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LEFT ADJOINT OF PULLBACK Cat^1-GROUPS

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Abstract

In [1] we define the pullback Cat^1-groups and showed that the category of pullback Cat^1-groups is equivalent to the category of pullback crossed modules. In this paper we proved that the pullback Cat^1-group has a left adjoint which is the induced Cat^1-group. We also give the left adjoint construction.

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1. Introduction

Crossed modules are usefully regarded as 2-dimensional forms of groups. They were introduced by J. H. C. Whitehead in [13], and have powerful topological applications [5, 6, 7, 12]. Loday in [8] showed that the category of crossed modules is equivalent to that of cat^1-groups. We implemented crossed modules and cat^1-groups structures to the computed using the group theory language GAP [10] as a package in [11]. We also enumerated cat^1-groups of low order and group order 41-47 in [2] and [1] using this program package XMOD.

Our aim is to define pullback cat^1-groups and to show that the equivalence between cat^1-groups and crossed modules due to Loday [5] takes pullback cat^1-groups to the pullback crossed modules defined by Brown and Higgins in [3].

2. Pre-cat^1-groups and Pullback cat^1-groups

A crossed module $\chi = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary of* χ*, together with an action* $\alpha : R \to Aut(S)$ *satisfying, for all* $s, s' \in S$ *and* $r \in R$*,*

$$\begin{aligned} \mathbf{XM1}: \quad \partial(s^r) &= r^{-1}(\partial s)r \\ \mathbf{XM2}: \quad s^{\partial s'} &= s'^{-1} s s'. \end{aligned}$$

The standard examples of crossed modules are:

1. *Any homomorphism* $\partial : S \to R$ *of abelian groups with* R *acting trivially on* S *may be regarded as a crossed module.*
2. *A conjugation crossed module is an inclusion of a normal subgroup* $S \trianglelefteq R$*, where* R *acts on* S *by conjugation.*
3. *A central extension crossed module has as boundary a surjection* $\partial : S \to R$ *with central kernel, where* $r \in R$ *acts on* S *by conwugation with* $\partial^{-1} r$*.*
4. *An automorphism crossed module has as its range a subgroup* R *of the automorphism group Aut (S) of* S *which contains the inner automorphism group of* S*. The boundary maps* $s \in S$ *to the inner automorphism of* S *by* s*.*
5. *An R-module crossed module has an* R*-module as source and* ∂ *as the zero map.*
6. *The direct product* $\chi_1 \times \chi_2$ *of two crossed modules has source* $S_1 \times S_2$*, range* $R_1 \times R_2$ *and boundary* $\partial_1 \times \partial_2$*, with* R_1, R_2 *acting trivially on* S_2, S_1 *respectively.*
7. *An important motivating topological example of crossed module due to Whitehead [12] is the boundary* $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ *from the second relative homotopy group of a based pair* (X, A, x) *of topological spaces, with the usual action of the fundamental group* $\pi_1(A, x)$*.*

A morphism between two crossed modules $\chi = (\partial : S \to R)$ *and* $\chi' = (\partial' : S' \to R')$ *is a pair* (σ, ρ)*, where* $\sigma : S \to S'$ *and* $\rho : R \to R'$ *are homomorphisms satisfying*

$$\partial'\sigma = \rho\partial, \; \sigma(s^r) = (\sigma s)^{\rho\tau}.$$

In [8], Loday reformulated the notion of a crossed modules as a cat^1 *-group, namely a group* G *with a pair of homomorphisms* $t, h : G \to G$ *having a common image* R *and satisfying certain axioms. We find it convenient to define a pre-*cat^1*-group* $\mathcal{C} = (e; t, h : G \to R)$ *as a group* G *with two surjections* $t, h : G \to R$ *and an embedding* $e : R \to G$ *satisfying:*

$$\textbf{CAT1}: \quad te = he = id_R.$$

The pre-cat^1*-group* $\mathcal{C} = (e, t, h : G \to R)$ *is a* cat^1*-group if it also satisfies*

$$\textbf{CAT2}: \quad [ker t, ker h] = \{1_G\}.$$

The maps t, h *are often called the source and target, but we choose to call them tail and head of* $\mathcal{C}$*, because source is the GAP term for the domain of a function.*

A morphism $\mathcal{C} \to \mathcal{C}'$ *of* cat^1*-groups is a pair* (γ, ρ) *where* $\gamma : G \to G'$ *and* $\rho : R \to R'$ *are homomorphisms satisfying*

$$h'\gamma = \rho h, \; t'\gamma = \rho t, \; e'\rho = \gamma e.$$

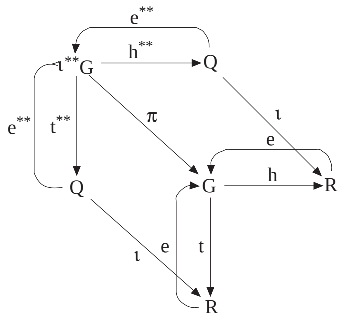
To any pre-cat^1*-group* $\mathcal{P}$ *there is a canonically associated a* cat^1*-group* $\mathcal{C}$*, obtained by quotienting the soruce group by the Peiffer subgroup [ker t, ker h].*

The corresponding functor is denoted

$$\textbf{ass} : \textit{(pre-}cat^1\textit{-groups)} \to \textit{(}cat^1\textit{-groups)}, \tag{0.1}$$

and is clearly the identity when restricted to cat^1*-group [6]*

A pullback cat^1*-group is defined in [1] as follows.*



Let $\mathcal{C} = (e; t, h : G \to R)$ be a cat^1-group and let $\imath : Q \to R$ be a group homomorphism. Define $e^{**}; t^{**}, h^{**} : \imath^{**}G \to Q$ to be the pullback of G where

$$\imath^{**}G = \{(q_1, g, q_2) \in Q \times G \times | \imath q_1 = tg, \imath q_2 = hg\},$$

$t^{**}(q_1, g, q_2) = q_1, h^{**}(q_1, g, q_2) = q_2$ and $e^{**}(q) = (q, e\imath q, q)$. Multiplication in $\imath^{**}G$ is componentwise. The pair $(\pi, \imath)$ is a morphism of cat^1-groups where $\pi : \imath^{**}G \to G, (q_1, g, q_2) \mapsto g$.

Proposition 2.1 *[1] If $\imath^{*}\chi$ is the pullback of the crossed module χ over $\imath : Q \to R$ and if $\mathcal{C}, \mathcal{D}$ are the cat^1-groups obtained from $\chi, \imath^{*}\chi$ respectively, then $\mathcal{D} \cong \imath^{**}\mathcal{C}$.*

3. Construction of the left adjoint

Proposition 3.1. *The category of cat^1-groups is co-complete.*

Proof. Let $F : C \to$ (cat^1-groups). We wish to construct colim F. For each object c of C, we write

$$F(c) = (e_c; t_c, h_c : F_1(c) \to F_0(c)).$$

Then we form

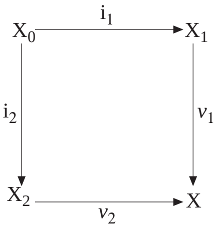
$$\begin{aligned} F' &= (i'; t', h' : \mathrm{colim}_c F_1(c) \to \mathrm{colim}_c F_0(c)) \\ &= (e'; t', h' : F_1' \to F_0'), \end{aligned}$$

where F'_1, F'_0 are the colimits in the category of groups, so that $t'e' = h'e' = 1$. So F' is a pre-cat^1-groups. The required colimit is the the associated cat^1-group **ass** F' (see (1) on page 4),

$$\mathbf{ass}F' = (e''; t'', h'' : F'_1/[\ker t', \ker h'] \to F'_0).$$

□

We recall the definition of pushouts in a general category. Suppose we are given a commutative diagram of morphisms in a category **C**:



Recall [9] that (v_1, v_2) is pushout of (i_1, i_2), and also that the above square is a pushout square, if the following property holds: if $f_1 : X_1 \to H$, $f_2 : X_2 \to H$ are morphisms such that $f_1 i_1 = f_2 i_2$ then there is a unique map $f : X \to H$ such that $fv_1 = f_1$, $fv_2 = f_2$.

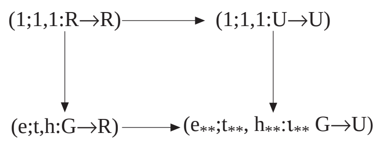
As usual, this property characterizes the pair (v_1, v_2) up to an automorphism of X. For this reason, it is common to make an abuse of language and refer to X as the pushout of (i_1, i_2). In this case, we write

$$X = X_2 *_{X_0} X_1,$$

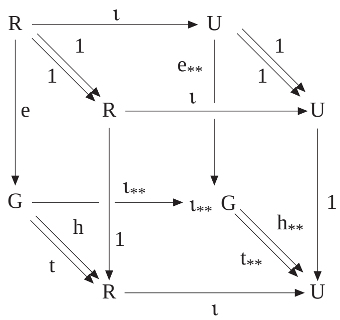
where $*_{X_0}$ is used to suggest a free product.

Proposition 3.2. *The functor $\imath^{**}$: Cat^1 $Grp/U \to Cat^1\, Grp/R$ has a left adjoint $\imath_{**}$: Cat^1 grp/R→ Cat^1 Grp/U.*

Proof. We can give the left adjoint construction as follows. Let $\mathcal{C} = (e; t, h : G \to R)$ be at cat^1-group over R and $\imath : R \to U$ is a morphism of groups. Then the induced cat^1-group is $\imath_{**}\mathcal{C} = (e_{**}; t_{**}.h_{**} : \imath_{**}G \to U)$ is given by the pushout



we draw the above diagram as a three dimensional diagram as follows



in the category of cat^1-groups. For computational purposes note that by the previous proposition 3.1, $\imath_{**}G = (G *_R U)/[ker\ t_{**}, ker\ h_{**}]$, where $*_R$ denotes coproduct of groups, that is, a free product with amalgamation over R. □

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