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COUNTABLE DENSE HOMOGENEOUS BITOPOLOGICAL SPACES

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Abstract

In this paper we shall introduce the concept of being countable dense homogeneous bitopological spaces and define several kinds of this concept. We shall give some results concerning these bitopological spaces and their relations. Also, we shall prove that all of these bitopological spaces satisfying the axioms p-T_0 and p-T_1.

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1. Introduction

Countable dense homogeneous spaces were introduced by Bennett [1]. Recall that a topological space (X, τ) is called countable dense homogeneous (CDH) iff X is separable and, if D_1 and D_2 are two countable dense subsets of X, then there is a homeomorphism h: X$\rightarrow$X such that h(D_1)=D_2.

In 1963, Kelly [4] introduced the concept of bitopological spaces. A set X equipped with two topologies is called a bitopological space.

Let X be any set. By $\tau_{cof}, \tau_{dis}, \tau_{ind}$ and $\tau_u, \tau_{1.r}, \tau_{r.r.}$ (in the case X=IR), we mean the cofinite, the discrete, the indiscrete, the usual Euclidean, the left ray, and the right ray topologies, respectively. Let (X,τ) be a topological space, A$\subseteq$X. By τ_A we mean the relative topology on A. If (X, τ_1, τ_2) is a bitopological space and A$\subseteq$X, cl_i(A) will denote the closure of A with respect to τ_i; i= 1, 2. A subset D in (X, τ_1, τ_2) is called dense if cl_1(D)=cl_2(D)=X. A bitopological space (X, τ_1, τ_2) is called separable if both topological spaces (X, τ_1) and (X, τ_2) are separable. For a set A, we shall denote the cardinality of A by $\mid A\mid$.

We shall use p- to denote pairwise for instance, p-T_i stands for pairwise T_i. For terminology not defined here one may consult Bennett [1] and Kelley [4].

Let us start with the following definitions.

DEFINITION 1.1 *Let f: (X, τ_1, τ_2)→(Y,σ_1, σ_2) be a map from a bitopological space (X, τ_1, τ_2) to a bitopological space (Y, σ_1, σ_2).*
a) f is called continuous (open, closed, homeomorphism) iff the maps f: (X, τ_1)→(Y, σ_1) and f: (X, τ_2)→(Y, σ_2) are continuous (open, closed, homeomorphism respectively).
b) f is called p-continuous iff for each $U \in \sigma_1 \cup \sigma_2, f^{-1}(U) \in \tau_1 \cup \tau_2$.
c) f is called p-homeomorphism iff f is bijection, p-continuous and $f^{-1} : (Y,\sigma_1,\sigma_2) \rightarrow (X,\tau_1,\tau_2)$ is p-continuous.

DEFINITION 1.2. *Let (X,τ_1, τ_2) and (Y,σ_1, σ_2) be bitopological spaces. A map f: (X,τ_1, τ_2) →(Y,σ_1, σ_2) is called p_1-continuous (p_1-open, p_1-closed, p_1-homeomorphism) iff the maps f: (X,τ_1)→(Y,σ_2) and f: (X,τ_2)→(Y,σ_1) are continuous (open, closed, homeomorphism respectively).*

The concept of p_1-that was defined in Definition 1.2 depends heavily on the order of the topologies, that is (X,τ_1, τ_2) is different from (X,τ_2, τ_1). For instance, ($\mathbb{R}$, τ_u, τ_{cof}) is p_1-homeomorphic to ($\mathbb{R}$, τ_{cof}, τ_u) but it is not p_1-homeomorphic to itself. However, a bitological space (X,τ_1, τ_2) is p_1-homeomorphic to itself if and only if (X, τ_1) is homeomorphic to (X, τ_2).

DEFINITION 1.3. *A subset D in a bitopological space (X,τ_1, τ_2) is called p-dense iff $cl_1(cl_2(D)) = cl_2(cl_1(D)) = X$. A bitopological space (X,τ_1, τ_2) is called p- separable if it has a countable p-dense subset.*

It is obvious that if one of the topological spaces (X, τ_1), and (X, τ_2) is separable then (X,τ_1, τ_2) is p- separable. The converse need not be true, in fact one may construct an example of a p- separable bitopological space (X,τ_1, τ_2) in which both (X, τ_1) and (X, τ_2) are not separable.

DEFINITION 1.4. *A bitopological space (X,τ_1,τ_2) is called p-T_0 iff for any distinct points x, y in X, there exists a set $U\in \tau_1 \cup \tau_2$ such that ($x\in U$ and $y \notin U$) or ($x\notin U$ and $y\in U$), that is, U contains only one of x and y.*

DEFINITION 1.5. *A bitopological space (X,τ_1,τ_2) is called p-T_1 iff for any distinct points x, y in X, there are two sets $U, V \in \tau_1 \cup \tau_2$ such that U contains x and does not contain y, and V contains y and does not contain x.*

2. CDH BITOPOLOGICAL SPACES

In this section we shall introduce several kinds of countable dense homogeneous bitopological spaces, and then we shall give some results concerning these spaces. This section will include necessary examples to describe the impossible between these bitopological spaces.

DEFINITION 2.1. *A bitopological space (X,τ_1,τ_2) is called p-CDH iff*

(i) X is p-separable.

(ii) If A and B are countable p-dense subsets of X, then there exists a p-homeomorphism h: $(X,\tau_1,\tau_2)\rightarrow(X,\tau_1,\tau_2)$ such that h(A)=B.

DEFINITION 2.2. *A bitopological space (X,τ_1,τ_2) is called p_1-CDH iff*

(i) X is p-separable.

(ii) If A and B are countable p-dense subsets of X, then there exists a p_1-homeomorphism h: $(X,\tau_1,\tau_2)\rightarrow(X,\tau_1,\tau_2)$ such that h(A)=B.

Another type of countable dense homogeneous bitopological spaces is the following.

DEFINITION 2.3. *A bitopological space (X,τ_1,τ_2) is called p_2-CDH iff*

(i) X is p-separable.

(ii) If A_i are countable τ_i-dense subsets of X, i=1,2, then there is a p_1-homeomorphism h: $(X,\tau_1,\tau_2)\rightarrow(X,\tau_1,\tau_2)$ such that $h(A_1)=A_2$.

In the next examples we shall see that not every p_2-CDH is p-CDH, and not every p-CDH is P_2-CDH. Before this it is helpful to give the following result.

Theorem 2.4. *If (X,τ_1,τ_2) is a p_2-CDH bitopological space then the topological spaces (X,τ_1) and (X,τ_2) are homeomorphic and (X,τ_1) is CHD.*

Proof. If (X,τ_1,τ_2) is p_2-CDH, then (X,τ_1), (X,τ_2) have countable dense subsets D_1, D_2, respectively; hence there exists a homeomorphism h: $(X,\tau_1)\to(X,\tau_2)$ such that $h(D_1)=D_2$.So (X,τ_1) is homeomorphic to (X,τ_2). Now, let K_1 and K_2 be two countable dense subsets in (X,τ_1). Since (X,τ_1) is homeomorphic to (X,τ_2), there exists a homeomorphism h_1: $(X,\tau_1)\to(X,\tau_2)$. So, $h_1(K_1)$ is a countable τ_2-dense set. Hence there exists a homeomorphism h_2:$(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ with $h_2(h_1(K_1))=K_2$. Therefore the composition h= h_2oh_1 : $(X,\ \tau_1)\to(X,\ \tau_1)$ is a homeomorphism and $h(K_1)=(h_2oh_1)(K_1)=K_2$.

The converse of Theorem 2.4 is not true as we shall see in the next example. □

EXAMPLE 2.5.

Let X = $\mathbb{R}\setminus\{-1,1\}$ and define the bases β_1 , β_2 on X as follows:

$$\begin{aligned}\beta_1 \;&=\; \{a,b]\subseteq(-\infty,-1):a<b\}\cup\{U\subseteq(-1,1):(-1,1)\setminus U\,isfinite\}\cup\{(c,d)\\ &\subseteq(1,\infty):c>d\},\end{aligned}$$

$$\begin{aligned}\beta_2 \;&=\; \{U\subseteq(-\infty,-1):(-\infty,-1)\setminus Uisfinite\}\cup\{(e,f)\subseteq(-1,1):e<f\}\cup\{(c,d]\\ &\subseteq(1,\infty):c<d\}.\end{aligned}$$

Then β_1 and β_2 are bases for some topologies $\tau_1=\tau(\beta_1)$ and $\tau_2=\tau(\beta_2)$ on X. Hence (X,τ_1) is homeomorphic to (X,τ_2) and (X,τ_1) is CDH, but (X,τ_1,τ_2) is not p_2-CDH.

For this, the rationals Q is τ_1-dense and τ_2-dense. If there exists a p_1-homeomorphism h: $(X,\tau_1,\tau_2)\to(X,\tau_1,\tau_2)$ with h(Q)=Q, then the maps h: $(X,\tau_1)\to(X,\tau_2)$ and h:$(X,\tau_2)\to(X,\tau_1)$ are homeomorphisms. But the first map will send $(-\infty$,-1) homeomorphically onto $(1,\infty)$ and the second one will send $(-\infty$,-1) homeomorphically onto (-1, 1), a contradiction.

In the following examples we shall show that not every p_1-CDH is p_2-CDH, and not every p-CDH is p_2-CDH.

EXAMPLE 2.6.

Consider $(\mathbb{R},\ \tau_u, \tau_{dis})$. This bitopological space is p-CDH, because; if A and B are countable p-dense subsets, then they are countable dense in $(\mathbb{R}, \tau_u)$, Since $(\mathbb{R}, J_u)$ is CDH, there exists a homeomorphism h:$(\mathbb{R},\tau_u)\rightarrow(\mathbb{R},\tau_u)$ such that h(A) = B. But the same map h is a p-homeomorphism, i.e., h:$(\mathbb{R},\tau_u,\tau_{dis})\rightarrow$ $(\mathbb{R},\tau_u,\tau_{dis})$ is p-homeomorphism and h(A) = B. It is clear that Q is a countable p-dense set in $(\mathbb{R},\ \tau_u, \tau_{dis})$. Hence $(\mathbb{R},\ \tau_u, \tau_{dis})$ is p-CDH.

On the other hand, since $(\mathbb{R}, \tau_{dis})$ is not separable, hence $(\mathbb{R}, \tau_u, \tau_{dis})$ is not p_2-CDH.

EXAMPLE 2.7.

Let X=$\mathbb{R}\setminus \mid \{0\}$ and define the bases β_1, β_2 on X as follows:

$$\beta_1 = \{(a,b) : a < b \leq 0\} \cup \{\{x\} : x \in (0,\infty)\},$$
$$\beta_2 = \{\{y\} : y \in (-\infty,0)\} \cup \{(c,d) : 0 \leq c < d\}.$$

Then β_1 and β_2 are bases for some topologies $\tau_1 = \tau(\beta_1)$ and $\tau_2 = \tau(\beta_2)$ on X. Hence (X, τ_1, τ_2) is p_1-CDH: To see this, for the first condition, Q is a p-dense set in (X, τ_1, τ_2), in fact $cl_1 cl_2$(Q)=$cl_1[(Q \cap (-\infty,0)) \cup (0,\infty)] = \mathbb{R} \setminus \{0\}$ and $cl_2 cl_1$(Q)=$cl_2[(-\infty,0) \cup (Q \cap (0,\infty))] = \mathbb{R}\setminus 0$. Now, if D_1 and D_2 are two p-dense subsets, since $((-\infty,0),\tau_u)$ is CDH and $((-\infty,0),\tau_u)$ is homeomorphic to $((0,\infty),\tau_u)$ then there exists a homeomorphism $h_1 : ((-\infty,0),\tau_u) \rightarrow ((0,-\infty),\tau_u)$ such that h$(D_1 \cap (-\infty,0)) = D_2 \cap (\infty,0)$. Similarly, there exists a homeomorphism $h_2 : ((0,\infty),\tau_u) \rightarrow ((-\infty,0),\tau_u)$ such that $h_2(D_1 \cap (0,\infty)) = D_2 \cap (-\infty,0)$. Thus h=$h_1 \cup h_2 : (X,\tau_1,\tau_2) \rightarrow (X,\tau_1,\tau_2)$ is p_1-homeomorphism. Therefore (X,τ_1,τ_2) is p_1-CDH.

Since (X,τ_1) is not separable, hence (X,τ_1,τ_2) is not p_2-CDH.

Although Example 2.7 shows that not every p_1-CDH bitopological space is p_2-CDH bitopological space, we have the following theorem.

Theorem 2.8. *If (X, τ_1, τ_2) is a separable p_1-CDH bitopological space, then (X, τ_1, τ_2) is p_2-CDH.*

Proof. *To prove this, it is enough to check condition (ii) of Definition 2.3. Let D_i be τ_i-dense subsets of X, i=1, 1. Hence D_1 and D_2 are p-dense sets. Since (X, τ_1, τ_2) is p_1-CDH, so there exists a p_1-homeomorphism h:$(X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ such that h(D_1) = D_2.*

For the implications between p-CDH and p_1-CDH we have the following:

Theorem 2.9. *If (X, τ_1, τ_2) is p_1-CDH, then (X, τ_1, τ_2) is p-CDH.*

Proof. *First condition of Definition 2.1 is already satisfied. For condition (ii), let D_1, D_2 be two countable p-dense subsets of X. Since (X, τ_1, τ_2) is p_1-CDH, hence there exists a p_1-homeomorphism h:$(X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2)$ such that h(D_1)=D_2, but such h is also a p-homeomorphism.*

Example 2.6 shows that the converse of Theorem 2.9 is false. In fact $(\mathbb{R}, \tau_u, \tau_{dis})$ is not p_1-CDH, because; $(\mathbb{R}, \tau_u)$ is not homeomorphic to $(\mathbb{R}, \tau_{dis})$.

To complete the implications between p-CDH, p_1-CDH and p_2-CDH bitopological spaces, we have the following questions.

Proof.

QUESTIONS 2.10.

Is every p_2-CDH bitopological spaces p-CDH?

QUESTIONS 2.11.

Is every p_2-CDH bitopological spaces p_1-CDH?

At the end of this section we shall discuss the following question which may be in mind: is there any relation between the countable dense homogeneity of the bitopological spaces (X, τ_1, τ_2) and their minimal topology $(X, < \tau_1, \tau_2 >)$; where $< \tau_1, \tau_2 >$ is the minimal topology containing τ_1 and τ_2 as a subbase on X. In fact one may ask the following.

QUESTIONS 2.12.

Let (X, τ_1, τ_2) be bitopological space. Is it true that (X, τ_1, τ_2) is p-(p_1-, p_2) CDH if and only if (X, $< \tau_1, \tau_2 >$) is CDH?

Unfortunately, all implications in Question 2.12 are false. To see this, consider the following examples.

EXAMPLE 2.13.

The bitopological space (X, $\tau_{1.r.}, \tau_{r.r.}$) is not p-CDH, because; A = {1}, B = {0,1} are countable p-dense sets but there is no bijection between them. Also, since ($\mathbb{R}$, $\tau_{1.r}$) is not CDH, therefore ($\mathbb{R}, \tau_{1.r.}, \tau_{r.r.}$) is not p_2-CDH. Finally, ($\mathbb{R}, \tau_{1.r.}, \tau_{r.r.}$) is not p_1-CDH, because; it is separable. However, ($\mathbb{R}, < \tau_{1.r.}, \tau_{r.r.} >$)=($\mathbb{R}$, τ_u) is a CDH space.

Since ($\mathbb{R}$ $< \tau_u, \tau_{dis} >$) = ($\mathbb{R}$, τ_{dis}), which is not CDH, but ($\mathbb{R}\tau_u, \tau_{dis}$) is p-CDH. For p-CDH, take Example 2.7 hence ($\mathbb{R}\backslash \mid \{0\}, \tau_1, \tau_2$), which is p_1-CDH; but ($\mathbb{R} \mid \{0\}, < \tau_1, \tau_2 >$)=($\mathbb{R} \mid \{0\}, \tau_{dis},$) is not CDH.

Consider the following and last example.

EXAMPLE 2.14.

Consider the set $\mathbb{R}$ of reals, and define the following bases on it:

$$\begin{aligned} \beta_1 &= \{(a, b] : a, b \in \mathbb{R}, a < b\} and \\ \beta_2 &= \{[a, b) : a, b \in \mathbb{R}, a < b\}. \end{aligned}$$

Then it is easy to see that ($\mathbb{R}$, $\tau(\beta_1), \tau(\beta_2)$) is p_2-CDH, but ($\mathbb{R}$, $< \tau(\beta_1), \tau(\beta_2) >$)= ($\mathbb{R}$, τ_{dis}) is not CDH.

P-CDH AND P-SEPARATION AXIOMS

Fitzpatrick and Zhou [2] discussed CDH spaces in the context of T_1-spaces, Fitzpatrick, White and Zhou [3] showed that the assumption T_1 is redundant and they proved that every CDH space is T_1. In this section we are going to adopt this to bitopological spaces.

If D=$\{d_1, d_2, \ldots\}$ $\subseteq$X and E$\subseteq$D such that D=$\{d_{i_1}, d_{i_2}, \ldots\}$ where $i_1 < i_2 < \ldots$, then d_{i_1} is said to be the first element in E.

Lemma 3.1. *A bitopological space (X, τ_1, τ_2) is p-T_0 if and only if for every two distinct points x, y in X we have $cl_1\{x\} \neq cl_1\{y\}$ or $cl_2\{x\} \neq cl_2\{y\}$.*

Proof. *Straightforward.*

Theorem 3.2. *Every p_2-CDH bitopological space is p-T_1*

Proof. *If (X, τ_1, τ_2) is p_2-CDH, then (X, τ_1) and (X, τ_2) are CDH by Theorem 2.4 and hence they are T_1 [2]. Thus (X, τ_1, τ_2) is p-T_1, as well as p-T_0.*

Theorem 3.3. *If (X, τ_1, τ_2) p-CDH, then it is p-T_1.*

Proof. *We shall give the proof as a consequence of the following steps:*

(1) For each p-homeomorphism f: X$\rightarrow$X and A*subseteq*X, we have f(cl_1 A$\cap cl_2$A) =cl_1(fA))$\cap cl_2$(f(A)).

(2) Let D be a countable p-dense subset of X, choose a sequence $x_n \epsilon$ D such that $x_n : n \notin \mathbb{N}$ is p-dense and $x_n \notin cl_1\{x_m\} \cap cl_2\{x_m\}$ for each n>m. Observe that if there is no such an infinite sequence then X has a finite p-dense subset, which is impossible.

(3) Suppose there exist two distinct points x, y in X, such that x$\in$ $cl_1\{y\} \cap cl_2\{y\}$ and y$\in$ $cl_1\{x\}, \cap cl_2\{x\}$, let D_1=$\{x_n$: n$\in \mathbb{N}\}$, $D_2 \cup \{x, y\}$, so D_1, D_2 are two countable p-dense sets, but there is no p-homeomorphism f such that f(D_1)=(D_2) are two countable p-dense sets, but there is no p-homeomorphism f such that f(D_1)=D_2 so if x$\neq$y and x$\in$ $cl_1\{y\} \cap cl_2\{y\}$ then y $\notin cl_1\{x\} \cap cl_2\{x\}$.

(4) Let y $\in$X, if cl$_1$ {y}$\cap cl_2$ {y} is infinite, choose an infinite sequence $y_n \in cl_1$ {y}$\cap cl_2$ {y}, D_1=$\{x_n$: n $\in \mathbb{N}\}$ and D_2 = $D_1 \cup$\{y\}$\cup\{y_n$: n $\in \mathbb{N}\}$. So there is a p-homeomorphism f,

such that f(D_2)=D_1, then f(y)=x_n for some n $\in \mathbb{N}$. Since $x_m \notin cl_1\{x_n\} \cap cl_2\{x_n\}$ for each m>n, then {f(y_n): n$\in \mathbb{N}$}$\subseteq f(cl_1\{y\} \cap cl_2\{y\}) = cl_1\{x_n\}) \cap cl_2\{x_n\}$, so $\{ f(y_n) : n \in \mathbb{N}\} \subseteq \{x_1, x_2, \ldots, x_n\}$, which is a contradiction, so for each y$\in$X, cl_1 {y}$\cap cl_2$ {y} must be finite.

(5) Let x$\in$ X, then D_1={x_n : $n \in \mathbb{N}$} and $D_2 = D_1 \cup$ {x} are two countable p-dense sets, so there is a p-homeomorphism f such that $f(D_2) = D_1$, then f(x)=x_n for some n $\in \mathbb{N}$. By (1), $f(cl_1\{x\} \cap cl_2\{x\}) = cl_1\{x_n\} \cap cl_2\{x_n\}$.

Hence to complete the proof it is sufficient to show that card $cl_1\{x_n\} \cap cl_2\{x_n\}$=1 for all n$\in \mathbb{N}$. Suppose that there exists n $\in \mathbb{N}$ such that card $cl_1\{x_n\} \cap cl_2\{x_n\}$=k where k>1. Let A=$x_j$: card($cl_1\{x_j\} \cap cl_2\{x_j\}) \geq k$. B={$x_n$: n$\in \mathbb{N}$}$\setminus$ ($cl_1\{A\} \cap cl_2$ {A}), $D_1 = A \cup B$ and $D_2 = D_1 \cup$ {y}, where y$\in cl_1\{x_n\} \cap cl_2\{x_n\}$ and y$\neq x_n$. Observe that D_1 and D_2 are p-dense and card $cl_1\{y\} \cap cl_2\{y\} < k$. Suppose f is p-homeomorphism satisfying f$(D_1) = D_2$, then we have f(A)=A and if there exists $x_i \in$ B such that f(x_i)=y and f(x_m)=x_n for some $x_m \in$ A, since $x_i \notin cl_1\{x_m\} \cap cl_2\{x_m\}$ then $y \notin cl_1\{x_n\} \cap cl_1\{x_n\}$ which is a contradiction.

Corollary 3.4. *If (X, τ_1, τ_2) is p_1-CDH then it is p-T_1.*

proof. *Let (X, τ_1, τ_2) be a p_1-CDH bitopological space, hence it is p-CDH, by Theorem 3.3, it is p-T_1*

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