

Simply connected symplectic 4-manifolds with positive signature

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Dedicated to Rob Kirby on the occasion of his 60th birthday.

1. Introduction

Recent advances in understanding the topology of symplectic 4-manifolds gave us the hope to characterize symplectic 4-manifolds — at least up to homeomorphism. By a theorem of Gompf [G1] any finitely presented group can be given as the fundamental group of a closed symplectic 4-manifold. Therefore in the following we will concentrate only on simply connected manifolds. Combining results of Donaldson [D] and Freedman [F] it can be shown that a simply connected symplectic 4-manifold X is determined by its Euler characteristic $e(X)$, signature $\sigma(X)$ and its spin property — up to homeomorphism. This observation naturally raises the question: Which pairs $(e, \sigma) \in \mathbb{Z} \times \mathbb{Z}$ can be represented as (Euler characteristic, signature) of a simply connected, symplectic 4-manifold. (For the present purpose we disregard the spinness of the 4-manifolds; we hope to return to the discussion of spin 4-manifolds in a future project.) These type of questions are usually called “geography questions”. For historic reasons, in 4-manifold geography one records $\chi_h(X) = \frac{1}{4}(e(X) + \sigma(X))$ and $c_1^2(X) = 3\sigma(X) + 2e(X)$ rather than $e(X)$ and $\sigma(X)$.

Since blowing up and down can be performed within the symplectic category, and for the blown up manifold X' we have $\chi_h(X') = \chi_h(X)$ and $c_1^2(X') = c_1^2(X) - 1$, we restrict our attention to *minimal* symplectic 4-manifolds, i.e., for those which do not contain (-1) -spheres.

Remark 1.1. As a consequence of the work of Taubes [T], [K], a simply connected, minimal symplectic 4-manifold X is irreducible, i.e., if X decomposes as a connected sum $X_1 \# X_2$, then either X_1 or X_2 is homeomorphic to the 4-sphere S^4 . Consequently, the geography of minimal, simply connected, symplectic 4-manifolds forms a part of the geography problem of irreducible, simply connected 4-manifolds.

Easy to see that the simple connectivity of X implies that $\chi_h(X) \geq 1$. As a consequence of Taubes’ work [T], [K], we also know that for a minimal symplectic 4-manifold $c_1^2(X) \geq 0$. Hence the geography problem reads as follows:

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Question 1.2. Which pairs $(a, b) \in \mathbb{N} \times \mathbb{N}$ correspond to minimal, simply connected, symplectic 4-manifolds X as $a = \chi_h(X)$ and $b = c_1^2(X)$?

Combining earlier results [FS1], [G1], [P], [S1], one can see that pairs satisfying $0 \leq b \leq 8a$ (with at most finitely many exceptions) do correspond to simply connected, minimal symplectic 4-manifolds. The region $b > 8a$ (or equivalently, when the signature σ is positive), however, turned out to be more mysterious. The famous Bogomolov-Miyaoka-Yau inequality asserts, that if S is a Kähler surface then $c_1^2(S) \leq 9\chi_h(S)$ holds. Consequently, the two natural questions arising in this context are the following

q1: Which pairs (a, b) satisfying $8a < b \leq 9a$ correspond to simply connected, minimal symplectic 4-manifolds?

q2: Is there a bound — similar to the Bogomolov-Miyaoka-Yau inequality above — for c_1^2 of a symplectic 4-manifold in terms of its topological data, e.g., its holomorphic Euler characteristic χ_h ?

In this note we will concentrate on the (partial) answer of **q1**. Most examples of symplectic 4-manifolds with positive signature originate from complex geometry [Ch1], [Ch2], [H], [MT], [So], [PPX]; these examples either have large fundamental group or satisfy $c_1^2(S) \leq 8\frac{6}{10}\chi_h(S)$. (There is a unique simply connected, complex surface satisfying $c_1^2 = 9\chi_h$ — the complex projective plane $\mathbb{C}P^2$. In the above remark we disregarded this trivial example.) In the following we will prove the following statement.

Theorem 1.3. *There are simply connected, minimal, symplectic 4-manifolds C_n for which $c_1^2(C_n)/\chi_h(C_n) \rightarrow 9$ as $n \rightarrow \infty$.*

For related examples and constructions see also [FS2], [S2], [Sz].

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2. Manifolds on the Bogomolov-Miyaoka-Yau line

Suppose that Σ_2 is the Riemann surface of genus 2 and $\gamma: \Sigma_2 \rightarrow \Sigma_2$ is a map with exactly 3 fixed points and $\gamma^5 = \text{id}_{\Sigma_2}$. (We will prove the existence of such γ in Section 4, cf. also [Ch2], [GS] or [S2].) It is easy to see that $\cup_{i=0}^4 \text{Graph}(\gamma^i) \subset \Sigma_2 \times \Sigma_2$ consists of 5 curves F_0, \dots, F_4 (each of genus 2) and the homology class $\sum_{i=0}^4 [F_i]$ is divisible by 5. The desingularization of the 5-fold (cyclic) branched cover of $\Sigma_2 \times \Sigma_2$ (branched along $\cup_{i=0}^4 F_i$) will be denoted by H_1 . Easy computation shows the following.

Lemma 2.1. *The Euler characteristic $e(H_1)$ of H_1 is equal to 75, its signature is $\sigma(H_1) = 25$, hence $c_1^2(H_1) = 225$ and $\chi_h(H_1) = \frac{1}{12}(c_2(H_1) + c_1^2(H_1)) = 25$. Consequently $c_1^2(H_1) = 9\chi_h(H_1)$, so H_1 is on the Bogomolov-Miyaoka-Yau line. The composition $H_1 \rightarrow \Sigma_2 \times \Sigma_2 \xrightarrow{\text{pr}} \Sigma_2$ provides a Lefschetz fibration on H_1 with fibers of genus 16. There are 3 singular fibers in this fibration, and the inverse image of $F_0 \subset \Sigma_2 \times \Sigma_2$ (the diagonal) provides a section of $H_1 \rightarrow \Sigma_2$: an embedded Riemann surface T (of genus 2) intersecting each fiber in a single point. Since $[F_0]^2 = -2$, the self-intersection of T is -1 . \square*

Suppose that $\varphi_n: \Sigma_{n+1} \rightarrow \Sigma_2$ is an n -fold (unbranched) cover of Σ_2 — here Σ_{n+1} denotes the Riemann surface of genus $n+1$. Pulling back the branched cover $H_1 \rightarrow \Sigma_2 \times \Sigma_2$ via the map $\varphi_n \times \varphi_n: \Sigma_{n+1} \times \Sigma_{n+1} \rightarrow \Sigma_2 \times \Sigma_2$, we get a manifold $H(n^2)$, which is a 5-fold branched cover of $\Sigma_{n+1} \times \Sigma_{n+1}$ and an n^2 -fold (unbranched) cover of H_1 .

Proposition 2.2. *The Euler characteristic of $H(n^2)$ is equal to $75n^2$, $\sigma(H(n^2)) = 25n^2$, hence $c_1^2(H(n^2)) = 225n^2$ and $\chi_h(H(n^2)) = 25n^2$. The map $H(n^2) \rightarrow \Sigma_{n+1} \times \Sigma_{n+1} \xrightarrow{\text{pr}} \Sigma_{n+1}$ provides a Lefschetz fibration on $H(n^2)$; now the genus of the generic fiber is $15n+1$. The inverse image of a section of H_1 gives a section of $H(n^2) \rightarrow \Sigma_{n+1}$; the corresponding submanifold is a Riemann surface of genus $(n+1)$ with self-intersection $-n$. \square*

3. Construction of the 4-manifolds C_n

Our goal in the following is to reduce the fundamental group of $H(n^2)$ by various operations to achieve a simply connected, symplectic 4-manifold.

It is known that for a Lefschetz fibration $f: M^4 \rightarrow B^2$ with generic connected fiber F , then

$$\pi_1(F) \rightarrow \pi_1(M) \rightarrow \pi_1(B) \rightarrow 0$$

is an exact sequence (see, e.g., [GS]). Fix now points p_1, \dots, p_{30n+4} in $\mathbb{C}\mathbb{P}^1 = \Sigma_0$ and q_1, q_2 in the 2-dimensional torus $T^2 = \Sigma_1$. Specify the curve $B_n = \cup_{i=1}^{30n+4} T^2 \times \{p_i\} \cup \cup_{j=1}^2 \{q_j\} \times \mathbb{C}\mathbb{P}^1 \subset T^2 \times \mathbb{C}\mathbb{P}^1$, take the double branched cover of $T^2 \times \mathbb{C}\mathbb{P}^1$ branched along B_n and desingularize the resulting complex surface. It is easy to see that the resulting smooth 4-manifold Z_n admits a genus- $(15n+1)$ Lefschetz fibration $Z_n \rightarrow T^2$ with two simply connected fibers (corresponding the fiber components of the branch locus B_n) — Compose the branched cover map $Z_n \rightarrow T^2 \times \mathbb{C}\mathbb{P}^1$ with the projection $T^2 \times \mathbb{C}\mathbb{P}^1 \rightarrow T^2$. Define X_n as the fiber sum of $H(n^2)$ and Z_n .

Proposition 3.1. *The 4-manifold X_n admits a genus- $(15n+1)$ Lefschetz fibration with a section T_n of genus $(n+2)$ and self-intersection $-(n+1)$. Furthermore, X_n can be equipped with a symplectic structure such that T_n is a symplectic submanifold. The map $X_n \rightarrow T_n$ (mapping an element $x \in F_y$ of a fiber $F_y \subset X_n$ to $F_y \cap T_n$) induces an isomorphism between the fundamental groups.*

Proof. Since the fiber sum of two Lefschetz fibrations is a Lefschetz fibration again, the first statement is obvious. By gluing a section of $H(n^2)$ and a section of Z_n (which is a torus of self-intersection -1) together we get T_n . Applying the construction of Gompf [G2], [GS], the Lefschetz fibered 4-manifold X_n can be endowed with an appropriate symplectic structure — this symplectic structure can be chosen in a way that $T_n \subset X_n$ becomes a symplectic submanifold. Since the fibration on X_n admits simply connected fibers (the introduction of these singular fibers is the reason of taking the above fiber sum), the inclusion of the generic fiber $F \hookrightarrow X_n$ induces the trivial homomorphism on the fundamental groups. Hence the exact sequence described earlier in this section reduces to an isomorphism between the fundamental groups of the total and the base space of the Lefschetz fibration, proving the last assertion of the proposition. \square

It is easy to see that $e(Z_n) = 120n + 12$ and $\sigma(Z_n) = -60n - 8$, hence

Lemma 3.2. *The Euler characteristic $e(X_n) = 75n^2 + 180n + 12$ and the signature $\sigma(X_n) = 25n^2 - 60n - 8$. Consequently $\chi_h(X_n) = 25n^2 + 30n + 1$ and $c_1^2(X_n) = 225n^2 + 180n$. \square*

For constructing the desired simply connected 4-manifolds we need to describe one more construction. Fix distinct points $p_1, \dots, p_6 \in \mathbb{C}\mathbb{P}^1$ and consider the curve $B = \bigcup_{i=1}^6 \mathbb{C}\mathbb{P}^1 \times \{p_i\} \cup \bigcup_{j=1}^2 \{p_j\} \times \mathbb{C}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. The desingularization of the double branched cover of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ branched along B — the resulting complex surface will be denoted by $X(1, 3)$ — admits a genus-2 Lefschetz fibration over $\mathbb{C}\mathbb{P}^1$ with a section of self-intersection -1 . Standard argument shows [GS] that $X(1, 3) \approx \mathbb{C}\mathbb{P}^2 \# 13\overline{\mathbb{C}\mathbb{P}^2}$, hence the signature and the Euler characteristic of $X(1, 3)$ are easy to determine. Consider now k copies of the generic fiber and one copy of the section in the fibration $X(1, 3) \rightarrow \mathbb{C}\mathbb{P}^1$. By resolving the k transverse double points we get an embedded surface G_k of genus $2k$ and self-intersection $2k - 1$. The section and the fibers are obviously symplectic submanifolds of the symplectic (in fact, Kähler) 4-manifold $X(1, 3)$; consequently we can perform the resolution in a way that $G_k \subset X(1, 3)$ is symplectic for each k .

Lemma 3.3. *The complement $X(1, 3) - \nu G_k$ of the tubular neighborhood of G_k in $X(1, 3)$ is simply connected.*

Proof. The 4-manifold $X(1, 3)$ is simply connected because it admits a Lefschetz fibration with at least one simply connected fiber over $\mathbb{C}\mathbb{P}^1$, cf. the exact sequence at the beginning of Section 3. The same simply connected fiber (intersecting G_k in a single point) shows that the normal circle of $G_k \subset X(1, 3)$ is nullhomotopic in $X(1, 3) - \nu G_k$, which proves the lemma. \square

Consider now the pairs (X_{2n}, T_{2n}) and $(X(1, 3), G_{n+1})$. Both pairs consist of a symplectic 4-manifold together with a symplectic submanifold. Moreover, the genera of T_{2n} and G_{n+1} are both equal to $2n + 2$ and $[G_{n+1}]^2 = 2n + 1 = -[T_{2n}]^2$. Applying the symplectic normal sum operation introduced by Gompf [G1], we get a symplectic 4-manifold $C_n = (X_{2n} - \nu T_{2n}) \cup (X(1, 3) - \nu G_{n+1})$. Elementary computation shows that $e(C_n) = 300n^2 + 368n + 32$ and $\sigma(C_n) = 100n^2 - 120n - 20$, consequently

Theorem 3.4. *The symplectic 4-manifold C_n is simply connected and $\chi_h(C_n) = 100n^2 + 62n + 3$, $c_1^2(C_n) = 900n^2 + 376n + 4$.*

Proof. The facts that $\pi_1(T_{2n}) \rightarrow \pi_1(X_{2n})$ is an isomorphism and that $\pi_1(X(1, 3) - \nu G_{n+1}) = 1$ combined with the Seifert-Van Kampen theorem now imply that $\pi_1(C_n) = 1$. The rest of the theorem is just elementary computation. \square

Proof of Theorem 1.3. Since C_n is simply connected and symplectic, moreover $c_1^2(C_n)/\chi_h(C_n) = (900n^2 + 376n + 4)/(100n^2 + 62n + 3) \rightarrow 9$, the theorem follows. \square

Remark 3.5. Similar strategy can be carried out by replacing Z_n and $X(1, 3)$ with manifolds having smaller Euler characteristics and signatures. In this way we might have smaller linear terms in the expressions of $\chi_h(C_n)$ and $c_1^2(C_n)$, but the endresult will not be changed significantly.

4. Appendix

Finally we show the existence of the map $\gamma: \Sigma_2 \rightarrow \Sigma_2$ required at the beginning of Section 2: Take the (singular) curve $A = \{[x_0 : x_1 : x_2] \in \mathbb{C}\mathbb{P}^2 \mid x_0^5 - x_1^3 x_2(x_1 + x_2) = 0\}$ in $\mathbb{C}\mathbb{P}^2$ and blow up $\mathbb{C}\mathbb{P}^2$ in $[0 : 0 : 1]$ (the singular point of the curve A). The proper transform \tilde{A} still has one singular point, but the proper transform D of an additional blow-up will be smooth. Hence we have found a smooth curve D in $\mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$; restricting the blow-down map $\mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2} \rightarrow \mathbb{C}\mathbb{P}^2$ to D and composing it with the projection $\mathbb{C}\mathbb{P}^2 - [1 : 0 : 0] \rightarrow \{x_0 = 0\} \approx \mathbb{C}\mathbb{P}^1$ (mapping $[x_0 : x_1 : x_2]$ to $[0 : x_1 : x_2]$) we get a map $\varphi: D \rightarrow \mathbb{C}\mathbb{P}^1$. This map is simply an explicit description of the 5-fold cyclic branched cover $D \rightarrow \mathbb{C}\mathbb{P}^1$ branched in three points $Q_1, Q_2, Q_3 \in \mathbb{C}\mathbb{P}^1$. Consequently we have a \mathbb{Z}_5 -action (the generator is denoted by $\gamma: D \rightarrow D$) on D ; the fixed points of γ are the inverse images of Q_i ($i = 1, 2, 3$) (still denoted by Q_i in D). An easy application of the adjunction formula shows that D has genus 2, consequently the above map γ acts on Σ_2 as required.

References

- [Ch1] Z. Chen, *On the geography of surfaces — simply connected minimal surfaces with positive index*, Math. Ann. **126** (1987), 141–164.
- [Ch2] Z. Chen, *The existence of algebraic surfaces with preassigned Chern numbers*, Math. Z. **206** (1991), 241–254.
- [D] S. Donaldson, *An application of gauge theory to four dimensional topology*, J. Diff. Geom. **18** (1983), 279–315.
- [FS1] R. Fintushel and R. Stern, *Surgery in Cusp Neighborhoods and the Geography of Irreducible 4-Manifolds*, Invent. Math. **117** (1994), 455–523.
- [FS2] R. Fintushel and R. Stern, personal communication.
- [F] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.
- [G1] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995), 527–595.
- [G2] R. Gompf, *Lecture at MSRI*.
- [GS] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, book in preparation.
- [H] F. Hirzebruch, *Chern numbers of algebraic surfaces*, Math. Ann. **266** (1984), 351–356.
- [K] D. Kotschick, *The Seiberg-Witten invariants of symplectic four-manifolds [after C. H. Taubes]*, Seminare Bourbaki 48 ème année (1995-96) n° 812.
- [MT] B. Moishezon and M. Teicher, *Simply connected algebraic surfaces of positive index*, Invent. Math. **89** (1987), 601–644.
- [P] J. Park, PhD thesis, Michigan State University, 1995.

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- [PPX] U. Persson, C. Peters and G. Xiao, *Geography of spin surfaces*, Topology **35** (1996), 845–862.
- [So] A. Sommese, *On the density of ratios of Chern numbers of algebraic surfaces*, Math. Ann. **268** (1984), 207–221.
- [S1] A. Stipsicz, *A note on the geography of symplectic manifolds*, Turkish J. Math. **20** (1996), 135–139.
- [S2] A. Stipsicz, *Simply connected 4-manifolds near the Bogomolov-Miyaoka-Yau line*, Math. Res. Lett. **5** (1998), 723–730.
- [Sz] Z. Szabó, personal communication.
- [T] C. Taubes, *Seiberg-Witten and Gromov invariants*, Geometry and Physics (Aarhus, 1995) (Lecture Notes in Pure and Applied Math., 184) Dekker, New York (1997), 591–601.

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