops $\gamma_i : S^1 \to \Sigma$. Moreover,

$$[\gamma_1,\ldots,\gamma_d] \sim [c,\ldots,c,\gamma_1\cdots\gamma_d].$$

Since the ordering of the γ_i is immaterial it follows that $\pi_1(S^d\Sigma)$ is abelian. If $\gamma : S^1 \to \Sigma$ is not homologous to zero then there is a cohomology class $\alpha \in H^1(\Sigma;\mathbb{Z})$ such that $\langle\alpha,[\gamma]\rangle = 1$. This gives rise to a cohomology class on $S^d\Sigma$ which pairs nontrivially with $[c,\ldots,c,\gamma]$. Hence $\pi_1(S^d\Sigma) = H_1(\Sigma;\mathbb{Z})$.

We prove that, for $d \geq 2$, there exists a pair $(J,A) \in \mathcal{J}(\Sigma) \times \mathcal{A}(E)$ such that

$$\dim^c \ker \bar{\partial}_{J,A} \geq 2.$$

(This is also well known.) Think of $\mathbb{C}P^1$ as the space of complex lines in $\mathbb{C}^2$ and denote by $H \to \mathbb{C}P^1$ the tautological bundle whose fibre over a line $\ell \in \mathbb{C}P^1$ is the dual space $\ell^* = \mathrm{Hom}(\ell,\mathbb{C})$. Then a holomorphic section of H has the form $s(\ell) = \phi|_\ell$ where $\phi \in \mathrm{Hom}(\mathbb{C}^2,\mathbb{C})$. This space has evidently dimension 2. Now choose a branched covering $u : \Sigma \to \mathbb{C}P^1$ of degree $d \geq 2$. Then the pullback bundle $E = u^*H \to \Sigma$ has degree d. Choose $A \in \mathcal{A}(E)$ to be the pullback of the tautological connection on H and $J \in \mathcal{J}(\Sigma)$

to be the pullback of the standard complex structure on $\mathbb{C}P^1$. Then the kernel of $\bar{\partial}_{J,A}$ has dimension at least 2.

Suppose that $d \geq 2$. We prove that $\tilde{\mathcal{M}}_{J,d}$ is simply connected for every J and every τ. By Theorem 5.1 it suffices to prove this for some J. Consider the homotopy exact sequence of the fibration (13). It has the form

$$\pi_1(\mathcal{G}) \to \pi_1(\tilde{\mathcal{M}}_{\Sigma,d}) \to \pi_1(\mathcal{M}_{\Sigma,d}) \to \pi_0(\mathcal{G}) \to 0. \tag{14}$$

We have proved that $\pi_1(\mathcal{M}_{\Sigma,d}) \cong \mathbb{Z}^{2g}$ whenever $d \geq 2$. Since $\pi_0(\mathcal{G}) \cong \mathbb{Z}^{2g}$ and the homomorphism $\pi_1(\mathcal{M}_{\Sigma,d}) \to \pi_0(\mathcal{G})$ is surjective it follows that this homomorphism is injective. Hence the homomorphism $\pi_1(\tilde{\mathcal{M}}_{\Sigma,d}) \to \pi_1(\mathcal{M}_{\Sigma,d})$ is zero. Now $\pi_1(\mathcal{G}) = \mathbb{Z}$ and the image of the homomorphism $\pi_1(\mathcal{G}) \to \pi_1(\tilde{\mathcal{M}}_{\Sigma,d})$ is generated by the loop

$$S^1 \to \tilde{\mathcal{M}}_{\Sigma,d}(J,\tau) : e^{it} \mapsto (A, e^{it}\Theta_0).$$

We have proved that, for $d \geq 2$, there exists a complex structure $J \in \mathcal{J}(\Sigma)$ and a connection $A \in \mathcal{A}(E)$ such that $\dim^c \ker \bar{\partial}_{J,A} \geq 2$. For this choice the aforementioned loop is obviously contractible. Hence the homomorphism $\pi_1(\mathcal{G}) \to \pi_1(\tilde{\mathcal{M}}_{\Sigma,d})$ is zero for some J and, by Theorem 5.1 it is zero for every J. Hence the exact sequence (14) shows that $\tilde{\mathcal{M}}_{\Sigma,d}$ is simply connected. □

7. Symplectic fixed points

Theorem 5.1 shows how to construct a homomorphism of symplectic mapping class groups

$$\mathrm{Diff}(\Sigma,\omega)/\mathrm{Ham}(\Sigma,\omega) \longrightarrow \mathrm{Diff}(\mathcal{M}(J,\tau),\Omega)/\mathrm{Ham}(\mathcal{M}(J,\tau),\Omega).$$

Here $\mathrm{Diff}(\Sigma,\omega)$ denotes the group of orientation and area preserving diffeomorphisms of Σ and $\mathrm{Ham}(\Sigma,\omega)$ denotes the subgroup of Hamiltonian symplectomorphisms. Let $f \in \mathrm{Diff}(\Sigma,\omega)$ and choose a lift $\tilde{f}$ of f to a unitary automorphism of E. Any two such lifts $\tilde{f}, \tilde{f}' : E \to E$ are related by

$$\tilde{f}' = m(u) \circ \tilde{f} = \tilde{f} \circ m(u \circ f)$$

for some $u \in \mathcal{G}$, where $m(u) : E \to E$ denotes the obvious action of u. Let $\mathbb{R} \to \mathcal{J}(\Sigma) : t \mapsto J_t$ be a smooth family of complex structures such that

$$J_{t+1} = f^* J_t.$$

Denote by $\psi_t : \mathcal{M}(J_0,\tau) \to \mathcal{M}(J_t,\tau)$ the symplectomorphisms induced by the solutions of (7) and (8) with $\tau_t = \tau$ and $\sigma_t = 0$. Then the symplectomorphism

$$\phi_{d,f} = \phi_{d,f,\{J_t\}} := {\psi_1}^{-1} \circ \tilde{f}^* : \mathcal{M}(J_0,\tau) \to \mathcal{M}(J_0,\tau)$$

is independent of the choice of the lift $\tilde{f}$ and, by Theorem 5.1, its Hamiltonian isotopy class is independent of the path $\{J_t\}$.

We examine the components of the path space $\Omega_{\phi_{d,f}}$. Denote by $\tilde{\mathcal{P}}_{d,\tilde{f}}$ the space of all smooth paths $\mathbb{R} \to \mathcal{A}(E) \times C^\infty(\Sigma, E) : t \mapsto (A(t), \Theta_0(t))$ that satisfy $[A(t), \Theta_0(t)] \in \mathcal{M}(J_t, \tau)$ and the periodicity condition

$$A(t+1) = \tilde{f}^* A(t), \qquad \Theta_0(t+1) = \tilde{f}^* \Theta_0(t).$$

The group $\mathcal{G}_f$ of gauge transformations $\mathbb{R} \to \mathcal{G} : t \mapsto u(t)$ that satisfy

$$u(t+t) = u(t) \circ f$$

acts on this space and the quotient will be denoted by

$$\mathcal{P}_{d,\tilde{f}} = \tilde{\mathcal{P}}_{d,\tilde{f}} / \mathcal{G}_f.$$

This space can be naturally identified with a subset of $\Omega_{\phi_{d,f}}$ via the map that assigns to every path $t \mapsto [A(t), \Theta_0(t)]$ in $\mathcal{P}_{d,\tilde{f}}$ the path $\gamma : \mathbb{R} \to \mathcal{M}(J_0, \tau)$ given by $\gamma(t) = {\psi_t}^{-1}([A(t), \Theta_0(t)])$. Evidently the set $\Omega_{\phi_{d,f}}$ is the union of the sets $\mathcal{P}_{d,\tilde{f}}$ over all unitary lifts of f. The next lemma shows that each set $\mathcal{P}_{d,\tilde{f}}$ is a component of $\Omega_{\phi_{d,f}}$ and that

$$\pi_0(\Omega_{\phi_{d,f}}) \cong \frac{H^1(\Sigma; \mathbb{Z})}{\mathrm{im}(\mathbb{1} - f^*)}.$$

This identification is not canonical.

Lemma 7.1. *Suppose that $d \geq 2$. Then, for every unitary lift $\tilde{f} : E \to E$ of f, the space $\mathcal{P}_{d,\tilde{f}}$ is a connected component of $\Omega_{\phi_{d,f}}$. Two such lifts $\tilde{f}$ and $\tilde{f}'$ determine the same component if and only if there exists a $u \in \mathcal{G}$ such that $\tilde{f}' = \tilde{f} \circ m(u)$ and*

$$\left[\frac{u^{-1}du}{2\pi i}\right] \in \mathrm{im}(\mathbb{1} - f^*) \subset H^1(\Sigma; \mathbb{Z}). \tag{15}$$

Proof. By Proposition 6.1, the space of all solutions of the vortex equations (4) is simply connected. Hence $\tilde{\mathcal{P}}_{d,\tilde{f}}$ is connected and hence, so is $\mathcal{P}_{d,\tilde{f}}$. Now let $\tilde{f}$ and $\tilde{f}'$ be two unitary lifts of f. Then the following are equivalent.

(i): $\mathcal{P}_{d,\tilde{f}} = \mathcal{P}_{d,\tilde{f}'}$.
(ii): $\mathcal{P}_{d,\tilde{f}} \cap \mathcal{P}_{d,\tilde{f}'} \neq \emptyset$.
(iii): There exists a $u \in \mathcal{G}$ that satisfies $\tilde{f}' = \tilde{f} \circ m(u)$ and (15)

We prove that (iii) implies (i). Suppose that $u : \Sigma \to S^1$ satisfies (15) and choose a closed 1-form $\sigma \in \Omega^1(\Sigma)$ with integer periods such that the 1-form $u^{-1}du/2\pi i - \sigma + f^*\sigma$ is exact. Choose $v : \Sigma \to S^1$ such that $v^{-1}dv/2\pi i = \sigma$. Then $(v \circ f)u : \Sigma \to S^1$ is homotopic to v. Hence there exists a path $\mathbb{R} \to \mathcal{G} : t \mapsto v(t)$ such that $v(0) = v$ and

$$v(t+1) = (v(t) \circ f)u.$$

Let $t \mapsto (A(t), \Theta_0(t))$ be a path in $\tilde{\mathcal{P}}_{d,\tilde{f}}$ and denote

$$A'(t) = v(t)^* A(t), \qquad \Theta_0'(t) = v(t)^{-1}\Theta_0(t), \qquad \tilde{f}' = \tilde{f} \circ m(u).$$

Then

$$\begin{array}{rcl} A'(t+1) & = & v(t+1)^*A(t+1) \\ & = & v(t+1)^*\tilde{f}^*A(t) \\ & = & u^*(v(t)\circ f)^*\tilde{f}^*A(t) \\ & = & u^*\tilde{f}^*v(t)^*A(t) \\ & = & \tilde{f}'^*A'(t). \end{array}$$

A similar identity holds with $A(t)$ replaced by $\Theta_0(t)$. This shows that the path $t \mapsto (A'(t), \Theta_0'(t))$ lies in $\tilde{\mathcal{P}}_{d,\tilde{f}'}$. Thus we have proved that there is a bijection

$$\tilde{\mathcal{P}}_{d,\tilde{f}} \to \tilde{\mathcal{P}}_{d,\tilde{f}'} : \{A(t), \Theta_0(t)\}_t \mapsto \{v(t)^*A(t), v(t)^{-1}\Theta_0(t)\}_t.$$

This proves (i). That (i) implies (ii) is obvious since $\mathcal{P}_{d,\tilde{f}} \neq \emptyset$. That (ii) implies (iii) follows by reversing the arguments in the proof that (iii) implies (i). This step is left as an exercise to the reader. □

A fixed point of $\phi_{d,f}$ in the class $\mathcal{P}_{d,\tilde{f}}$ can be represented by a path

$$\begin{array}{rcl} \mathbb{R} & \to & \mathcal{A}(E) \times C^\infty(\Sigma, i\mathbb{R}) \times C^\infty(\Sigma, E) \times \Omega^{0,1}(\Sigma, E) \\ t & \mapsto & (A(t), \Psi(t), \Theta_0(t), \Theta_1(t)) \end{array}$$

that satisfies the equations

$$\bar\partial_{J_t,A}\Theta_0 = 0, \qquad *iF_A + \frac{|\Theta_0|^2}{2} = \tau, \tag{16}$$

$$*_t(\dot{A} - d\Psi) = i\mathrm{Im}\,\langle\Theta_0, \Theta_1\rangle, \qquad i(\dot{\Theta}_0 + \Psi\Theta_0) = \bar\partial_{J,A}{}^*\Theta_1, \tag{17}$$

$$\bar\partial_{J_t,A}\,\bar\partial_{J_t,A}{}^*\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 = \frac{1}{2}(\partial_{J_t,A}\Theta_0)\circ\dot{J}_t, \tag{18}$$

and the periodicity condition

$$\begin{array}{ll} A(t+1) = \tilde{f}^*A(t), & \Psi(t+1) = \Psi(t)\circ f, \\ \Theta_0(t+1) = \tilde{f}^*\Theta_0(t), & \Theta_1(t+1) = \tilde{f}^*\Theta_1(t). \end{array} \tag{19}$$

Here (16) asserts that $[A(t), \Theta_0(t)] \in \mathcal{M}(J_t, \tau)$ for every t, (17) and (18) assert that the path $t \mapsto [A(t), \Theta_0(t)]$ is horizontal with respect to the universal connection, and (19) asserts that the path $t \mapsto [A(t), \Theta_0(t)]$ belongs to $\mathcal{P}_{d,\tilde{f}}$. Two such paths represent the same fixed point if and only if they are related by

$$(A, \Psi, \Theta_0, \Theta_1) \mapsto (B + u^{-1}du, \Psi + u^{-1}\dot{u}, u^{-1}\Theta_0, u^{-1}\Theta_1)$$

for some $u \in \mathcal{G}_f$.

8. Mapping tori

We examine the Seiberg-Witten equations on a mapping torus. As before, let Σ be a compact oriented smooth 2-manifold of genus g equipped with a volume form ω. Let $f \in \mathrm{Diff}(\Sigma, \omega)$ and denote by

$$Y_f = \mathbb{R} \times \Sigma / \sim$$

the mapping torus. The equivalence relation is given by

$$(t+1, z) \sim (t, f(z)).$$

Choose a smooth function $\mathbb{R} \to \mathcal{J}(\Sigma)$ such that $J_{t+1} = f^* J_t$ and denote by

$$\langle \cdot, \cdot \rangle_t = \omega(\cdot, J_t \cdot) + i\omega(\cdot, \cdot)$$

the Hermitian form on $T\Sigma$ induced by J_t and ω. Such a family of complex structures determines a metric on Y_f and a spin^c structure.

The canonical spin^c structure

The canonical spin^c structure on Y_f, determined by the family $\{J_t\}$ of almost complex structures, will be denoted by $\gamma_f : TY_f \to \mathrm{End}(W_f)$. The Hermitian rank-2 bundle $W_f \to Y_f$ is given by

$$W_f = \Big\{ (t, z, \Theta_0, \Theta_1) \,|\, t \in \mathbb{R},\, z \in \Sigma,\, \Theta_0 \in \mathbb{C},\, \Theta_1 \in \Lambda^{0,1}_{J_t} T_z^* \Sigma \Big\} \Big/ \sim .$$

The equivalence relation is

$$(t+1, z, \Theta_0, \Theta_1) \sim (t, f(z), \Theta_0, \Theta_1 \circ df(z)^{-1})$$

and γ_f has the form

$$\gamma_f(t, z; \tau, \zeta) \begin{pmatrix} \Theta_0 \\ \Theta_1 \end{pmatrix} = \begin{pmatrix} -i\tau\Theta_0 - \sqrt{2}\Theta_1(\zeta) \\ i\tau\Theta_1 + \langle \cdot, \zeta \rangle_t \Theta_0 / \sqrt{2} \end{pmatrix}$$

for $t, \tau \in \mathbb{R}$ and $\zeta \in T_z\Sigma$. This structure is isomorphic to γ_v in Example 3.1 for the vector field $v = \partial/\partial t$. To see this identify $T\Sigma$ with the bundle $\Lambda^{0,1}T^*\Sigma$ via $\theta_1 \mapsto \Theta_1 = -\langle \cdot, \theta_1 \rangle / \sqrt{2}$.

Lemma 8.1. *Let* $\eta = \eta_2 - \eta_1 \wedge dt \in \Omega^2(Y_f, i\mathbb{R})$, *i.e.* $\eta_2(t) \in \Omega^2(\Sigma, i\mathbb{R})$ *and* $\eta_1(t) \in \Omega^1(\Sigma, i\mathbb{R})$ *satisfy* $\eta_i(t+1) = f^*\eta_i(t)$. *Then*

$$\gamma_f(*_3(\eta_2 - \eta_1 \wedge dt)) = (\Theta\Theta^*)_0$$

if and only if

$$*i\eta_2 + \frac{|\Theta_0|^2 - |\Theta_1|^2}{2} = 0, \qquad *\eta_1 - i\sqrt{2}\,\mathrm{Im}\,\langle \Theta_0, \Theta_1 \rangle = 0.$$

Proof. The Hodge $*$-operator on 2-forms on Y_f is given by

$$*_3(\eta_2 - \eta_1 \wedge dt) = *_2\eta_2 dt + *_2\eta_1,$$

where $*_2$ denotes the Hodge $*$-operator on Σ. Let $v : \Sigma \to T\Sigma$ be the vector field dual to $\mathrm{Im}\,\eta_1$. Then Jv is dual to $*\mathrm{Im}\,\eta_1 = -\mathrm{Im}\,\eta_1 \circ J$ and

$$\theta_1(Jv) = \langle \eta_1^{0,1}, \theta_1 \rangle, \qquad \langle \cdot, Jv \rangle = 2\eta_1^{0,1}$$

Hence

$$\begin{aligned}
\gamma_f(*_3(\eta_2 - \eta_1 \wedge dt)) \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} &= \gamma_f((*_2\eta_2)dt + *_2\eta_1) \begin{pmatrix} \theta_0 \\ \theta_1 \end{pmatrix} \\
&= \begin{pmatrix} -i(*_2\eta_2)\theta_0 - i\sqrt{2}\theta_1(Jv) \\ i(*_2\eta_2)\theta_1 + i\langle \cdot, Jv \rangle \theta_0/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} -(*_2 i\eta_2)\theta_0 - i\sqrt{2}\langle \eta_1^{0,1}, \theta_1 \rangle \\ (*_2 i\eta_2)\theta_1 + i\sqrt{2}\eta_1^{0,1}\theta_0 \end{pmatrix}.
\end{aligned}$$

Compare this with the formula

$$(\Theta\Theta^*)_0\theta = \begin{pmatrix} \lambda\theta_0 + \langle \Theta_1, \theta_1 \rangle \Theta_0 \\ -\lambda\theta_1 + \langle \Theta_0, \theta_0 \rangle \Theta_1 \end{pmatrix}, \qquad \lambda = \frac{|\Theta_0|^2 - |\Theta_1|^2}{2}$$

to obtain $*i\eta_2 + \lambda = 0$ and

$$\langle \Theta_0, \Theta_1 \rangle = i\sqrt{2}\eta_1^{0,1} = i\eta_1/\sqrt{2} - \eta_1 \circ J/\sqrt{2} = i\eta_1/\sqrt{2} + *\eta_1/\sqrt{2}$$

Since η_1 is an imaginary valued 1-form, this is equivalent to $i\mathrm{Im}\,\langle \Theta_0, \Theta_1 \rangle = *\eta_1/\sqrt{2}$. This proves the lemma. □

The canonical spin^c connection

Computation in local coordinates shows that the vertical tangent bundle of the fibration $Y_f \to S^1$ is invariant under the Levi-Civita connection. The direct sum of this bundle with $\mathbb{C}$ is isomorphic to W_f and this gives rise to a spin^c-connection $\nabla = \nabla_f$ on W_f. In explicit terms ∇_f agrees with the Levi-Civita connection of the metric $\omega(\cdot, J_t\cdot)$ over each slice $\{t\} \times \Sigma$ and the covariant derivative in the direction $\partial/\partial t$ is given by

$$\nabla_t\Theta_1 = \dot{\Theta}_1 + \frac{1}{2}\Theta_1 \circ J\dot{J}.$$

If Θ_1 is of type $(0,1)$ then so is $\nabla_t\Theta_1$. Let A_f denote the Hermitian connection on $\det(W_f)^{1/2}$ induced by ∇_f. The curvature of A_f is the 2-form

$$F_{A_f} = -\frac{iK_t}{2}\omega - \frac{\alpha_t}{2} \wedge dt,$$

where $K_t : \Sigma \to \mathbb{R}$ denotes the Gauss curvature of the metric $\omega(\cdot, J_t\cdot)$ and $\alpha_t \in \Omega^1(\Sigma, i\mathbb{R})$ is defined by

$$(\mathrm{Im}\,\alpha_t)J = \dot{\nabla} + \frac{1}{2}J\nabla\dot{J}.$$

The Seiberg-Witten equations

Let $E \to \Sigma$ be a Hermitian line bundle and choose a lift $\tilde{f} : E \to E$ of f to a unitary automorphism of E:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ \Sigma \arrow[r, "f"] & \Sigma \end{tikzcd} .$$

Such a lift determines a Hermitian line bundle $E_{\tilde{f}} = \mathbb{R} \times E_f / \sim$ over Y_f where

$$(t+1, z, \theta_0) \sim (t, f(z), \tilde{f}(z)\theta_0).$$

A connection on $E_{\tilde{f}}$ has the form $A(t) + \Psi(t)\,dt$ where $A(t) \in \mathcal{A}(E)$ and $\Psi(t) \in \Omega^0(\Sigma, i\mathbb{R})$ satisfy (19). The curvature of this connection is given by

$$F_{A+\Psi\,dt} = F_A - (\dot{A} - d\Psi) \wedge dt.$$

Now consider the twisted spin^c structure

$$\gamma_{d,\tilde{f}} : TY_f \to \mathrm{End}(W_{d,\tilde{f}}), \qquad W_{d,\tilde{f}} = W_f \otimes E_{\tilde{f}}.$$

The Dirac operator on the Riemann surface with the standard spin^c structure is equal to the Cauchy-Riemann operator determined by J and multiplied by a factor $\sqrt{2}$ (cf. [26, Theorem 6.17]). Abbreviate

$$\nabla_t\Theta_0 = \dot{\Theta}_0 + \Psi\Theta_0, \qquad \nabla_t\Theta_1 = \dot{\Theta}_1 + \Psi\Theta_1 + \frac{1}{2}\Theta_1 \circ J\dot{J}$$

For $\Theta_0 = \Theta_0(t) \in C^\infty(\Sigma, E)$ and $\Theta_1 = \Theta_1(t) \in \Omega^{0,1}(\Sigma, E)$. Then the Dirac equations for the twisted spin^c structure have the form

$$-i\nabla_t\Theta_0 + \sqrt{2}\,\bar{\partial}_{J,A}{}^*\Theta_1 = 0, \qquad i\nabla_t\Theta_1 + \sqrt{2}\bar{\partial}_{J,A}\Theta_0 = 0. \tag{20}$$

By Lemma 8.1, the second equation in (2) decomposes as

$$*i(F_A + \eta_2) + \frac{K_t}{2} + \frac{|\Theta_0|^2 - |\Theta_1|^2}{2} = 0, \tag{21}$$

$$*_t\left(\dot{A} - d\Psi + \frac{\alpha_t}{2} + \eta_1\right) = i\sqrt{2}\mathrm{Im}\,\langle\Theta_0, \Theta_1\rangle. \tag{22}$$

Here $\eta = \eta_2 - \eta_1 \wedge dt \in \Omega^2(Y_f, i\mathbb{R})$ is the perturbation. Together with the periodicity conditions (19) these are the Seiberg-Witten equations on Y_f for the spin^c structure $\gamma_{d,\tilde{f}}$. The goal is now to relate the solutions of these equations to those of (16), (17), (18), and (19) which correspond to the fixed points of $\phi_{d,f}$ in the class $\mathcal{P}_{d,\tilde{f}}$.

As a first step we choose a perturbation

$$\eta = \eta_2 - \eta_1 \wedge dt, \qquad \eta_2 = i\left(\frac{\tau}{2} + \frac{K_t}{2}\right)\omega, \qquad \eta_1 = -\frac{\alpha_t}{2}.$$

If τ is independent of t then this form is closed. Next we would like to get rid of the various factors $\sqrt{2}$. For this it is convenient to rename Θ_0 and the metric on Σ by:

$$\Theta_0{}^{\mathrm{new}} = \sqrt{2}\Theta_0{}^{\mathrm{old}}, \qquad \omega^{\mathrm{new}} = \frac{1}{2}\omega^{\mathrm{old}}, \qquad K_t{}^{\mathrm{new}} = 2K_t{}^{\mathrm{old}}.$$

Then the Hodge $*$-operator on 1-forms (on Σ) remains unchanged, the Hodge $*$-operators on 2-forms are related by $*^{\mathrm{new}} = 2*^{\mathrm{old}}$, and the norm of a 1-form in the new metric is by a factor $\sqrt{2}$ bigger. Moreover, the product $K_t\omega$ and the 1-form α_t are invariant under this scaling. All this is just change in notation and the Seiberg-Witten equations now have the following form.

$$i\nabla_t\Theta_0 = \bar{\partial}_{J,A}{}^*\Theta_1, \qquad -i\nabla_t\Theta_1 = \bar{\partial}_{J,A}\Theta_0, \tag{23}$$

$$*iF_A + \frac{|\Theta_0|^2 - |\Theta_1|^2}{2} = \tau, \tag{24}$$

$$*_t\left(\dot{A} - d\Psi\right) = i\mathrm{Im}\,\langle\Theta_0, \Theta_1\rangle. \tag{25}$$

The comparison between (23), (24), (25) and (16), (17), (18) involves an adiabatic limit argument.

The Chern-Simons-Dirac functional

Fix a path of connections $A_0(t) \in \mathcal{A}(E)$ such that $A_0(t+1) = \tilde{f}^*A_0(t)$. Consider the Chern-Simons-Dirac functional on Y_f with the spin^c structure $\gamma_{d,\tilde{f}}$, the basepoint $A_f + A_0$, the perturbation $\eta = i\tau\omega/2 - F_{A_f}$, and the above renaming of ω and Θ_0. This functional has the form

$$\begin{aligned}
\mathcal{CSD}_\tau(A, \Psi, \Theta) \quad &= \quad \frac{1}{2}\int_0^1\int_\Sigma (A - A_0)\wedge(\dot{A} + \dot{A}_0)\,dt \\
&\quad - \int_0^1\int_\Sigma\Big(\Psi(F_A + i\tau\omega) + \mathrm{Re}\,\langle\Theta_1, \bar{\partial}_{J_t,A}\Theta_0\rangle\omega\Big)\,dt \\
&\quad + \frac{1}{2}\int_0^1\int_\Sigma\Big(\mathrm{Re}\,\langle i\nabla_t\Theta_0, \Theta_0\rangle - \mathrm{Re}\,\langle i\nabla_t\Theta_1, \Theta_1\rangle\Big)\omega\,dt
\end{aligned}$$

If $\Theta_1 = 0$ and $(A(t), \Theta_0(t)) \in \tilde{\mathcal{M}}(J_t, \tau)$ then

$$\mathcal{CSD}_\tau(A, \Psi, \Theta) = \frac{1}{2}\int_0^1\int_\Sigma\Big((A - A_0)\wedge(\dot{A} + \dot{A}_0) + \mathrm{Re}\,\langle i\dot{\Theta}_0, \Theta_0\rangle\omega\Big)\,dt.$$

This is the symplectic action of the path $t \mapsto [A(t), \Theta_0(t)]$.

9. Adiabatic limits

The main idea is to change the parameters in the equations (23), (24), and (25). We multiply the metric on Σ by a small constant ε^2 and simultaneously divide τ by the the same constant:

$$\omega_\varepsilon = \varepsilon^2\omega, \qquad \tau_\varepsilon = \varepsilon^{-2}\tau.$$

This does not affect the product $\tau\omega$ and hence the original perturbation η remains unchanged. The new equations have the form

$$i\nabla_t\Theta_0 = \varepsilon^{-2}\,\bar\partial_{J,A}{}^*\Theta_1, \qquad -i\nabla_t\Theta_1 = \bar\partial_{J,A}\Theta_0, \tag{26}$$

$$\varepsilon^{-2} * iF_A + \frac{|\Theta_0|^2 - \varepsilon^{-2}|\Theta_1|^2}{2} = \varepsilon^{-2}\tau, \tag{27}$$

$$*_t\left(\dot A - d\Psi\right) = i\mathrm{Im}\,\langle\Theta_0,\Theta_1\rangle. \tag{28}$$

Here the Hodge $*$-operators are to be understood with respect to the old metric and the dependence of ε is made explicit. Now it is convenient to rename the variables Θ_0 and Θ_1 by

$$\Theta_0{}^{\mathrm{new}} = \varepsilon\Theta_0{}^{\mathrm{old}}, \qquad \Theta_1{}^{\mathrm{new}} = \varepsilon^{-1}\Theta_1{}^{\mathrm{old}}.$$

Then the Seiberg-Witten equations (26), (27), and (28) translate into the form

$$i\nabla_t\Theta_0 = \bar\partial_{J_t,A}{}^*\Theta_1, \qquad -i\nabla_t\Theta_1 = \varepsilon^{-2}\bar\partial_{J_t,A}\Theta_0, \tag{29}$$

$$\varepsilon^{-2}\left(*iF_A + \frac{|\Theta_0|^2}{2} - \tau\right) = \frac{|\Theta_1|^2}{2}, \tag{30}$$

$$*_t\left(\dot A - d\Psi\right) = i\mathrm{Im}\,\langle\Theta_0,\Theta_1\rangle. \tag{31}$$

This already looks promising. The first equation in (29) and (31) are reminiscent of the equations for parallel transport in (17) and the other two equations give the vortex equations in the limit $\varepsilon \to 0$. The crucial point is to control the behaviour of Θ_1 and its derivatives in the small ε limit. The first step in this direction is the following observation, which relates the section Θ_1 in the Seiberg-Witten equations to the variable Θ_1 in (18).

Lemma 9.1. *Every solution of (29), (30), and (31) satisfies*

$$\bar\partial_{J_t,A}\,\bar\partial_{J_t,A}{}^*\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 - \frac{1}{2}(\partial_{J_t,A}\Theta_0)\circ\dot J_t = \varepsilon^2\nabla_t\nabla_t\Theta_1. \tag{32}$$

Proof. First recall that

$$\nabla_t\bar\partial_{J_t,A}\Theta_0 = \frac{d}{dt}(\bar\partial_{J_t,A}\Theta_0) + \Psi\bar\partial_{J_t,A}\Theta_0 + \frac{1}{2}(\bar\partial_{J_t,A}\Theta_0)\circ J\dot J.$$

Since

$$id_A\Theta_0 = (\partial_{J,A}\Theta_0)\circ J - (\bar\partial_{J,A}\Theta_0)\circ J$$

this gives the commutator identity

$$\nabla_t\bar\partial_{J_t,A}\Theta_0 - \bar\partial_{J_t,A}\nabla_t\Theta_0 = (\dot A - d\Psi)^{0,1}\Theta_0 + \frac{1}{2}(\partial_{J_t,A}\Theta_0)\circ J_t\dot J_t. \tag{33}$$

Moreover, (31) is equivalent to

$$i(\dot A - d\Psi)^{0,1} = \frac{1}{2}\langle\Theta_0,\Theta_1\rangle.$$

Hence

$$\begin{aligned}\bar\partial_{J_t,A}\,\bar\partial_{J_t,A}{}^*\Theta_1 &= i\bar\partial_{J_t,A}\nabla_t\Theta_0\\ &= i\nabla_t\bar\partial_{J_t,A}\Theta_0 - i(\dot A - d\Psi)^{0,1}\Theta_0 - \frac{i}{2}(\partial_{J_t,A}\Theta_0)\circ J_t\dot J_t\\ &= \varepsilon^2\nabla_t\nabla_t\Theta_1 - \frac{1}{2}\langle\Theta_0,\Theta_1\rangle\Theta_0 + \frac{1}{2}(\partial_{J_t,A}\Theta_0)\circ \dot J_t.\end{aligned}$$

This proves the lemma. □

Remark 9.1. It is interesting to consider the special case of the product

$$Y = S^1\times\Sigma$$

with the product metric and the product spin^c structure

$$\gamma_d : TY \to \mathrm{End}(W_d),$$

where $W_d = S^1\times(E\oplus\Lambda^{0,1}T^*\Sigma\otimes E)$ and $E\to\Sigma$ is a Hermitian line bundle of degree d. In this case J can be chosen independent of t, the adiabatic limit is not required, and (32) with $\varepsilon = 1$ takes the form

$$\bar\partial_{J_t,A}\,\bar\partial_{J_t,A}{}^*\Theta_1 - \nabla_t\nabla_t\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 = 0.$$

Take the inner product with Θ_1 and integrate to obtain

$$\int_0^1\int_\Sigma\left(|\,\bar\partial_A{}^*\Theta_1|^2 + |\nabla_t\Theta_1|^2 + \frac{1}{2}\,|\Theta_0|^2\,|\Theta_1|^2\right)\omega\,dt = 0.$$

This implies that either $\Theta_0\equiv 0$ or $\Theta_1\equiv 0$. Since the mean value of $\tau - *iF_A$ is positive it follows that $\Theta_1\equiv 0$. Moreover, by choosing an appropriate gauge transformation, we may assume without loss of generality that $\Psi(t) = 0$ for all t. Then it follows that $A(t) = A$ and $\Theta_0(t) = \Theta_0$ are independent of t and satisfy the vortex equations. In other words, the moduli space of solutions of the Seiberg-Witten equations over $S^1\times\Sigma$ can be identified with the symmetric product and a standard perturbation argument now shows that

$$\mathrm{SW}(S^1\times\Sigma,\gamma_d) = \chi(S^d\Sigma) = \mathcal{T}(S^1\times\Sigma, e_d). \tag{34}$$

All the other invariants are zero and this proves Theorem 1.1 in the product case. A similar argument works whenever some iterate of f is the identity.

The proof of Theorem 1.1 in the general case is considerably deeper. It is obvious from (30) that the square of the L^2-norm of Θ_0 is bounded below by twice the mean value of $\tau - *iF_A$. Hence one can introduce $\Theta_1'(t) \in \Omega^{0,1}_{J_t}(\Sigma, E)$ as the unique solution of (18). In particular, one has to prove that the difference $\Theta_1 - \Theta_1'$ converges to zero as $\varepsilon \to 0$. This requires some pointwise estimates on the functions Θ_0, Θ_1, Θ_1' and their derivatives that are reminiscent of some of the estimates that appear in the work of Taubes [29, 30]. This is related to the convergence question. From the other side one needs a singular perturbation result which asserts that near every nondegenerate solution of (16), (17), (18), and (19) (corresponding to a fixed point of $\phi_{d,f}$ in the class $\mathcal{P}_{d,\tilde{f}}$) there is, for $\varepsilon > 0$ sufficiently small, a solution of the Seiberg-Witten equations (29), (30), and (31) that satisfies the same periodicity condition (19) (contributing to the Seiberg-Witten invariant $\mathrm{SW}(Y_f, \gamma_{d,\tilde{f}})$). Once the one-to-one correspondence between gauge equivalence classes of solutions has been established, one needs to compare the fixed point index with μ^{SW}. This amounts to a comparison of the spectral flows. The full details of the proof will appear elsewhere.

10. Floer homology

There is a 4-dimensional version of the adiabatic limit argument. After the appropriate choices of perturbation, change in parameters, and scaling the Seiberg-Witten equations over the tube $\mathbb{R} \times Y_f$ take the form

$$\nabla_s\Theta_0 + i\nabla_t\Theta_0 = \bar{\partial}_{J_t,A}{}^*\Theta_1, \qquad \nabla_s\Theta_1 - i\nabla_t\Theta_1 = \varepsilon^{-2}\bar{\partial}_{J_t,A}\Theta_0. \tag{35}$$

$$\varepsilon^{-2}\left(*iF_A + \frac{|\Theta_0|^2}{2} - \tau\right) = \frac{|\Theta_1|^2}{2} + i(\partial_t\Phi - \partial_s\Psi), \tag{36}$$

$$(\partial_s A - d\Phi) + *_t(\partial_t A - d\Psi) = i\mathrm{Im}\,\langle\Theta_0, \Theta_1\rangle. \tag{37}$$

Here s is the real parameter and $A + \Phi\,ds + \Psi\,dt$ is the connection on the bundle $\mathbb{R} \times E_{\tilde{f}} \to \mathbb{R} \times Y_f$. In the adiabatic limit $\varepsilon \to 0$ the solutions of these equations degenerate to holomorphic curves in the moduli space $\mathcal{M}_{\Sigma,d}(J, \tau) \cong S^d\Sigma$. Explicitly, the limit equations have the form

$$\bar{\partial}_{J_t,A}\Theta_0 = 0, \qquad *iF_A + |\Theta_0|^2/2 = \tau, \tag{38}$$

$$(\partial_s A - d\Phi) + *_t(\partial_t A - d\Psi) = i\mathrm{Im}\,\langle\Theta_0, \Theta_1\rangle, \tag{39}$$

$$\nabla_s\Theta_0 + i\nabla_t\Theta_0 = \bar{\partial}_{J_t,A}{}^*\Theta_1, \tag{40}$$

$$\bar{\partial}_{J_t,A}\,\bar{\partial}_{J_t,A}{}^*\Theta_1 + \frac{|\Theta_0|^2}{2}\Theta_1 = \frac{1}{2}(\partial_{J_t,A}\Theta_0) \circ \dot{J}_t. \tag{41}$$

The small ε analysis should now give rise to a proof of the following analogue of the Atiyah-Floer conjecture [1, 3, 4, 5].

Conjecture 10.1. *For every* $f \in \mathrm{Diff}(\Sigma, \omega)$ *and every lift* $\tilde{f}$ *of* f *to a unitary automorphism of a line bundle* $E \to \Sigma$ *of degree* d *there is a natural isomorphism between Seiberg-Witten and symplectic Floer homologies*

$$\mathrm{HF}^{\mathrm{SW}}(Y_f, \gamma_{d,\tilde{f}}) \to \mathrm{HF}^{\mathrm{symp}}(\phi_{d,f}, \mathcal{P}_{d,\tilde{f}}).$$

These isomorphisms intertwine the natural product structures:

$$\begin{tikzcd}
\mathrm{HF}^{\mathrm{SW}}(Y_f, \gamma_{d,\tilde{f}}) \arrow[d] & \otimes & \mathrm{HF}^{\mathrm{SW}}(Y_g, \gamma_{d,\tilde{g}}) \arrow[d] \arrow[r] & \mathrm{HF}^{\mathrm{SW}}(Y_{fg}, \gamma_{d,\tilde{f}\tilde{g}}) \arrow[d] \\
\mathrm{HF}^{\mathrm{symp}}(\phi_{d,f}, \mathcal{P}_{d,\tilde{f}}) & \otimes & \mathrm{HF}^{\mathrm{symp}}(\phi_{d,g}, \mathcal{P}_{d,\tilde{g}}) \arrow[r] & \mathrm{HF}^{\mathrm{symp}}(\phi_{d,fg}, \mathcal{P}_{d,\tilde{f}\tilde{g}})
\end{tikzcd}.$$

Theorem 2.1 asserts that the Seiberg-Witten and the symplectic Floer homology groups have the same Euler characteristic. The comparison of the spectral flows shows in fact that they can be modeled on the same chain complex. The adiabatic limit argument should prove that the boundary operators agree for ε sufficiently small.

One of the difficulties in the proof of Conjecture 10.1 lies in the presence of holomorphic spheres with negative Chern number. Such spheres exist in $\mathcal{M}_{\Sigma,d}$ whenever the genus g and the degree d satisfy

$$\frac{g}{2} + 1 < d < g - 1. \tag{42}$$

In this case the new approaches to Floer homology in the presence of holomorphic spheres with negative Chern number are required (cf. Fukaya–Ono [10], Liu–Tian [18], Ruan [24], and Hofer–Salamon [13, 25]). If (42) does not hold then the standard theory applies (cf. [6, 7, 8, 12, 21, 22, 28, 25, 27]). In this case the proof of Conjecture 10.1 should be quite analogous to the proof of the Atiyah-Floer conjecture for mapping tori in Dostoglou–Salamon [3, 4, 5].

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References

[1] M.F. Atiyah, *New invariants of three and four dimensional manifolds,* Proc. Symp. Pure Math. **48** (1988).

[2] S.K. Donaldson, *The Seiberg-Witten equations and* 4*-manifold topology,* Bulletin Amer. Math. Soc. **33** (1996), 45–70.

[3] S. Dostoglou and D.A. Salamon, *Instanton homology and symplectic fixed points,* in Symplectic Geometry, edited by D. Salamon LMS Lecture Notes Series **192**, Cambridge University Press, 1993, pp. 57–93.

[4] S. Dostoglou and D.A. Salamon, *Cauchy-Riemann operators, self-duality, and the spectral flow,* in First European Congress of Mathematics, Volume I, Invited Lectures (Part 1), edited by A. Joseph, F. Mignot, F. Murat, B. Prum, R. Rentschler, Birkhäuser Verlag, Progress in Mathematics, **Vol. 119**, 1994, pp. 511–545.

[5] S. Dostoglou and D.A. Salamon, *Self-dual instantons and holomorphic curves,* Annals of Mathematics **139** (1994), 581–640.

[6] A. Floer, *Symplectic fixed points and holomorphic spheres,* Comm. Math. Phys. **120** (1989), 575–611.

[7] A. Floer and H. Hofer, *Coherent orientations for periodic orbit problems in symplectic geometry,* Math. Zeit. **212** (1993), 13–38.

[8] A. Floer, H. Hofer, and D. Salamon, *Transversality in elliptic Morse theory for the symplectic action,* Duke Math. Journal (1996).

[9] K. Froyshov, *The Seiberg-Witten equations and* 4*-manifolds with boundary,* to appear in Math. Res. Letters.

[10] F. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariants for general symplectic manifolds,* Preprint, February 1996.

[11] O. García-Prada, *A direct existence proof for the vortex equations over a compact Riemann surface,* Bull. London Math. Soc. **26** (1994), 88–96.

[12] H. Hofer and D.A. Salamon, *Floer homology and Novikov rings,* in The Floer Memorial Volume, edited by H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, Birkhäuser, 1995, pp 483–524.

[13] H. Hofer and D.A. Salamon, *Rational Floer homology and the general Arnold conjecture,* in preparation.

[14] M. Hutchings and Y.-J. Lee, *Circle valued Morse theory, Reidemeister torsion, and Seiberg-Witten invariants of* 3*-manifolds,* to appear in Topology.

[15] M. Hutchings and Y.-J. Lee, *Circle valued Morse theory and Reidemeister torsion,* to appear in Math. Research Letters.

[16] E.-N. Ionel and T. H. Parker, *Gromov invariants and symplectic maps,* Preprint, March 1997.

[17] P. Kronheimer and T.S. Mrowka, *The genus of embedded surfaces in the projective plane,* Math. Res. Letters **1** (1994), 797–808.

[18] G. Liu and G. Tian, *Floer Homology and Arnold Conjecture,* Preprint, August 1996, revised May 1997.

[19] D. McDuff and D. Salamon, *Introduction To Symplectic Topology*, 2nd edition, Oxford University Press, 1998.

[20] G. Meng and C.Taubes, *SW = Milnor Torsion,* Preprint, 1996.

[21] S. Piunikhin, D. Salamon, and M. Schwarz, *Symplectic Floer-Donaldson theory and quantum cohomology,* to appear in Proceedings of the Symposium on Symplectic Geometry, held at the Isaac Newton Institute in Cambridge in 1994, edited by C.B. Thomas, LMS Lecture Note Series, Cambridge University Press.

[22] M. Pozniak, *Floer homology, Novikov rings, and clean intersections,* PhD thesis, University of Warwick, 1994.

[23] J.W. Robbin and D.A. Salamon, *The spectral flow and the Maslov index,* Bulletin of the LMS **27** (1995), 1–33.

[24] Y. Ruan, *Virtual neighbourhoods and monopole equations,* Preprint, March 1996.

[25] D.A. Salamon, *Lectures on Floer homology,* Notes for the IAS/Park City Graduate Summer School on Symplectic Geometry and Topology, August 1997.

[26] D.A. Salamon, *Spin Geometry and Seiberg-Witten invariants*, to appear in Birkhäuser Verlag.

[27] D.A. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index,* Comm. Pure Appl. Math. **45** (1992), 1303–1360.

[28] P. Seidel, *Floer homology and symplectic isotopy,* PhD thesis, Oxford, 1997.

[29] C.H. Taubes, *The Seiberg-Witten and the Gromov invariants,* Math. Res. Letters **2** (1995), 221–238.

[30] C.H. Taubes, SW $\Longrightarrow$ Gr*: From the Seiberg-Witten equations to pseudoholomorphic curves,* J. Amer. Math. Soc. **9** (1996), 845–918.

[31] C.H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension* 4*,* Preprint, Harvard, 1996.

[32] C.H. Taubes, Gr $\Longrightarrow$ SW*: From pseudoholomorphic curves to the Seiberg-Witten invariants,* Preprint, Harvard, 1996.

[33] V. Turaev, *Torsion invariants of Spin*^{c} *structures on* 3*-manifolds,* Math. Res. Letters **4** (1997), 679–695.

[34] V. Turaev, Personal communication, February 1999.
[35] E. Witten, *Monopoles and* 4*-manifolds,* Math. Res. Letters **1** (1994), 769–796.

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